Optimal Insurance Pricing, Reinsurance, and Investment for a Jump Diffusion Risk Process under a Competitive Market

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Abstract

We extend previous research by considering the role of reinsurance in hedging underwriting risk, pricing risk, and investment risk. We consider a stochastic dynamic optimization model applied to the problem of insurance pricing under a competitive insurance market with a jump diffusion risk process. Our model seeks to maximize the expected utility of the insurer’s terminal wealth, incorporating the interaction of a stochastic process for the insurance price evolution, reinsurance, investment strategy, and the possible hedging effect between insurance liabilities and investment risk. We solve this optimization problem by constructing a Hamilton–Jacobi–Bellman (HJB) equation.

Keywords: stochastic dynamic optimization, insurance demand, insurance price, investment portfolio, jump diffusion

Introduction

Determining the insurer’s optimal investment strategy, as well as the joint investment and reinsurance strategy, has received considerable attention in the literature, including Browne (1995), Hipp and Plum (2000), Liu and Yang (2004), Browne (2000), Korn (2005), Korn and Seifried (2009), Mataramvura and Åksendal (2008), Promislow and Young (2005), Zhang and Siu (2009) and Cao and Wan (2009). Mao et al. (2013) review the related literature.

Schmidli (2002) considers a classical model with drift and discusses the optimal decision on investment and reinsurance strategy by establishing the objective function of minimizing the ruin probability via solving a Hamilton-Jacobi-Bellman equation.
Luo, Taksar, and Tsoi (2008) consider a problem of optimal reinsurance and investment for an insurer whose surplus is governed by a linear diffusion. Their main goal is to find the optimal investment and reinsurance strategy which minimizes the probability of ruin. Edoli and Runggaldier (2010) study the optimization of reinsurance and investment. They assume that the arrival of a claim and the change of the price of the underlying asset(s) corresponds to a Poisson point process. The objective is to maximize the expected total utility. And as a special case, they also discuss the maximizing exponential utility functions whereby negative values of the risk process are penalized.

Lin and Li (2011) consider an optimal reinsurance-investment problem of an insurer whose surplus process follows a jump-diffusion model. The dynamics of the risky asset are governed by a constant elasticity of variance model to incorporate conditional heteroscedasticity. The objective of the insurer is to choose an optimal insurance-investment strategy so as to maximize the expected exponential utility of terminal wealth. Eisenberg and Schmidli (2011) study optimal control of a classical risk model and its diffusion approximation. They assume that the individual claims are reinsured fully or partially and they also assume the insurer is allowed to invest in a riskless asset with some constant interest rate. The objective is to minimize the discounted capital injections. They find explicit optimal solutions by solving an HJB equation. Liu, Yiu, Siu and Ching (2013) study an optimal investment–reinsurance problem for an insurer who faces dynamic risk constraint in a Markovian regime-switching environment.
The issue of optimal setting of insurance price has also been studied extensively. The works cited above address the issue of finding the optimal investment portfolio of the insurers, or investment-reinsurance strategy under the assumption that the price of the insurance product is given, and stated per unit of exposure. However, in reality the risks and returns of investment portfolios vary, and they may affect the price of insurance. Additionally, the quantity of insurance demanded will affect the insurance price and, under some circumstances, the risk of liabilities assumed may be balanced with the investment risk. Taylor (1986) developed a theory in which the premium rate may be optimally determined by considering the effects of competition. Emms (2005) studied determination of premium using optimal control theory by maximizing the terminal wealth of an insurer and considering the insurance demand function. Emms (2007) extended his research by calculating the premium using dynamic programming with the objective of maximizing the expected utility of the insurer’s terminal wealth. Emms, Haberman, and Savoulli (2007) studied optimal premium pricing of insurance in a competitive market using approximation method and simulation of sample paths. They assumed that the market average price is a diffusion process with the premium as a control function, with the objective of maximizing the insurer’s expected total wealth over a finite time horizon. Emms and Haberman (2009) determined the optimal premium strategy in a competitive market using a deterministic general insurance model.

However, their research does not consider the effect of the insurer’s investment strategy on the price of insurance, nor the effect of the insurance price as a stochastic
process on the investment strategy, and the possible hedging effect, which was represented by increasing the number of policies written to hedge investment risk. Mao et al. (2013) and Carson et al. (2013) incorporate investment risk and the market average price uncertainty in the pricing decision of an insurer. They assume that the price, the investment, and the insured losses are stochastic processes, while simultaneously considering the effect of demand on price, assuming constant price elasticity of demand.\(^1\)

Because reinsurance is important in insurer operations, we extend the above approach by considering the role of reinsurance in hedging underwriting risk, pricing risk, and investment risk. We consider reinsurance not only as a hedge against underwriting risks, but as a hedge to pricing and investment risk. For simplicity, we assume that all insurers sell insurance based on the same price process, and this actually results in a perfectly competitive market, with all insurers charging the same price. We construct a Hamilton–Jacobi–Bellman (HJB) equation and determine the optimal price of insurance and optimal investment strategy by solving it.

The paper is organized as follows: Section 2 presents the models. In Section 3, the HJB equation is given and the optimal price and investment portfolio of the insurer are determined. Section 4 carries out sensitivity analysis, and in Section 5 we give our conclusions.

