Optimal Prevention for Correlated Risks

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Abstract
This paper analyzes optimal prevention expenditures in a situation of multiple correlated risks. We focus on probability reduction (self-protection). This renders correlation endogenous so that we measure dependence as the relative deviation of the probability of joint losses from the uncorrelated case. If prevention concerns only one risk, introducing a negatively correlated exogenous risk increases the level of prevention expenditures. If prevention expenditures may be invested for both risks, a substitution effect arises due to the competition for resources. Under decreasing returns on self-protection we find that increased dependence increases overall prevention expenditures, but not necessarily prevention expenditures for each risk due to differences in prevention efficiency. Similar results are found when considering the impact of more severe losses. We derive policy implications from our results.

Keywords: prevention · correlation · multiple risks

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1 Introduction

Prevention is an ex-ante activity that reduces the probability of a loss.\footnote{Following Ehrlich and Becker (1972)'s terminology, this activity refers also to self-protection. Throughout the paper we use both terms synonymously.} The economic analysis of prevention started with the seminal work of Ehrlich and Becker (1972) and has led since then to a flourishing literature (Courbage, Rey, and Treich, 2013). The canonical model, developed along several dimensions, assumes that the decision-maker (DM) faces only one risk to be mitigated using prevention measures. However, it often happens that individuals and firms face multiple risks that they can address with prevention activities. As an illustration, consider a DM confronted simultaneously with a risk of fire and a risk of burglary on his property. He can reduce the probability of loss due to fire by investing in fire-proof materials and the probability of illegal entry by investing in locks and alarms or by hiring a watchman. The question we ask in this paper is how the decision to prevent one risk interacts with the decision to prevent other risks. More particularly, we wonder how the characteristics of the multiple risks and the efficiency of alternative prevention instruments influence the decision to invest in a specific portfolio of prevention measures. For instance, if we consider two risks and if the dependence between the two risks increases, will the DM increase his total investment and the level of both prevention activities? How will he alter the composition of his portfolio of prevention measures? Our analysis can also help to predict the effect of imposing mandatory prevention expenditures for one risk on the decision to prevent other risks.

While the literature on economic decisions in a multiple risk setting is quite abundant, only a few papers have recently addressed the study of self-protection in the presence of multiple risks.\footnote{Note, however, the contributions of Briys, Schlesinger, and v. d. Schuleamburg (1991) and Schlesinger (1992) considering the reliability of prevention expenses. There is only one risk in the endowment, but the possible failure of prevention measures introduces an additional (multiplicative) risk.} These papers look at either the relation between self-protection and risk aversion in the presence of an independent zero-mean background risk (Dachraoui, Dionne, Eeckhoudt, and Godfroid, 2004), or the impact of an independent zero-mean background risk on self-protection activities in a one-period model (Lee, 2012) or a two-period model (Eeckhoudt, Huang, and Tzeng, 2012; Courbage and Rey, 2012). Our paper differs markedly from the above literature in two ways. First, we consider correlated risks, not an independent background risk. Second, we consider the decision to prevent both risks simultaneously, and not merely the influence of the second risk on optimal prevention for the first risk.

Our approach and our motivation are more closely related to the literature on optimal insurance purchasing under multiple correlated risks (Doherty and Schlesinger, 1983b; vd Schulenburg, 1986; Briys, Kahane, and Kroll, 1988) but with quite different results. The reason is that more insurance for one risk provides the same benefit as insurance for another risk: additional money if an insured loss arises. This is not so with self-protection. Different self-protection measures provide different marginal benefits and these measures are in addition subject to the law of diminishing returns. The focus cannot be on whether the risk is completely removed (full in-
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Insurance) or only partially alleviated (partial insurance) since complete elimination of the risk is not an option, in general.

To carry out our analysis, we rely on recent works that model prevention activities in a two-period expected utility framework. The idea of a two-period model stems from the fact that the decision to engage in prevention activities precedes its effect on the risk, whereas the one-period model implicitly assumes that the decision to engage in prevention and its effect on the risk are simultaneous. Such a way to model prevention was introduced by Menegatti (2009) who investigates the role of prudence in the decision to develop prevention. It was used further to address the link between saving and prevention (Menegatti and Rebessi, 2011), the effect of the introduction of an independent background risk on the decision to prevent a first risk (Eeckhoudt, Huang, and Tzeng, 2012; Courbage and Rey, 2012), or in the context of preventive care (Menegatti, 2013).

Our analysis proceeds along the following lines. First, we look at how the characteristics of one risk affect the decision to lower the probability of loss from the other risk. We show that introducing a second correlated risk increases prevention for the endogenous risk, independently of the sign of correlation. We also show that if the overall risk faced by the DM is higher, through either a higher loss or a higher probability for the second risk, or a higher interdependence between the two risks, then the DM carries out more prevention with respect to the first risk. Secondly, when we consider the decision to prevent both risks simultaneously, we can define conditions under which the level of prevention directed at one risk is higher than the level of prevention directed at the other risk. Thirdly, we show that both prevention measures are substitutes, with important implications for regulation in the field of risk management. Our results do not replicate the results obtained in the literature on compulsory insurance in a multiple risk setting (vd Schulenburg, 1986) where the sign of correlation - positive or negative - play a major role. Fourthly, we consider the effect of interdependence on the willingness to spend resources on self-protection. It turns out that a higher dependence increases overall prevention expenditure, as expected, but the impact is not the same on both prevention instruments. It may happen that one kind of expense increases and the other is reduced. Similarly, an increase in the amount of loss for one risk increases overall prevention expenditure, but does not necessarily increase prevention expenses for the risk directly affected. These effects have not been explicitly addressed in the literature on insurance with multiple risks, although it is commonly considered, from the results of these models, that an increase in dependence or an increase in one possible loss should have a positive influence on the purchase of market insurance, if the insurance premium is loaded.

The paper is organized as follows. In the next section, we introduce the general model. In section 3, we consider only one endogeneous risk and investigate how the the decision to prevent this risk is influenced by the properties of the second risk. In section 4, we consider the two risks to be endogeneous and compare the optimal level of prevention for each of them. Then we develop some comparative statics to address the interdependency between the two preventive activities. A short conclusion is provided in the last section.
2 The General Model

Let us consider a decision-maker confronted with two risks. The first risk shall be given as $\tilde{\epsilon}$ with outcomes $-l$ and 0, i.e., the individual might suffer a loss of $l > 0$. The probability of loss is given by $p$. The second risk is symmetric in structure and is given by $\tilde{\zeta}$ with outcomes $-g$ and 0, i.e., the individual might suffer a loss of $g > 0$. The loss probability shall be denoted as $q$ for the risk $\tilde{\zeta}$.