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\(^1\) A commonly used assumption in economic literature is that of a linear demand function. Of course this assumption is restrictive, and represents simplification of the complexity of economic interaction of buyers and sellers, but it has the advantage of being intuitively appealing and giving a range of elasticities of demand.
2. The Models

In a manner analogous to that of Mao et al. (2013), we assume that the insurance company can invest its wealth in one risk-free asset and one risky asset, which can be traded continuously over time, without any transaction costs or taxes. Let \( \{W_t(t); t \in [0, T]\} \) be a standard Brownian motion on a filtered probability space, and \( \mathcal{F}_t \) is the \( P \)-augmentation of the natural filtration. We then assume that the price process of risky investment, i.e., \( S(t) \) evolves over time according to the following constant variance model:

\[
dS(t) = \mu S(t) dt + \sigma S(t) dW_t(t), \quad \text{with } S(0) = s_0, \tag{1}
\]

where \( \mu \) is the drift of the process and \( \sigma \) is the volatility of the process. The price of the risk-free asset is assumed to evolve according to the equation

\[
 dB(t) = r B(t) dt , \tag{2}
\]

where \( r \) is the risk-free interest rate, with \( r < \mu \). The average market price of insurance \( \bar{p} \) (per unit of exposure) is assumed to follow a geometric Brownian motion satisfying

\[
d \bar{p}(t) = \bar{p}(t) \left( \mu_p dt + \sigma_p dW_3(t) \right), \tag{3}
\]

where \( \mu_p \) and \( \sigma_p \) are appropriate drift and volatility parameters. We also assume that the price \( p \) for insurance charged by a specific company is determined from the value of \( \bar{p} \) via

\[
p(t) = k \bar{p}(t), \tag{4}
\]

with \( k \) being a company-specific fixed parameter. We also assume that surplus of the insurer without reinsurance and investment at time \( t \) follows the following jump
diffusion stochastic process: \( R(t) = x + p(t) - \sum_{i=1}^{N(t)} Y_i + \gamma W_3(t) \) \hspace{1cm} (5)

where the constant \( x \) is the initial surplus, \( p(t) \) is the price of insurance products, \( \{N(t); t \in [0, T]\} \) is a poisson process with a constant intensity \( \lambda \), where \( \lambda > 0 \) and \( N(t) \) represents the number of claims up to time \( t \), \( \{Y_i, i = 1, 2, \ldots\} \) is a sequence of independent and identically distributed nonnegative random variables with a common distribution \( G(y) \), having a finite mean \( \nu \), where \( Y_i \) is the amount of the \( i \)th claim, \( \{W_3(t); t \in [0, T]\} \) be a standard Brownian motion on a filtered probability space, and \( \mathcal{F}_i \) is the \( P \)-augmentation of the natural filtration and the constant \( \theta \) represents the volatility parameter or the diffusion parameter.

The price of insurance product affects the quantity demanded, and the relationship is expressed by the demand function. Mao et al. (2013) discuss the dynamic pricing of insurance under the assumption of constant price elasticity demand. The linear demand functions are also often used in economics literature (Lilien and Kotler, 1983, Mankiw, 2008), and we use it here in the form

\[ p(t) = D - Eq(t) \] \hspace{1cm} (6)

where \( D \) and \( E \) are constants, \( q(t) \) is the demand for insurance product at time \( t \), and \( p(t) \) satisfies its stochastic differential equation:

\[ dp(t) = k p(t) \left( \mu_p dt + \sigma_p \, dW_3(t) \right) = p(t) \left( \mu_p dt + \sigma_p \, dW_3(t) \right). \]

We also assume that insurance contracts are independent of each other.

Let \( \{X(t); t \in [0, T]\} \) be the wealth process of the insurance company considered. The wealth process of the insurer who sells the quantity of insurance
contracts, $q(t)$, at time $t$ and reinsurances the proportion of $1 - a(t)$ to the ceded company can be described by the following stochastic differential equations:

$$dX(t) = \left( p(t)q(t)(a(t)(1 + \eta) - \eta) + \pi(t)(\mu - r) + X(t)r \right) dt + \sigma \pi(t) dW_1(t) + \theta dW_2(t) - d \left( q(t) \cdot a(t) \cdot \sum_{i=1}^{N(t)} Y_i \right)$$

$$dp(t) = \tilde{p}(t) \left( \mu_p dt + \sigma_p dW_3(t) \right)$$

where $\{W_1(t), W_3(t) \mid \mathcal{F}_t, t \geq 0\}$ are two correlated standard Brownian motions on a filtered probability space, and $\mathcal{F}$ is the $P$-augmentation of the natural filtration, $a(t)$ is the proportion of retention at time $t$ and $\eta$ is the ratio of reinsurance cost. Furthermore, the correlation matrix is

$$\left( \begin{array}{cc} 1 & \rho_{31} \\ \rho_{31} & 1 \end{array} \right)$$

$W_2(t)$ is an independent stochastic process and $\pi(t)$ is allocation of the investment portfolio to the risky investment at time $t$. We assume that $\pi(t)$ may be less than zero, i.e., short selling is permitted. We also assume that borrowing money is allowed, i.e., $\pi(t)$ can be larger than $X(t)$, with the risk-free rate being the borrowing cost.

3. HJB Equation and its Solutions for Optimal Investment and Optimal Price of Insurance

We study the optimal pricing of insurance product and optimal determining of investment strategy of the insurer under the assumption that quantity demanded of the insurance products is a linear function of the price.

We formulate the problem of maximizing the expected utility of the terminal wealth. Given the initial values of time, $t_0$, the initial wealth of the insurer, $X_0$, the
objective functional over the class of admissible controls \( A_{t_0, X_0, p_0} \) is given by

\[
J(t_0 = 0, X_0 = x, p_0 = p_0; (\pi, p)) = E_{t_0, X_0 = x, p_0 = p_0} \left( U(X(T)) \right)
\]  

(8)

The optimization problem can is to find the value function \( V(t, X, p) \) and solutions \( (\pi, p) \in A_{t_0, X_0, p_0} \) that satisfy the condition

\[
V(t, X, p) = \sup_{(\pi, p) \in A_{t_0, X_0, p_0}} \left( J(t, X, p; (\pi, p)) \right)
\]  

(9)

It is not difficult to show that \( V(t, X, p) \) is a Markov process. For any twice continuously differentiable function \( h \in C^{1,2}(O) \cap C(O) \), where

\[
O := (0, T) \cap (0, x) \cap (0, x) \quad \text{and} \quad \overline{O} \quad \text{denotes the closure of} \quad O, \quad \text{there exits partial differential operator} \quad L^{a, p}(h(t, x, p)):
\]

\[
L^{a, p}(h(t, x, p)) = \frac{\partial h}{\partial t} + \frac{p(D - p)}{E} (a(1 + \eta) - \nu + rz) \frac{\partial h}{\partial x} + \frac{D - p}{E} \mu \frac{\partial h}{\partial p} + \\
\left( \sigma^2 \pi^2 + \theta^2 \right) \frac{\partial^2 h}{\partial x^2} + \left( \rho_1 \pi \sigma \sqrt{\frac{D - p}{E}} \right) p \frac{\partial^2 h}{\partial x \partial p} + \left( \rho_2 \right) \frac{D - p}{E} \frac{\partial^2 h}{\partial p^2} + \\
\lambda \frac{D - p}{E} \int V(t, p, x - ay, \frac{D - p}{E}) - V(t, p, x) G \left( \frac{D - p}{E} \right) dy
\]

(10)

We have the following verification theorem.

Theorem 1: Suppose that there exists a function \( \phi_p(t, x, p) \in C^{1,2}(O) \cap C(O) \) and a control (i.e., an optimal solution) \( (\pi^*, p^*) \in A \) such that

1. \( L^{a, p} \left( \phi(t, x, p) \right) \geq 0 \) for all \( (\pi, p) \in A \) and \( (t, x) \in O \);
2. \( L^{a, \pi^*, p^*} \left( \phi(t, x, p) \right) = 0 \) for all \( (t, x) \in O \);
3. For all \( (\pi, p) \in A \): \( \lim_{t \to T} \phi(t, x, p) = U(X^\pi, p(T)) \);
4. Let \( \mathcal{K} \) denote the set of stopping times \( \tau \leq T \). The family \( \left\{ \phi(\tau, x, p) \right\}_{\tau \in \mathcal{K}} \) is uniformly integral for all \( (t, x, p) \in O, (\pi, p) \in A \).
Then \( \phi(t, \ x \ \overline{p}) = V(t, \ x \ \overline{p}) \), and \((\pi^*, a^*, p^*)\) is an optimal control.

Now, let us formulate the optimal investment and pricing problem of maximizing the expected utility of the terminal wealth of insurance company. We consider the case of exponential utility, i.e., we assume that the utility function of the investor is an exponential utility function defined by:

\[
U(x) = -e^{-\gamma x},
\]

where \( \gamma \) is a positive constant and represents the coefficient of absolute risk aversion. This assumption, similarly to our linear demand assumption, is also somewhat restrictive, but it is reasonable and has been use in insurance literature before. In order to obtain the optimal value function \( V(t, \ X, \ \overline{p}) \) and the optimal strategy \((\pi^*, a^*, p^*)\), we only need to solve the following HJB equation (see, e.g., Flaming and Rishel (1975)):

\[
\begin{cases}
\sup_{(\pi, a, \bar{p}) \in A} L_{\pi, a, \bar{p}} \left( V(t, x, \overline{p}) \right) = 0, \\
V(t, x, \overline{p}) = -\frac{e^{-\gamma(x + \overline{p}q(a(1+\eta) - \eta))}}{\gamma},
\end{cases}
\]

(12)

To solve the above HJB equation, we try a solution of the following form:

\[
\phi'(t, \ x, \ \overline{p}) = -\frac{\exp(-\gamma(x + \overline{p}q(a(1+\eta) - \eta)))e^{(T-t)} + f(t)}{\gamma},
\]

(13)

where \( f(t) \) is an undetermined function with \( f(T) = 0 \). We take partial derivatives of the first and second order of (13) with respect of \( x, \ t \) and \( \overline{p} \), and obtain the...
following:

\[
\frac{\partial \phi^1}{\partial x} = -\phi^1 \gamma e^{(T-t)}, \\
\frac{\partial \phi^1}{\partial t} = \frac{\partial \phi^1}{\partial t} \left( r(x + \overline{p} q) \gamma e^{(T-t)} - \mu^p \overline{p} \left( q - \frac{k \overline{p}}{E} \right) \gamma e^{(T-t)} + f_i \right), \\
\frac{\partial \phi^1}{\partial \overline{p}} = -\phi^1 \gamma e^{(T-t)} \left( \frac{D - 2k \overline{p}}{E} \right) (a(1 + \eta) - \eta), \\
\frac{\partial \phi^1}{\partial \overline{x}^2} = \phi^1 \left( \gamma e^{(T-t)} \right)^2, \\
\frac{\partial \phi^1}{\partial \overline{p} \overline{x}} = \phi^1 \left( \gamma e^{(T-t)} \right)^2 \left( \frac{D - 2k \overline{p}}{E} \right) (a(1 + \eta) - \eta), \\
\frac{\partial \phi^1}{\partial \overline{p} \overline{x}} = \gamma e^{(T-t)} \left( \frac{D - 2k \overline{p}}{E} \right)^2 (a(1 + \eta) - \eta) \gamma e^{(T-t)} + \frac{2k}{E} (a(1 + \eta) - \eta). \\
\]

\[V(t, p, x - ay) - V(t, p, x) = [e^{\phi^1 \gamma e^{(T-t)}} - 1]V\]