Regarding the joint distribution we allow for correlation between the two risks meaning that the distribution of $(\tilde{\epsilon}, \tilde{\zeta})$ need not necessarily be given by the product distribution of the two. To parametrize this situation of interdependence we introduce the parameter $k$ that describes to what extent the probability of occurrence of both losses deviates from the mere product of the individual loss probabilities. We call $k$ a measure of (inter)dependence and assume it to be exogenous to the analysis. With this convention, four states of nature are possible with their respective probabilities:
- Both losses of $l$ and $g$ with probability $kpq$,
- only the loss of $l$ with probability $p(1-kq)$,
- only the loss of $g$ with probability $q(1-kp)$,
- and no loss at all with probability $1-p-q+kpq$.

Doherty and Schlesinger (1983a) use a related framework to analyze the impact of correlated uninsurable background risks on the demand for insurance against insurable foreground risks, Rey (2003) extends their approach to the study of a nonpecuniary correlated background risk, and Courbage and Rey (2007) use it to study the implications of correlation between a financial and a nonpecuniary risk on precautionary savings. In this sense our modeling structure is standard for the analysis of correlation in a multiple risk setting.

Under the assumptions stated above, the correlation coefficient between the two risks is given by
$$
\tau(\tilde{\epsilon}, \tilde{\zeta}) = (k - 1) \sqrt{\frac{p}{1-p} \cdot \frac{q}{1-q}}.
$$
Hence, a parameter of $k = 1$ represents uncorrelated risks, $k > 1$ positively correlated risks, and $k < 1$ negatively correlated risks.\(^3\) Second, the strength of correlation is monotonically linked to the size of $k$ for given loss probabilities $p$ and $q$. As we endogenize $p$ and $q$ via self-protection later, this parameter measures the exogenous part of correlation. Looking at the occurrence probabilities of each event we can determine thresholds for $k$ to ensure that this parametrization always renders probabilities between 0 and 1. It is easy to see that this is the case for choices of $k$ in the interval
$$\left[\max\left(0, \frac{p + q - 1}{pq}\right), \min\left(\frac{1}{p}, \frac{1}{q}\right)\right].$$

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\(^3\) Examples are floods and damages from wind which are positively correlated risks ($k > 1$). In a firm, credit risk and workers’ injuries are negatively correlated risks, due to their dependence on the business cycle ($k < 1$).
In this sense, the plausible range for values of \( k \) depends on both individual loss probabilities.

Regarding the time structure of our model we assume two periods in time, \( t_1 \) and \( t_2 \). We make this assumption as we follow some recent works on prevention by modeling prevention in a two-period setting (see Menegatti, 2009). We assume that utility be separable across time and assume first-period preferences to be given by the vNM utility function \( u \) and second-period preferences to be given by the vNM utility function \( v \). We assume that the utility is increasing in wealth in each period (\( u'(\cdot) > 0 \) and \( v'(\cdot) > 0 \)) and the individual is risk-averse in both periods (\( u''(\cdot) < 0 \) and \( v''(\cdot) < 0 \)). Therefore, the individual's expected utility reads as:

\[
\mathbb{E} u(w_0) + \mathbb{E} v(w_2 + \tilde{\epsilon} + \tilde{\zeta}).
\]

To compress notation we define \( w_{NN} := w_2, w_{LN} := w_2 - l, w_{NL} := w_2 - g \) and \( w_{LL} := w_2 - l - g \), where subscript \( N \) denotes “no-loss” and subscript \( L \) denotes “loss”. The first letter of the subscript refers to the first risk, i.e., \( \tilde{\epsilon} \), the second letter of the subscript refers to the second risk, i.e., \( \tilde{\zeta} \). With this notation, expected utility becomes

\[
\mathbb{E} u(w_0) + (1 - p - q + kpq)v(w_{NN}) + p(1 - kq)v(w_{LN})
+ q(1 - kp)v(w_{NL}) + kpqv(w_{LL}).
\]

Furthermore, we use the conventions \( \alpha := v(w_{NN}) - v(w_{LN}), \beta := v(w_{NL}) - v(w_{LL}), \gamma := v(w_{NN}) - v(w_{NL}) \) and \( \delta := v(w_{LN}) - v(w_{LL}) \). Note that \( \alpha - \beta = \gamma - \delta \), and under risk aversion this quantity is negative.

### 3 One Exogenous Risk

In this section, we study how the characteristics of one risk affect the decision to lower the probability of loss from the other risk. We look at a situation where one of the two risks is endogenous, as the individual can invest in self-protection. Prevention expenditures of \( x \) at \( t_1 \) lead to a reduction of the loss probability to \( p(x) \) in the next period \( t_2 \). The individual’s maximization problem is therefore given as

\[
\max_x \left\{ u(w_0 - x) + (1 - p(x) - q + kp(x)q)v(w_{NN}) + p(x)(1 - kq)v(w_{LN})
+ q(1 - kp(x))v(w_{NL}) + kp(x)qv(w_{LL}) \right\},
\]

with associated first-order condition

\[
T_x = -u'(w_1) - p'(1 - kq)(v(w_{NN}) - v(w_{LN})) - kp'q(v(w_{NL}) - v(w_{LL}))
= -u'(w_1) - p'(1 - kq)\alpha - kp'q\beta = 0,
\]

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4 To focus on self-protection, we do not include savings into the analysis. For a joint analysis of self-protection and saving in an intertemporal framework refer to Menegatti and Rebessi (2011). They, however, only consider one source of risk whereas we focus on multiple risks.
where $T$ denotes the objective function and subscript $x$ indicates the respective derivative. $w_1$ is shorthand for the (endogenous) wealth level at $t_1$, i.e., $w_1 = w_0 - x$, and the second-order condition holds under risk aversion and standard assumptions on prevention technology, i.e., $p' < 0$, $p'' > 0$. Let $x^*$ denote the solution to the first-order condition, i.e., the optimal level of self-protection expenditures.

The first term of the FOC represents the marginal cost of prevention, i.e., the loss of first-period utility due to self-protection expenditures. The second and third terms represent the marginal benefit of prevention, i.e., the expected gain of second period utility due to a reduction in the loss probability $p$. More precisely, the second term is the marginal benefit of prevention conditional on the loss $g$ not occurring while the third term is the marginal benefit conditional on the loss $g$ occurring.