Substituting the above trial function, equation (4), and equations (14), into equation (11) yields:

\[L^{x,p} \left( \phi^1 (t, x, \overline{p}) \right) = \phi^1 (t, x, \overline{p}) - \left[ f_i - \gamma e^{(T-t)} \left( \frac{p(D - p)}{E} \right) (a(1 + \eta) - \eta) + (\mu - r) p - \left( \frac{rp}{k} D - p \frac{\mu^p (D - 2p)}{E} \right) \right] (a(1 + \eta) - \eta) + \frac{1}{2} \left( \sigma^2 \pi^2 + \theta^2 \right) \left( \gamma e^{(T-t)} \right)^2 + \left( \rho^2 \pi^2 \sigma^2 \sigma^2 \right) p \left( \gamma e^{(T-t)} \right)^2 \left( \frac{D - 2p}{E} \right) (a(1 + \eta) - \eta) - \mu^p \overline{p} \left( \frac{D - p}{E} \right) \left( \frac{D - 2p}{E} \right) (a(1 + \eta) - \eta) \gamma e^{(T-t)} + \frac{1}{2} \rho^2 \left( \frac{D - p}{E} \right) \left( \frac{D - 2p}{E} \right) \left( \gamma e^{(T-t)} \right)^2 (a(1 + \eta) - \eta) \gamma e^{(T-t)} + \frac{2k}{E} (a(1 + \eta) - \eta) + \lambda \frac{D - p}{E} \left( e^{\phi^1 \gamma e^{(T-t)}} - 1 \right) G(dy)\]

(15)

We further simplify our analysis and assume that insurance market is a competitive market and all companies sell insurance for same price, that is,
\[
p(t) = \frac{\sum_{i=1}^{N} p_i(t)}{N} = \frac{\sum_{i=1}^{N} kp_i(t)}{N} = k \bar{p}(t), \text{ implying that } k = 1. \text{ The insurance price } p \text{ in our model is assumed to be equal to the average price } \bar{p}. \text{ Given that, we can assume that } \frac{2k}{E} \text{ in equation (15) is equal to } \frac{2}{E}. \text{ This allows us to set}
\]
\[
\frac{rp}{k} \cdot \frac{D - p}{E} = r_1 \frac{D - p}{E} \text{ in order to simplify the calculations. The first order conditions for optimal control } \left( \pi^*(t), a^*(t), p^*(t) \right) \text{ are:}
\]
\[
- \left( \frac{D - 2p}{E} \left( a(1 + \eta) - \eta \right) + \frac{pD - p)(1 + \eta)}{E} \right) \frac{\partial a}{\partial \eta} + \left( \frac{D - 2p}{E} \right) \left( a(1 + \eta) - \eta \right) + \\
+ \mu \left( \frac{2p(D - p)}{E^2} \left( \frac{D - 2p}{E} \right)^2 \right) \left( a(1 + \eta) - \eta \right) + \mu \frac{D - p}{E} \left( a(1 + \eta) - \eta \right) + \\
+ \frac{1}{2} \sigma^2 \left( \frac{D - 2p}{E} \right)^2 \left( a(1 + \eta) - \eta \right) \gamma e^{(T-t)} + \frac{1}{2} \right) \frac{\partial a}{\partial \eta} + \\
+ p^3 \left( \frac{D - p}{E} \right)^2 \left( a(1 + \eta) - \eta \right) \gamma e^{(T-t)} + \frac{1}{E} \right) \frac{\partial a}{\partial \eta} + \\
+ \lambda \left( \frac{D - p}{E} \gamma e^{(T-t)} - \frac{1}{E} \right) \int_0^{+\infty} \gamma e^{(T-t)} G(dy) = 0
\]

(16)

Where
\[
\pi(t) = \frac{1}{\sigma} \left( \frac{\mu - r}{\sigma \gamma e^{(T-t)}} - \rho \sqrt{\frac{D - p}{E}} \sqrt{p \sigma_p \cdot \frac{D - 2p}{E} (a(1+\eta) - \eta)} \right) 
\]

\[
\frac{\partial \pi}{\partial \rho} = -\frac{1}{\sigma} \rho \sigma_p \sigma_p \left( \frac{D - p}{2} \sqrt{\frac{D - p}{E}} \cdot \frac{D - 2p}{E} - \frac{D - p}{E} \cdot \frac{D - 2p}{E} \right) (a(1+\eta) - \eta) 
\]

\[
a(t) = \frac{C(\eta \left( \frac{D - 2p}{E} \right)^{-1/E} + B - A - D_1}{(1+\eta)C \left( \frac{D - 2p}{E} \right)^2} 
\]

where

\[
A = \left( \rho \sigma_p \sigma_p \sqrt{\frac{D - p}{E}} \cdot \frac{D - 2p}{E} (1+\eta) \right), \\
B = \mu_p \rho \cdot \frac{D - p}{E} \cdot \frac{D - 2p}{E} (1+\eta), \quad \text{and} \\
C = p^2 \cdot \frac{D - p}{E} \cdot \sigma_p \gamma e^{(T-t)} (1+\eta). 
\]

Since

\[
D_1 = \lambda \frac{D - p}{E} \int_0^{a\gamma e^{(T-t)}} yG(dy) = \lambda \frac{D - p}{E} \cdot \frac{1}{(a\gamma e^{(T-t)} - 1)^2}, 
\]

when \(0 < a < 1\)

where we assume that the distribution of claim size is exponential with parameter \(\gamma\)

and the density function of it is \(g(y) = e^{-y}\), then by putting equation (20) into (19),

we get the implicit expression of \(a\) as follows

\[
a(t) = \frac{C \left( \eta \left( \frac{D - 2p}{E} \right)^{-1/E} + B - A - D_1 \right)}{(1+\eta)C \left( \frac{D - 2p}{E} \right)^2} 
\]
\[ \frac{\partial A}{\partial p} = (1 + \eta) \left( \gamma e^{r(T-t)} \right) \left\{ -\frac{D - 2p}{E} \rho_3 \sigma \frac{\partial \pi}{\partial p} \sqrt{\frac{D - p}{E}} - \frac{1}{2E} \frac{\pi}{\sqrt{D - p}} E \right\}, \]
\[ \frac{\partial B}{\partial p} = \mu_p (1 + \eta) \left( \frac{p(-3D + 4p) + D - p}{E^2} + \frac{2D - 3p}{E} \right), \]
\[ \frac{\partial C}{\partial p} = \sigma_p \gamma \sqrt{r(T-t)} (1 + \eta) \left( \frac{2D - 3p}{E} \right) \]
\[ \frac{\partial D}{\partial p} = \lambda \left( \frac{1}{E} - \gamma e^{r(T-t)} \right) \left( \frac{D - p}{E} \right) \int_0^{\infty} e^{ray^{r(T-t)}} G(dy), \]

where \[ \int_0^{\infty} e^{ray^{r(T-t)}} G(dy) = \frac{1}{(\alpha y e^{r(T-t)} - 1)^2} \]

then

\[
\frac{\partial a}{\partial p} = \frac{\left[ C \left( \frac{D - 2p}{E} \right) - \left( 1 + \eta \right) E \left( \alpha y e^{r(T-t)} - 1 \right) \right]}{\left( 1 + \eta \right) \left[ C \left( \frac{D - 2p}{E} \right) + \lambda y e^{r(T-t)} \frac{D - p}{E} \right]}. \]

(22)

Substituting (17) through (22) into (16), we get the implicit function of \( p^*(t) \), but we cannot get the explicit solutions of \( (\pi^*(t), a^*(t), p^*(t)) \) by standard analytical methods. With the help of Matlab and solve system equations of implicit functions (17) and (21), (18) and (22) by interative method, we can get the optimal solution of \( p^*(t) \) at any given values of time \( t, 0 \leq t \leq T \). Substituting \( p^*(t) \) into (6),(18) and (19) we then obtain the optimal solutions of \( q^*(t), a^*, \) and \( \pi^* \) \( t = t_0, t_0 + \Delta t, ..., T \). When \( a^* > 1 \), we set \( a^* = 1 \). The function \( f(t) \) is determined by the following differential equation:
\[ f_i = \gamma e^{(T-t)} \left( p^* \frac{D - p^*}{E} (a^* (1 + \eta) - \eta) + (\mu - r) \pi^* - rp^* \frac{D - p^*}{E} \right) \]

\[ - \frac{1}{2} \left( \theta^2 + \sigma^2 \pi^* \right) \left( \gamma e^{(T-t)} \right)^2 \]

\[ - \left( \rho \pi \sqrt{\frac{D - p^*}{E} \sigma^2} \right) p^* \left( \gamma e^{(T-t)} \right)^2 \frac{D - 2p^*}{E} \left( a^* (1 + \eta) - \eta \right) \]

\[ + \mu_p p^* \frac{D - p^* D - 2p^*}{E} \gamma e^{(T-t)} (a^* (1 + \eta) - \eta) \]

\[ - \frac{1}{2} \left( p^* \right)^2 \frac{D - p^*}{E} \sigma^2 p^* e^{(T-t)} \left( \frac{D - 2p^*}{E} \right)^2 \frac{2}{E} \left( a^* (1 + \eta) - \eta \right)^2 \]

\[ + \lambda \frac{D - p^*}{E} \int_0^{\infty} \left[ e^{\alpha^* p^* y^{(T-t)}} - 1 \right] G(dy) \]

(23)

We show now that our solution is a global optimum based on the implicit function theorem.

Substituting equations (17), (18), (19), (20) into equation (16) and writing it as

\[ G(t, p) = 0, \]

we can find the Jacobian as:

\[ (DG)(c, d) = \left[ \frac{\partial G}{\partial t}(c, d) \quad \frac{\partial G}{\partial k}(c, d) \right], \]

where \((c, d)\) is a point which satisfies \(G(c, d) = 0\). Since \(G(t, p)\) is a combination of elementary functions, \(\frac{\partial G}{\partial t}\) and \(\frac{\partial G}{\partial p}\) are differentiable in the domain of function \(G(t, p)\). By the implicit function theorem, we see that we can write the form

\[ p = p(t) \]

for all points where \(\frac{\partial G}{\partial p} \neq 0\). And the resulting solution of \(p^*(t)\) is sufficiently smooth so that a verification theorem can be applied.