A straightforward question is how the presence of the exogenous risk influences self-protection expenditures on the endogenous risk. Therefore, we can compare the optimal decision under the absence of the exogenous risk to the situation characterized above. If the second risk is not present, the DM chooses to maximize

$$V(x) = u(w_0 - x) + p(x)v(w_2 - l) + (1 - p(x))v(w_2),$$

with associated first-order condition

$$V_x = -u'(w_0 - x) - p'(x)\alpha = 0,$$

where $\bar{x}$ represents optimal self-protection expenditures in the one-risk case. Note that the second-order condition holds generically in the single-risk case considered here. Next, we evaluate the first-order expression in the two-risk situation at the optimal self-protection expenditures in the one-risk situation. This yields

$$T_x(\bar{x}) = -u'(w_0 - \bar{x}) - p'(\bar{x})(1 - kq)\alpha - p'(\bar{x})kq\beta$$

$$= p'(\bar{x})\alpha - p'(\bar{x})\alpha + p'(\bar{x})kq(\alpha - \beta)$$

$$= p'(\bar{x})kq(\alpha - \beta) > 0.$$

Due to the second-order condition this implies that the introduction of the second risk increases self-protection expenditures on the first risk. This is summarized in the following proposition.

**Proposition 1.** Optimal self-protection expenditures for risk $\tilde{\epsilon}$ increase when the exogenous risk $\tilde{\zeta}$ is introduced for any $k > 0$.

It is not surprising that the dependence parameter $k$, positively linked to the correlation coefficient for given $p$ and $q$, has a positive influence on the difference between $x^*$ and $\bar{x}$. If $k$ increases, $T_x$ increases and self-protection expenses increase (see Proposition 2 below). What is more surprising is to observe that $x^* > \bar{x}$, for all $k > 0$, independently of the sign of correlation. It is well-known in the financial and insurance literature dealing with multiple risks that a negative correlation coefficient has a mitigating impact on the overall risk faced by the
DM. For example, Schlesinger and Doherty (1985) note that the willingness to insure a given insurable risk is reduced if a second negatively correlated risk introduces a strong diversification effect in the final wealth. In the limit, the demand for insurance is zero if the second risk offsets completely the insurable risk so that final wealth is non-random. This is a case of “homemade hedging” (see vd Schulenburg, 1986, Proposition 2). We could thus expect that a risk-averse DM would be prepared to spend less on self-protection for one risk if a second, negatively correlated risk introduces “homemade hedging” in his final wealth. As the result above shows, this is not true. Even with perfect negative correlation between the two risks it turns out that the optimal prevention expenditure in the two-risk case is not lower than in the one risk case.

We can, however, intuitively explain this effect by utilizing the decomposition of the marginal benefit into the component conditional on the loss $g$ not occurring and the component conditional on the loss $g$ occurring. The fact that the correlated second risk is introduced has two effects on the marginal benefit of preventing the first risk. The first one is that for positive $k$ the marginal benefit conditional on the loss $g$ not occurring is smaller than the marginal benefit from $x$ in the absence of the second risk. This is due to the fact that the probability of the condition has to be taken into account so the marginal benefit is decreased. Furthermore, the marginal benefit conditional on the loss $g$ occurring needs to be added and here it pays more to prevent the loss of $l$ knowing that $g$ has already occurred. This increases the marginal benefit. A priori the overall effect is indeterminate. We have assumed risk aversion though, and this implies that a given difference in income yields a larger difference in utility levels at low initial income than at high initial income. This is a direct consequence of diminishing marginal utility. For our context it implies that the positive effect on the marginal benefit when introducing the correlated second risk always outweighs the negative effect so that an increase in prevention expenditures take place. This reasoning is independent of the sign of correlation. Specifically, it also holds for actuarially fair prevention for which the expenditures coincide with the expected monetary benefit so that expected intertemporal consumption is constant. This sharpens the contrast to the case of insurance under negative correlation.

As a next step we study how properties of the risk $\tilde{\zeta}$ and specifically the extent of interdependence affect the selected level for the preventive investment. In this sense, we relate the riskiness of the second risk to the amount invested in self-protection for the first risk. The results are summarized in the following proposition.

5 In this case, a positive demand for insurance would break the compensating effect, with no impact on the expected wealth if the insurance premium is unloaded.

6 The limit case with $k = 0$ corresponds to the situation in which the increase and decrease of the marginal benefit just offset each other and prevention expenditures are unaffected by the introduction of the correlated second risk.

7 Note that the correlation coefficient $\tau$ is increasing (decreasing) in $p$ if and only if $k > 1$ ($k < 1$). For this reason, introducing a positively correlated second risk leads to a fall in $p$ and a decrease in correlation. Introducing a negatively correlated second risk leads also to a fall in $p$, but an increase in correlation. In this sense, introducing a correlated second risk induces regression towards no correlation.

8 In this case, the self-protection technology must be linear with $p'(x) = -1/l$. 

7
Proposition 2. The DM exerts more prevention on risk $\hat{\epsilon}$ with

i) a larger loss probability for risk $\hat{\zeta}$,

ii) a more severe loss for risk $\hat{\zeta}$,

iii) increased dependence between the two risks.

Proof. The second-order condition is satisfied, i.e., $T_{xx} < 0$. Hence the direction of the effect of marginal variations of exogenous parameters is given by the sign of the derivative of the first-order condition with respect to the exogenous parameter. This yields

\[
T_{xq} = kp'(\alpha - \beta) > 0,
\]
\[
T_{xg} = kp'q(v'(w_{NL}) - v'(w_{LL})) > 0,
\]
\[
T_{xk} = p'q(\alpha - \beta) > 0.
\]

First note that the marginal cost of prevention is not affected by the variations under consideration. Hence, we focus on the marginal benefit to rationalize the results. As said before, the marginal benefit has two components, one conditional on the loss $g$ not occurring and the other conditional on the loss $g$ occurring. When $q$ increases, it decreases the weight of the marginal benefit of prevention when the loss $g$ has not occurred, i.e., $p'\alpha$, and it increases the weight of the marginal benefit of prevention in the presence of the loss $g$, i.e., $p'\beta$. However, due to risk aversion, the utility difference between the states in which loss $g$ has not occurred ($\alpha$) is smaller than the utility difference between the states in which loss $g$ has occurred ($\beta$). The overall effect on the marginal benefit of prevention is then positive leading to an increase in prevention and explaining the positive sign of $T_{xq}$.