With the boundary condition of \(f(T) = 0\), where \(\pi^*, a^*\) and \(p^*\) satisfy the system of equations of (16), through (22). From the above, analysis, we obtain the
following:

Theorem 2: When the expected utility function of the terminal wealth of the insurer is exponential and the demand function for insurance products is linear, the optimal strategy \((\pi^*, a^*, p^*)\) is given by system of equations (16), (18) and (19), the optimal price \(q^*\) is determined as \(p^* = D - Eq^*\) and the optimal value function is:

\[
V(t, x, \bar{p}) = -\exp\left(-\gamma(x + \bar{p}q(a(1+\eta) - \eta))e^{(r-t)} + f(t)\right)/\gamma,
\]

where \(f(t)\) is given by equation (23).

From equation (17), we see that \(\pi^*\) is directly related to the drift rate \(\mu\) of the risky assets process, but inversely related to the volatility of return rate of risky assets \(\sigma\) and risk-free interest rate \(r\). Finally, \(\pi^*\) is inversely related to the coefficient of risk aversion \(\gamma\). More detailed numerical analysis of these relationships will be presented in the next section. It should be noted that the optimal solution \(\pi^*\) may be less than zero when \(D < 2p^*\) (see equation (17), and in that case the optimal investment strategy is to short the risky assets, while investing more than 100\% of the portfolio in the risk-free asset.

4. Numerical Analysis

In our analysis we assume the following values of the parameters:

\[
\mu = 0.10, \sigma = 0.10, \theta = 0.15, \rho_{13} = 0.2, \gamma = 2, T = 8, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1,
\]

\[
\mu_p = 0.05, \sigma_p = 0.15, r = 0.05.
\]

We obtain the following optimal solutions when time \(t\) takes values of 1 through 20.

Table 1 Optimal solutions when \(t\) takes values of 1 through 20.
From Table 1 we see that optimal price and optimal amount of risky assets increases with time. However, optimal quantity demanded and optimal proportion of retention decreases with time. An aggressive investment strategy corresponds with having less retention in order to hedge the investment risk. In the following, we will study the sensitivity of the model by varying the parameters.
4.1. Varying the parameters $\mu$, $\sigma$, $\mu_p$, $\sigma_p$ and $r$

Figure 1 (a) through Figure 1(o) display the change pattern in the values of the optimal price, $p^*$, the optimal risky assets, $\pi^*$ and the optimal retention $a$ when the return rate, $\mu$, the standard deviation of the risky assets, $\sigma$, the shift and volatility of insurance price, $\mu_p$, and $\sigma_p$ and the risk free interest rate $r$ change.

The curve surfaces in Figures 1 (a) (b) and (c) show that the optimal price and optimal proportion of retention do not change whatever value of $\mu$ is taken on and increasing the value of $\mu$ will slightly increase optimal amount of risky assets. Figure 1 (d) and (f) show that the optimal value of $p^*$ and the optimal proportion of retention $a^*$ has no change whatever value of volatility of risky assets $\sigma$ is taken on. Figure 1 (e) shows that the optimal value of $\pi^*$ decreases with the increase of volatility of risky assets, $\sigma$, and optimal value of $\pi^*$ is rather sensitive to the change of $\sigma$. The possible explanation can be deducted from equation (17), where the value of second term of its right hand side is much larger than the first term and $\mu$ only appears in the first term but $\sigma$ affects the results of both terms.

From Figure 1 (g) through (i) we know that increasing the shift of the insurance price $\mu_p$ or decreasing the volatility of insurance price $\sigma_p$ will increase optimal price and amount of risky assets and decrease optimal proportion of retention, which means lower price risk will promote the insurer to take more aggressive investment strategy. And the optimal solutions of $p^*$ and $\pi^*$ are very sensitive to the change of the shift and volatility of insurance price. But the optimal retention $a^*$ is not sensitive to the change of parameters of $\mu_p$ and $\sigma_p$. 

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From Figure 1 (m), (n) we find that the optimal price and the optimal amount of risky assets decrease as the increase of risk free interest rate increases and optimal solutions of $p^*$ and $\pi^*$ are rather sensitive to the change of $r$. From Figure 1(o), we find that the optimal retention $a^*$ slightly increases with the increase of $r$. The safety of higher risk free interest rate will promote the insurer to increase the retention.

$$r = 0.05, \sigma = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15, \gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1$$

**Figure 1 (a).** Change in the optimal price of insurance products $p^*$ as the means of return rate of risky assets $\mu$ and time $t$ change
Figure 1 (b). Change of the optimal risky assets of insurer $\pi^*$ as the means of return rate of risky assets $\mu$ and time $t$ change.

$r = 0.05, \sigma = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15, \\
\gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1$

Figure 1 (c). Optimal proportion of retention $a^*$ as the means of return rate of risky assets $\mu$ and time $t$ change.

$r = 0.05, \sigma = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15, \\
\gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1$
Figure 1 (d). Optimal price of insurance products $p^*$ as the volatility of risky assets $\sigma$ and time $t$ change.

$$r = 0.05, \mu = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15,$$
$$\gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1$$

Figure 1 (e). Optimal amount of risky assets $\pi^*$ as the volatility of risky assets $\sigma$ and time $t$ change.

$$r = 0.05, \mu = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15,$$
$$\gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1$$
Figure 1 (f). Optimal proportion of retention $a^*$ as the volatility of risky assets $\sigma$ and time $t$ change

$$r = 0.05, \mu = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15, \gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1$$

Figure 1 (g). Optimal price $p^*$ as the drift of insurance price $\mu_p$ and time $t$ change

$$r = 0.05, \mu = 0.10, \sigma = 0.10, \rho_{13} = 0.2, \sigma_p = 0.15, \gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1$$
Figure 1 (h). Optimal amount of risk assets $\pi^*$ as the drift of insurance price $\mu_p$ and time $t$ change.

$$r = 0.05, \mu = 0.10, \sigma = 0.10, \rho_{13} = 0.2, \sigma_p = 0.15,$$
$$\gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1$$

Figure 1 (i). Optimal proportion of retention $a^*$ as the drift of insurance price $\mu_p$ and time $t$ change.

$$r = 0.05, \mu = 0.10, \sigma = 0.10, \rho_{13} = 0.2, \sigma_p = 0.15,$$
$$\gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1$$
Figure 1 (j). Optimal price \( p^* \) as the volatility of insurance price \( \sigma_p \) and time \( t \) change

\[
\begin{align*}
\gamma &= 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1
\end{align*}
\]

Figure 1 (k). Optimal amount of risk assets \( \pi^* \) as the volatility of insurance price \( \sigma_p \) and time \( t \) change

\[
\begin{align*}
r &= 0.05, \mu = 0.10, \sigma = 0.10, \rho_{13} = 0.2, \mu_p = 0.05,
\gamma &= 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1
\end{align*}
\]
Figure 1 (l). Optimal proportion of retention $a^*$ as the volatility of insurance price $\sigma_p$ and time $t$ change.

\begin{align*}
    r &= 0.05, \mu = 0.10, \sigma = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \\
    \gamma &= 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1
\end{align*}

Figure 1 (m). Optimal price $p^*$ as the risk free interest rate $r$ and time $t$ change.

\begin{align*}
    \mu &= 0.10, \sigma = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15 \\
    \gamma &= 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1
\end{align*}
\[
\begin{align*}
\mu &= 0.10, \sigma = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15 \\
\gamma &= 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1
\end{align*}
\]

Figure 1 (n). Optimal amount of risky assets \( \pi^* \) as the risk free interest rate \( r \) and time \( t \) change

\[
\begin{align*}
\mu &= 0.10, \sigma = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15 \\
\gamma &= 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1
\end{align*}
\]

Figure 1 (o). Optimal proportion of retention \( a^* \) as the risk free interest rate \( r \) and time \( t \) change
4.3. Varying the parameters $\eta, \gamma, \lambda$ and $\rho_{13}$

In the following, we also study the sensitivity of the model by varying the parameters $\eta, \gamma, \lambda$ and $\rho_{13}$ in turn in Figure 2 (a), (b), (c), (d), (e), (f), (g), (h), (i), (j), (k) and (l). The plots show that the optimal price and the optimal risky assets in Figure 2 (a) and (b) unchange even though the ratio of the reinsurance cost changes. From Figure 2 (c) we find that optimal proportion of retention increases with the increase of the ratio of the reinsurance cost. Figures 2 ((d) and (e)) show that the coefficient of risk aversion, $\gamma$, will affect the optimal solutions of $p^*$, $\pi^*$ and $a^*$. Increasing the value of $\gamma$ will decrease the optimal price and decrease the proportion of the risky assets of the insurer. And the decrease of the optimal price will increase the quantity demanded so as to hedge the underwriting risk. From Figure 2 (f) we find that optimal proportion of retention slightly increases with the increase of $\gamma$. Because higher quantity demanded hedges the underwriting risk so that the insurer can increase the proportion of retention. From Figure 2 (g), (h), we find that optimal price and optimal amount of risky assets are directly related to the parameter $\lambda$, which means that higher claim size leads to higher optimal insurance price and higher funds which will promote the insurer to select more aggressive investment strategy. From Figure 2 (i) we find that increasing $\lambda$ will slightly decrease the optimal proportion of retention in order to decrease the underwriting risk. However, from equation (21) we know that there are several factors affecting the proportion of retention. Higher quantity demanded (the
optimal quantity demanded, \( \frac{D - p}{E} = 2.9482 \times 10^4 \) when \( t = 20 \) in our case) will automatically hedge underwriting risk so \( \lambda \) has very small effect on the retention proportion. However, it do not meaning that other factors will not affect the optimal proportion of retention. From Figure 2 (j), (k) and (l), we find that the correlation coefficient between insurance price and risky investment \( \rho_{13} \) has little influence on the optimal price and optimal retention proportion, but have great effect on the optimal investment strategy. Greater positive value of \( \rho_{13} \) will result in greater optimal amount invested in risky assets. Figure 2 (m) shows that negative value of \( \rho_{13} \) will result in short position of risky assets. Table 2 summarizes the change directions of optimal solutions when the parameters of \( \mu, \sigma, \mu_p, \sigma_p, r, \eta, \gamma, \lambda, \rho_{13} \) change.