An increase in the loss size of the second risk only affects the marginal benefit conditional on this loss occurring. Due to diminishing marginal utility an increase in $g$ increases the pain from suffering the loss $l$ also, because the utility difference of a given income difference is larger for low wealth levels. Consequently it is more beneficial to prevent the first loss knowing that the second loss has occurred. This explains the positive sign of $T_{xg}$.

Increasing interdependence shifts probability mass from the states of the world in which only one loss occurs to the states of the world where either both losses occur or neither does. Again, $p'q\alpha$ measures by how much the marginal benefit of preventing $l$ conditional on $g$ not occurring decreases due to an increase in dependence. The intuition is that with increased interdependence knowing that $g$ has not occurred makes it less likely for $l$ to occur. Similarly, $-p'q\beta$ measures the increase in the marginal benefit of preventing $l$ conditional on $g$ occurring due to increased dependence. Knowing that $g$ has occurred, increased interdependence makes it more likely that also $l$ will occur, so preventing this loss is more valuable. The overall effect is again positive due to the fact that risk aversion leads to a larger difference in utility conditional on $g$ having occurred than conditional on $g$ not having occurred. Obviously, even with only one endogenous
risk, the interdependence between the two risks has quite some influence on the optimal decision.

Proposition 2 says that if the overall risk faced by the individual is higher, through either a higher loss of risk $\tilde{\zeta}$, a higher loss probability of risk $\tilde{\zeta}$ or a higher dependence between the two risks, he carries out more prevention with respect to the risk $\tilde{\epsilon}$. In this sense a correlated risk impacts optimal decisions even if it is completely exogenous, but we do not observe any hedging effects. Note that in a situation of self-protection activities the resulting loss probabilities are endogenous and so is correlation as conventionally measured by the correlation coefficient. This makes it necessary to use alternative measures of interdependence to study exogenous variations in the dependence structure of the two distributions. We decided to use $k$ as it measures by how much the probability of the joint occurrence of both losses deviates from the uncorrelated benchmark case for any level of self-protection expenditures. We think this is appropriate given the endogeneity of correlation.\footnote{Note that \[ \frac{d\tau}{dk} = \sqrt{\frac{p}{1-p}} \cdot \frac{q}{1-q} \left(1 + \frac{k - 1}{2} \left[ \frac{1}{p(1-p)} \frac{\partial x}{\partial k} + \frac{1}{q(1-q)} \frac{\partial y}{\partial k} \right] \right), \] so that increased interdependence as measure by $k$ might lower or increase the correlation coefficient. When benchmarking against uncorrelated risks, i.e., for $k = 1$ the analogy is perfect meaning that marginal increases of $k$ are associated with marginal increases in the correlation coefficient and vice versa.}$

4 Both Risks Endogenous

We now assume that both $p$ and $q$ may be reduced by an investment $x$ and $y$, respectively, that takes place today. Moreover, we assume decreasing returns on self-protection. We define the (technical) rate of return on self-protection expenditures $x$ by

$$\rho(x) := \lim_{\Delta x \to 0} \left( \frac{p(x) - p(x + \Delta x)}{p(x)} \frac{1}{\Delta x} \right) = -\frac{p'(x)}{p(x)},$$

and analogously $\sigma(y)$ for loss prevention expenditures $y$.\footnote{$\rho(x)$ measures by how much the loss probability decreases when investing an additional dollar into loss prevention, normalized by the current loss probability. Alternatively, it may be called the absolute decay rate of the loss probability.}$

With this specification, the maximization problem is given by\footnote{As mentioned earlier, an alternative specification would be a model with separable cost of effort. If the disutility is an increasing and convex function of aggregate effort, all our results go through.}

$$\max_{x,y} \left\{ u(w_0 - x - y) + (1 - p(x) - q(y) + kp(x)q(y))v(w_{NN}) + p(x)(1 - kq(y))v(w_{LN}) + q(y)(1 - kp(x))v(w_{NL}) + kp(x)q(y)v(w_{LL}) \right\}. \tag{1}$$
Let $T$ denote the objective function and let $w_1$ again be shorthand for the (endogenous) wealth level at $t_1$. Then, associated first-order conditions are given as

$$
T_x = -u'(w_1) - p'(1-kq)(v(w_{NN}) - v(w_{LN})) - kp'q(v(w_{NL}) - v(w_{LL}))
= -u'(w_1) - p'(1-kq)(\alpha - kp'q\delta) = 0,
$$

$$
T_y = -u'(w_1) - q'(1-kp)(v(w_{NN}) - v(w_{NL})) - kpq'(v(w_{NL}) - v(w_{LL}))
= -u'(w_1) - q'(1-kp)\gamma - kpq\delta = 0
$$

(2)

The optimal expenditures on self-protection are denoted as $(x^*, y^*)$. Second-order conditions are assumed to hold for maximality. We first answer the question how the introduction of an endogenous second risk affects expenditures on self-protection devoted to the first risk. Applying Proposition 1 to the risk $\zeta$ with probability $q(y^*)$ and the given $k$ yields that also the introduction of an endogenous second risk raises optimal expenditures on the first risk for any positive $k$. Next we can study how expenditures for the risks $\tilde{\epsilon}$ and $\tilde{\zeta}$ compare to each other. Specifically, we investigate the conditions under which more money will be spent on preventing risk $\tilde{\epsilon}$ than risk $\tilde{\zeta}$. Intuitively, this depends on the rate of return of self-protection expenditures $x$ and $y$ and on the severity of the risks under consideration. The technical requirement for the comparison is provided in Gollier (2001), p. 151, but we will summarize it in the following lemma.

**Lemma 1.** Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a concave function in the variables $(x, y)$ and $(x^*, y^*)$ be the local maximum implicitly defined via the first-order conditions. For a given $\alpha \in \mathbb{R}$ it holds that $x^* > \alpha$ if and only if $f_x(\alpha, \hat{y}) > 0$ where $\hat{y}$ is the value that maximizes $f(\alpha, y)$.

**Proof.** See the appendix. 

We now apply Lemma 1 to the self-protection situation outlined above with two endogenous levels of expenditures. The technical result is summarized in the following proposition and will be utilized to develop some intuition afterwards.

**Proposition 3.** Prevention expenditures for risk $\tilde{\epsilon}$ will be larger than for risk $\tilde{\zeta}$ if and only if the marginal benefit of preventing $\tilde{\epsilon}$ is larger than the marginal benefit of preventing $\tilde{\zeta}$ when evaluated at $(y^*, \hat{y})$, with $\hat{y}$ solving $\max_y T(y^*, y)$. This comparison can be decomposed into a comparative efficiency component and a component comparing marginal benefits conditional on the other risk not occurring.

**Proof.** According to Lemma 1 we first carry out the maximization $\max_y T(y^*, y)$ to obtain optimal expenditures $\hat{y}$ if $x$ is set to $y^*$. $\hat{y}$ is defined via the first-order condition

$$
-u'(w_0 - y^* - \hat{y}) - q'(\hat{y})(1-kp(y^*))\gamma - kp(y^*)q'(\hat{y})\delta = 0.
$$

12 Under the assumptions self-protection technology $x$ providing a larger return on self-protection than self-protection technology $y$, decreasing returns on self-protection and under the assumption that $-q''/q' > -p'/p$ which we will introduce later to prove Proposition 6, one can show that the second-order conditions are satisfied, see the appendix. Therefore, the assumption seems to be innocuous.
Now the first-order expression with respect to $x$ evaluated at $(y^*, \hat{y})$ is given by

$$T_x(y^*, \hat{y}) = -u'(w_0 - y^* - \hat{y}) - p'(y^*)(1 - kq(\hat{y}))\alpha - kp'(y^*)q(\hat{y})\beta =$$

$$= q'(\hat{y})(1 - kp(y^*))\gamma + kp(y^*)q'(\hat{y})\delta - p'(y^*)(1 - kq(\hat{y}))\alpha - kp'(y^*)q(\hat{y})\beta$$

$$= MB_x(y^*, \hat{y}) - MB_y(y^*, \hat{y}),$$

where $MB_x$ denotes the marginal benefit arising from prevention expenditures $x$ and $MB_y$ the marginal benefit arising from prevention expenditures $y$.

We can gain more intuition by rearranging $T_x(y^*, \hat{y})$ in the following way:

$$(\alpha - \beta)k(p'(y^*)q(\hat{y}) - p(y^*)q'(\hat{y})) + (q'(\hat{y})\gamma - p'(y^*)\alpha).$$

The first summand describes a comparative efficiency component. Under the assumption that $\rho(y^*) > \sigma(\hat{y})$, it is positive indicating that more prevention should be carried out for risk $\tilde{\epsilon}$ than for risk $\tilde{\zeta}$. Furthermore, this effect is increasing in the amount of interdependence between the two risks ($k$) meaning that the higher the two risks are correlated the more weight has to be attached to the comparison of the efficiency of the two technologies. Lastly, this effect is also related to the comparison of the loss in utility when going from one loss to two losses ($\alpha - \beta$). The more severe it becomes to suffer the second loss the more relevant becomes this comparative efficiency effect.

However, it is not sufficient to have one technology more efficient than the other to arrive at an unambiguous result. The second summand relates the marginal benefit of prevention expenditures as if there was only one risk for each decision. This second summand is positive if and only if

$$-p'(y^*)(v(w_{NN} - v(w_{LN})) > -q'(\hat{y})(v(w_{NN}) - v(w_{NL})).$$

In short, the overall sign of $T_x(y^*, \hat{y})$ is ambiguous since the second component might be negative even if $x$ provides more prevention efficiency than $y$.

A special case occurs if both risks have the same loss size ($l = g$) and if the marginal loss probability for risk $\tilde{\epsilon}$ is always below the marginal loss probability for risk $\tilde{\zeta}$ ($p' < q'$). Then more expenditures will be devoted to preventing risk $\tilde{\epsilon}$. Another special case is given if the loss probabilities and the marginal loss probabilities at $y^*$ and $\hat{y}$ respectively are equal for both risks. Then the first component is nil due to equal returns to self-protection for both technologies. The second component prescribes that higher expenditures will be incurred for the risk with the larger loss size.

5 Comparative Statics

Next we study how an optimal bundle of prevention expenditures reacts to changes in exogenous variables. In order to look at the interaction between the two decisions we first look at the reaction functions, i.e., we investigate how the optimal expenditures to protect against one risk react to a change in the expenditures to protect against the other risk and inversely. The
joint optimum is obtained when both curves intersect. The following observation is of crucial
importance and will be utilized in the rest of the paper.

**Proposition 4.** Preventive expenditures for risk $\tilde{\epsilon}$ and for risk $\tilde{\zeta}$ are substitutes.

**Proof.** Differentiating $T_x$ with respect to $y$ and $T_y$ with respect to $x$ give:

$$T_{xy} = T_{yx} = u''(w_0 - x - y) + kp'q' \alpha < 0$$

for $u'' < 0$ and $v'' < 0$. This means that the optimal value of $x$ decreases following an exogenous increase of $y$ and conversely.

The intuition behind the substitution between the two prevention measures is twofold. First, there is competition for resources, i.e., exogenous increases of expenditures on one prevention measure reduce wealth to be spent at $t_1$ and thereby increases the marginal cost of prevention for the other measure. Second, the marginal benefit is affected. The marginal benefit conditional on the other loss not occurring is increased due to the fact that higher prevention expenditures to prevent the other loss make the state of the world in which it does not occur more likely. Similarly, the marginal benefit conditional on the other loss occurring is decreased because higher prevention expenditures to prevent the other loss make the state of the world in which it does occur less likely. The overall effect is negative, i.e., the second effect prevails, again due to diminishing marginal utility.

Note that this result is independent of the sign of correlation and has public policy implications. It implies that there are compensating reactions when the government imposes mandatory prevention expenditures on one kind of risk, for example fire, car accidents or elevator accidents. Individuals and firms react by cutting self-protection expenditures on other risks. Therefore, the policymaker needs to take these reactions into account when changing safety requirements for specific risks, as unintentional increases in the exposure to other risks might decrease or even outweigh the gains from increased safety for the first risk. This substitution result is in sharp contrast with comparable results in the demand for insurance with multiple risks where mandatory insurance requirements for one risk increase the insurance demand for the other risk if the two risks are negatively correlated (vd Schulenburg, 1986).

**5.1 Effect of Interdependence**

We utilize Proposition 4 to conduct comparative statics analysis. We start with dependence as it is the focus of the paper and we wonder how the optimal values of both $x$ and $y$ vary when the extent of interdependence is increased. Intuitively one could think that when risks become more positively correlated, the expenditures to reduce the probability of each one increase. To smooth the exposition we make the assumption that without loss of generality expenditures $x$ provide a larger return on self-protection than expenditures $y$. More precisely, as we study the impact of interdependence in this subsection, we take into account that each level of dependence $k$ implies

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13 An example is mandatory safety equipment that vehicle drivers in many countries have to carry with them.
optimal self-protection expenditures $x^*(k)$ and $y^*(k)$ according to the first-order conditions. Thus we can define the rate of return on self-protection as function of $k$, i.e., $\rho(k) \equiv \rho(x^*(k))$ and $\sigma(k) \equiv \sigma(y^*(k))$ and postulate that $\rho(k) > \sigma(k) \forall k$. This means that technology $x$ provides a larger return on self-protection than technology $y$ for all respective pairs of optimal self-protection expenditures.

Totally differentiating the two first order conditions with respect to $k$ and applying the implicit function rule yields:

$$\text{sgn} \left( \frac{dx}{dk} \right) = \text{sgn}(-T_{yy}T_{xk} + T_{xy}T_{yk}),$$

where

$$-T_{yy}T_{xk} + T_{xy}T_{yk} = (\alpha - \beta) \left[ -u''(p'q - pq') + (\alpha - \beta)kqq'((q')^2 - qq'') + \gamma p'qq' \right].$$

The sign of the first summand is determined by $(p'q - pq')$ and depends on the relative efficiency of the two prevention opportunities. It is positive under the assumption that the return on self-protection for expenditures $x$ is larger than for expenditures $y$. The sign of the second summand depends on the expression $(q')^2 - qq''$. It is positive whenever $-\frac{q''}{q'} < -\frac{q'}{q''}$, i.e., the assumption of decreasing returns on self-protection is sufficient as presented in section 4. The last term in squared brackets together with $(\alpha - \beta)$ is unambiguously positive and measures the direct impact of increased interdependence on self-protection expenditures $x$.

Hence, increased dependence between the two risks enhances prevention expenditures for the more efficient technology. From equation (3) we see that there are two competing effects at work. As demonstrated in Proposition 2, there is a direct effect that leads to more prevention with increased dependence, because the increase in marginal benefit of prevention on one risk conditional on the other risk occurring is larger than the decrease in marginal benefit of prevention on one risk conditional on the other risk not occurring. This is contained in $-T_{yy}T_{xk}$ which is unambiguously positive. However, an increase in prevention on one risk increases the marginal costs and lowers the marginal benefit for preventing the other risk, i.e., $T_{xy} < 0$, so there is a substitution effect, see Proposition 4. As demonstrated above the assumptions of $x$ providing a larger return on self-protection than $y$ and decreasing returns to self-protection are sufficient to guarantee that for the more efficient technology the direct effect prevails.

Similarly, we can determine how prevention expenditures for $q$ react to changes in our interdependence measure $k$. This yields

$$\text{sgn} \left( \frac{dy}{dk} \right) = \text{sgn}(T_{xy}T_{xk} - T_{xx}T_{yk}),$$

with

$$T_{xy}T_{xk} - T_{xx}T_{yk} = (\alpha - \beta) \left[ u''(p'q - pq') + (\alpha - \beta)kqq'((p')^2 - pp'') + \alpha p''q' \right].$$

We drop the arguments of utility to compress notation.
The analysis is as above. Note, however, that the first summand of the expression in squared brackets is negative as we assumed $y$ to provide a lower return on self-protection than $x$. This explains why the effect of increased dependence on $y$ is ambiguous.

Lastly, we can determine how overall prevention expenditures react to changes in the level of interdependence. Let $D$ be the determinant of the Hessian; then, after some simplifications and using the definitions given above

$$
\frac{d(x + y)}{dk} = \frac{dx}{dk} + \frac{dy}{dk} = -\frac{1}{D} (\alpha - \beta)^2 k p p' q q' \left[ \frac{\rho'}{\rho} + \frac{\sigma'}{\sigma} \right] - \frac{1}{D} (\alpha - \beta) p q [\alpha \sigma p'' + \gamma \rho q''],
$$

which is unambiguously positive. Note that the comparative efficiency effects cancel out each other so only the effect due to decreasing returns on self-protection and the direct effects remain. We summarize the analysis in the following proposition.

**Proposition 5.** Increasing dependence increases overall prevention expenditures under decreasing returns on self-protection. Individual prevention expenditures display a comparative efficiency effect, a positive effect due to decreasing returns of the other prevention technology and a positive direct effect. Therefore, expenditures on the more efficient technology increase, whereas the effect on the less efficient technology is ambiguous.

These results highlight the specificity of self-protection when considered in a multiple risks setting. The comparative efficiency of alternative prevention instruments plays a major role and may lead to observe a simultaneous increase in the use of one instrument and decrease in the use of the other instrument although the two risks have become more interdependent.

### 5.2 Impact of Loss Size

We now study the question how changes in the severity of loss affect the optimal amount of self-protection expenditures carried out by the agent. One might intuitively think that an increase in the size of the loss increases prevention expenditures for that risk, but decreases expenditures for the other risk due to substitution between the two prevention opportunities. This assumption is, however, misleading due to the fact that increasing the loss size of one risk also exerts a positive effect on prevention expenditures for the other risk, see Proposition 2.

Totally differentiating the first-order conditions with respect to $l$ and applying the implicit function theorem yields:

$$
\text{sgn} \left( \frac{dx}{dl} \right) = \text{sgn}(-T_{yy} T_{xl} + T_{xy} T_{yl}),
$$
where after some algebra

\[ -T_{yy}T_{xl} + T_{xy}T_{yl} = u''\left[k(v'_{LL} - v'_{LN})(p'q - pq') + p'v'_L \right] \\
+ k^2 pp'((\alpha - \beta)(v'_{LN} - v'_{LL}) [(q')^2 - qq''] \\
+ (kp'qq'' \gamma (v'_{LN} - v'_{LL})] + kpp'q''((\alpha - \beta)v'_{LN} - p'q'' \gamma v'_L). \]

The first term describes a comparative efficiency effect that favors the more efficient prevention technology. Under the assumption that \( x \) provides a larger return on self-protection than \( y \) this effect is positive indicating an upward effect on expenditures \( x \). The second term describes how changes in \( x \) affect the efficiency of prevention expenditures \( y \). Again, under the assumption of decreasing returns on self-protection this effect is positive. And the last term is unambiguously positive reflecting the initial intuition. Hence, we can conclude that an increase in the loss size \( l \) unambiguously increases loss prevention expenditures on \( x \) under the assumptions made. In this sense, if \( x \) provides a higher return on self-protection than \( y \) the direct positive effect dominates the negative substitution effect and therefore if risk \( \tilde{\epsilon} \) becomes more severe in terms of the loss size this unambiguously leads to an increase of the self-protection investment \( x \).

Next let us analyze how an increase in \( l \) affects prevention expenditures on \( y \). Proposition 2 gives rise to the conjecture that with a larger loss for risk \( \tilde{\epsilon} \) also prevention expenditures regarding risk \( \tilde{\zeta} \) should increase. However, due to substitution between the two types of expenditures matters are more complex. The implicit function rule implies

\[ \text{sgn} \left( \frac{dy}{dl} \right) = \text{sgn}(T_{xy}T_{xl} + T_{xx}T_{yl}), \]

which resolves to

\[ T_{xy}T_{xl} - T_{xx}T_{yl} = u''\left[k(p'q - pq')(v'_{LN} - v'_{LL}) - p'v'_L \right] \\
+ k^2 pp'((\alpha - \beta)(v'_{LN} - v'_{LL}) [(p')^2 - pp''] \\
- k(p')^2 p'((\alpha - \beta)v'_{LN} + kpp'' q'(v'_{LN} - v'_{LL}) \alpha. \]

The first term describes a comparative efficiency effect again which favors the more efficient prevention technology. Under the assumption that \( x \) provides a larger rate of return on self-protection than \( y \) this summand is negative. The second term describes how changes in \( y \) affect the efficiency of preventive effort \( x \). Under the assumption of decreasing returns on self-protection it is positive. The second last term is negative, whereas the last term is positive. Hence, the overall sign is ambiguous.

Finally we can again analyze how changes in the loss size \( l \) influence overall prevention expen-
ditures \( x + y \). We obtain that

\[
\frac{d(x + y)}{dl} = \frac{dx}{dl} + \frac{dy}{dl} = \frac{1}{D} k^2 (\alpha - \beta)(v'_{LN} - v'_{LL})pp'q'q'\left[\frac{\rho'}{\rho} + \sigma'\right] \\
- \frac{1}{D} kpq(v'_{LN} - v'_{LL}) (\rho q'' \gamma + \sigma p' \alpha) \\
- \frac{1}{D} kpp'q'q' (\alpha - \beta) v'_{LN} \left(-\frac{q''}{q'} + \frac{p'}{p} \right) - \frac{1}{D} D p' q'' v'_{LN} \gamma
\]

Note that the comparative efficiency effects cancel each other out again. The first summand here is positive due to the assumption of decreasing returns to self-protection. The second is also positive. The third one is positive under the assumption that \(-q''/q' > -p'/p\) and the fourth one is always positive. Hence, under suitable technical conditions overall prevention expenditures rise. We summarize our results in the following proposition.

**Proposition 6.** Under suitable technical conditions a higher loss size increases overall prevention expenditures. The single prevention expenditures display a comparative efficiency effect, a positive effect due to decreasing returns to prevention and a third effect that is positive for the direct prevention measure and ambiguous for the indirect one. If the direct prevention measure is more efficient a more severe loss for that risk is unambiguously associated with higher prevention expenditures on that measure.

Again, these results are original, compared to results obtained in previous literature on hedging/insuring against losses. In this literature an increased loss size has an unambiguously positive effect on the demand for hedging. The specificity of prevention lies in the relative efficiencies of different prevention technologies to preserve final wealth. As a result, increasing a loss size does not necessarily increase prevention expenditure to protect against that loss. In our model, increasing loss \( g \) does not necessarily increase expenditure on \( y \) if \( x \) is more efficient than \( y \).

### 6 Conclusion

This paper examines decision-makers’ expenditures to mitigate risks in a situation of multiple risks. Specifically, we investigate self-protection investments in the sense of Ehrlich and Becker (1972), i.e., expenses to reduce the probability of loss. Following recent contributions in the literature (Menegatti, 2009; Eeckhoudt, Huang, and Tzeng, 2012; Courbage and Rey, 2012) we model prevention as investment preceding the reduction of the loss probability.

We first identify how characteristics of one risk affect the amount devoted to lowering the loss probability of another risk. We find that introducing a second correlated risk increases prevention for the endogenous risk, independently of the sign of correlation. We also find that increased riskiness in terms of the loss probability or the loss severity of one risk or the extent of interdependence between the two risks are associated with larger loss prevention investments for the other risk. We then proceed by analyzing a situation in which loss prevention can be carried out for both risks. By comparing the marginal benefit for both prevention opportunities at a
specific level of investment we can decide for which risk the larger investment shall be incurred. This condition translates into two sub-conditions relating efficiency of prevention technology and the isolated benefit to one another.

We also find that both measures of prevention are substitutes. This has two reasons: First, by modeling prevention as investment preceding its benefit there is competition for resources between the two prevention opportunities. Second, the marginal benefit conditional on the other loss not occurring increases, whereas the marginal benefit conditional on the other loss occurring decreases, but the second effect prevails so that the overall marginal benefit decreases. This implies that exogenous increases of expenditures on one prevention measure due to regulation for example should be accompanied by endogenous decreases of expenditures on other prevention measures.

Finally, we conduct comparative statics analysis regarding our measure of interdependence and the size of potential losses. If dependence is increased, the marginal effect on the individual prevention measures can be decomposed into three components. There is a comparative efficiency component, an effect due to decreasing returns on self-protection and a positive direct effect. As the comparative efficiency effects exhibit opposite signs, overall prevention expenditures increase following an increase in interdependence. Furthermore, larger expenditures will be devoted to the more efficient technology, whereas the investment into the less efficient one might increase or decrease. When it comes to the size of the loss, we can again decompose the marginal effects into three components analogous to the study of dependence. The comparative efficiency effect favors the more efficient technology. If the prevention technology that is more efficient is associated with the increased loss, more resources will be incurred to prevent this risk. Moreover we identify suitable technical conditions that guarantee that overall expenditures increase. Again, the behavior of the less efficient technology is ambiguous.

Several extensions seem promising. A natural first thought would be to wonder whether and how the results obtained carry over to self-insurance or loss reduction. Following Ehrlich and Becker (1972) self-insurance refers to an investment that reduces the size of a given loss but leaves its probability of occurrence unaffected. It is well-known in the literature that self-insurance and self-protection may behave quite differently (Dionne and Eeckhoudt, 1985). Another avenue would be to investigate correlated risks that appear in different arguments of the utility function, i.e., a monetary risk flanked by a health risk. Tools of multivariate utility analysis (Rey, 2003; Lee, 2012) could then prove helpful to investigate the impact of correlation. It would also be interesting to analyze how comparative risk aversion impacts prevention expenses in a multiple risk setting by using results on risk aversion under multiple sources of risk (Kihlstrom, Romer, and Williams, 1981; Pratt, 1988). Finally, technological relationships between expenditures on self-protection can be richer in a multiple risk setting than in environments with only one risk. The reason is that expenditures to prevent one risk could not only affect the exposure to the other risk via interdependence, but one could also think of a direct benefit. In other words, it would be promising to investigate more complex overall prevention technologies.
References


Appendix

The Second-Order Conditions

This subsection is to prove that the assumption of self-protection technology $x$ providing larger returns than self-protection technology $y$, decreasing returns on self-protection together with the assumption of $-q''/q' > -p'/p$ are sufficient for the second-order conditions to hold. Consider the first-order conditions (2) that characterize the optimal bundle of self-protection expenditures. Then,

$$
T_{xx} = u'' - p''(1 - kpq)\alpha - kp''q\beta < 0,
$$

$$
T_{yy} = u'' - q''(1 - kpq)\gamma - kpp''q\delta < 0,
$$

and after some algebra

$$
det \text{Hess } T = T_{xx}T_{yy} - T_{xy}^2 = -u''[(1 - kp)q''\gamma + kpp''q\delta + p''(1 - kp)\alpha + kp''q\beta + 2kp'q'(\alpha - \beta)]
$$

$$
+ p''(1 - kp)q''(1 - kp)\alpha\gamma + kpp''q''(1 - kp)\delta + kp''(1 - kp)qq''\beta\gamma
$$

$$
+ k^2pp''qq''\beta\delta - k^2(p'q')^2(\alpha - \beta)^2.
$$

We want to find conditions under which the determinant of the Hessian is positive at an interior solution to guarantee maximality. The expression in square brackets can be rearranged to obtain

$$
p''\alpha + q''\gamma + k(\alpha - \beta)[2p'q' - pq'' - p''q].
$$

This is positive as long as $2p'q' - pq'' - p''q$ is negative which is equivalent to

$$
\frac{p''}{p'} \cdot \frac{q''}{q'} + \frac{p''}{p'} \cdot \frac{q}{q'} > 2.
$$

Now $-q''/q' > -p'/p$ is equivalent to having the first expression larger than 1 and the assumption that $x$ provides a larger return on self-protection than $y$ and that returns on self-protection are decreasing ensure that the second expression is larger than 1.

The remaining five terms of the determinant of the Hessian can be reorganized as follows:

$$
k^2pp''qq''(\alpha - \beta)^2 - kp''qq''\gamma(\alpha - \beta) - kpp''qq''\alpha(\gamma - \delta)
$$

$$
+ p''qq''\alpha\gamma - k^2(\alpha - \beta)^2(pp''qq'' - (p'q')^2).
$$

The first four of these are positive, the last one enters negative. Combining the first and the last of these and factoring out yields

$$
k^2(\alpha - \beta)^2(pp''qq'' - (p'q')^2),
$$

which is positive as long as $p''/p' \cdot q''/q' > p'/p \cdot q'/q$ which is ensured by decreasing returns on self-protection expenditures. This proves that the assumptions made to sign the comparative
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statics results in the paper are sufficient to guarantee that the second-order conditions for the maximization over both levels of self-protection expenditures are satisfied and that the solutions characterized by the first-order conditions are actually describing utility-maximizing choices.

Proof of Lemma 1

We consider the constrained maximization problem

$$\max_{x,y} f(x, y) \quad \text{s.t.} \quad x = \alpha.$$ 

The Lagrangian is given by

$$\mathcal{L}(x, y; \lambda) = f(x, y) + \lambda(\alpha - x) \quad (4)$$

and the constrained optimum is characterized by

$$\frac{\partial \mathcal{L}}{\partial x} = f_x - \lambda = 0,$$

$$\frac{\partial \mathcal{L}}{\partial y} = f_y = 0,$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = \alpha - x = 0.$$ 

It is given by \((\alpha, \hat{y})\) where \(\hat{y}\) maximizes \(f(\alpha, y)\). The associated Lagrange multiplier is denoted by \(\hat{\lambda}\) and equals \(f_x(\alpha, \hat{y})\).

Now let us assume \(\lambda\) to be exogenous to (4) and define the optimal \(x\) as function of \(\lambda\). Then obviously \(x(\hat{\lambda}) = \alpha\) and \(x(0) = x^*\). To see how variations in \(\lambda\) affect the optimal value of \(x\) we apply the implicit function rule to obtain

$$\left( \begin{array}{c} \frac{dx}{d\lambda} \\ \frac{dy}{d\lambda} \end{array} \right) = - \frac{1}{f_{xx}f_{yy} - f_{xy}^2} \left( \begin{array}{cc} f_{yy} & -f_{xy} \\ -f_{xy} & f_{xx} \end{array} \right) \cdot \left( \begin{array}{c} \frac{\partial^2 \mathcal{L}}{\partial x \partial \lambda} \\ \frac{\partial^2 \mathcal{L}}{\partial y \partial \lambda} \end{array} \right) = \left( \begin{array}{c} \frac{f_{yy}}{f_{xx}f_{yy} - f_{xy}^2} \\ -\frac{f_{xy}}{f_{xx}f_{yy} - f_{xy}^2} \end{array} \right).$$

Now due to the fundamental theorem of calculus we obtain

$$x^* - \alpha = x(0) - x(\hat{\lambda}) = \int_{\lambda}^{0} \frac{dx}{d\lambda} d\lambda = \int_{\lambda}^{0} \frac{f_{yy}}{f_{xx}f_{yy} - f_{xy}^2} d\lambda,$$

where the integrand is unambiguously negative due to concavity of \(f\), and \(\hat{\lambda}\) determines the direction of integration. Therefore, \(x^* > \alpha\) if and only if \(\hat{\lambda} = f_x(\alpha, \hat{y}) > 0\).