\[
\begin{array}{cccccccc}
\text{time} & 0 & 5 & 10 & 15 & 20 \\
\text{the ratio of reinsurance cost} & 0.08 & 0.1 & 0.12 & 0.14 & 0.16 \\
\end{array}
\]

\[
r = 0.05, \mu = 0.10, \sigma = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15, \gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1
\]

Figure 2 (a). Optimal price of insurance products \( p^* \) as the ratio of reinsurance cost \( \eta \) and time \( t \) change
$r = 0.05, \mu = 0.10, \sigma = 0.10, \rho_{12} = 0.2, \mu_p = 0.05, \sigma_p = 0.15$

$\gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-3}, \lambda = 1$

Figure 2 (b). Optimal amount of risk assets $\pi^*$ as the ratio of reinsurance cost $\eta$ and time $t$ change

$\sigma_p = 0.10, \mu = 0.10, \rho_{12} = -0.1, \rho_{13} = -0.2, \rho_{23} = 0.2, \mu_p = 0.05, p_1 = 0.1$

$\sigma_p = 0.10, \sigma = 0.10, \gamma = 2, T = 8, D = 0.22, E = 1.0 \times 10^{-3}$

Figure 2 (c). Optimal proportion of retention $\alpha^*$ as the risk-free interest rate $r$ and time $t$ change
\[ r = 0.05, \mu = 0.10, \sigma = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15 \]
\[ T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1 \]

Figure 2 (d). Optimal price of insurance products \( p^* \) as the coefficient of risk aversion \( \gamma \) and time \( t \) change

\[ r = 0.05, \mu = 0.10, \sigma = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15 \]
\[ T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1 \]

Figure 2 (e). Optimal amount of risk assets \( \pi^* \) as the coefficient of risk aversion \( \gamma \) and time \( t \) change
Figure 2 (f). Optimal proportion of retention $a^*$ as the coefficient of risk aversion $\gamma$ and time $t$ change

$$r = 0.05, \mu = 0.10, \sigma = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15$$

$$T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \lambda = 1, \eta = 0.1$$

Figure 2 (g). Optimal price of insurance products $p^*$ as the intensity of claim $\lambda$ and time $t$ change

$$r = 0.05, \mu = 0.10, \sigma = 0.10, \rho_{13} = 0.2, \mu_p = 0.05, \sigma_p = 0.15$$

$$\gamma = 2, T = 20, D = 0.5, E = 1.0 \times 10^{-5}, \eta = 0.1$$
Figure 2 (h). Optimal amount of risky assets $\pi^*$ as the the intensity of claim $\lambda$ and time $t$ change

Figure 2 (i). Optimal proportion of retention $p^*$ as the the intensity of claim $\lambda$ and time $t$ change
Figure 2 (j). Optimal price of insurance products $p^*$ as the correlation coefficient of $\rho_{13}$ and time $t$ change

\[ r = 0.05, \mu = 0.10, \sigma = 0.10, \mu_p = 0.05, \sigma_p = 0.15 \]
\[ \gamma = 2, T = 20, \lambda = 1, D = 0.5, E = 1.0 \times 10^{-5}, \eta = 0.1 \]

Figure 2 (k). Optimal amount of risky assets $\pi^*$ as the correlation coefficient of $\rho_{13}$ and time $t$ change ($\rho_{13} \geq 0$)
\[ r = 0.05, \mu = 0.10, \sigma = 0.10, \mu_p = 0.05, \sigma_p = 0.15 \]
\[ \gamma = 2, T = 20, \lambda = 1, D = 0.5, E = 1.0 \times 10^{-5}, \eta = 0.1 \]

**Figure 2 (l).** Optimal proportion of retention \( a^* \) as the correlation coefficient of \( \rho_{13} \) and time \( t \) change

\[ \rho_{13} \leq 0 \]

**Figure 2 (m).** Optimal amount of risky assets \( \pi^* \) as the correlation coefficient of \( \rho_{13} \) and time \( t \) change \( (\rho_{13} \leq 0) \)
Table 2. The direction of the change of the optimal solutions when the parameters change ((+) indicates the direction of the change of the optimal solution is the same as that of the parameters, while (-) indicates the direction of the change of the optimal solution is opposite to that of the parameters)

<table>
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<tr>
<th>parameters</th>
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<th>$q^*$</th>
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<th>$a^*$</th>
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<td>(+)slightly</td>
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<tr>
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<td>$t$</td>
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Conclusions

We have found the optimal price, reinsurance, and investment strategy for an insurer using stochastic optimal control theory. We establish a pricing model with the assumptions that the price, the investment, and the claim loss rate are stochastic processes, and we also assume that the insurance price is a linear function of the demand in a competitive insurance market. The objective of our model is to maximize the expected utility of terminal wealth of the insurer. We establish a HJB equation and obtain the optimal solutions of dynamic insurance price, reinsurance, and investment strategy using stochastic optimal control theory. We also study the sensitivity of the models by varying the parameters. The results in this paper also show that the shift
and volatility of insurance price, the parameter of claim size, and other parameters will affect the optimal price, reinsurance, and investment strategy. This work could have possible applications for insurance firm, which would like to seek optimal dynamic management strategy, and using a model similar to ours as a permanent enterprise risk management tool. It should be noted that regulatory developments, for example Solvency II in the European Union, encourage or even require such integrated risk management approach. Let us also note that our model can be further extended by considering more complex structure of risks, e.g., multiple risky asset classes, and relaxing our rather restrictive assumptions about the market price process, utility of wealth, and demand for insurance.

References:


