Early Default Risk and Surrender Risk: Impacts on Participating Life Insurance Policies

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July 14, 2014

Abstract

We study the fair valuation of participating life insurance policies with surrender guarantees when an early default mechanism is imposed by a regulator. An insurance company is forced to be liquidated once a solvency threshold is reached before maturity. The early default regulation affects the contracts’ value not only directly via changing the contracts’ payment streams but also indirectly via influencing policyholders’ surrender behaviors. In this paper, we endogenize surrender risk by assuming a representative policyholder’s surrender intensity bounded from below and from above and uncover the impacts of the regulation on the policyholders’ surrender decision making. A partial differential equation is derived to characterize the price of a participating policy and solved with the finite difference method. Finally, we discuss the impacts of early default regulation and insurance company’s investment strategies on the policyholder’s surrender behavior as well as on the contract value, which are dependent on policyholder’s rationality level.

Keywords: participating life insurance contracts, early default risk, surrender risk, partial rationality

JEL: G22; G28
1 Introduction

A typical participating life insurance contract provides policyholders a minimum interest rate guarantee and bonus payments upon death and upon survival which are linked to the performance of the insurance company. Usually, additional options are embedded in the contracts to increase their attraction to the policyholders, among which the most popular one is the surrender option. Surrender option entitles the policyholders the right to terminate their contracts prematurely and to obtain the surrender benefit promised by the insurance companies.

The policyholders may not necessarily receive the payments specified in their contracts even if they hold the contracts till the maturity date. If the insurance company does not have enough reserves to pay back its liabilities at the maturity, the policyholders cannot get more than what remains in the company. To protect the policyholders from collecting too little benefit as the insurance company declares bankruptcy at the maturity, regulatory authorities impose early default mechanisms to monitor insurance companies’ financial status and close them before it is too late so. For example, under Solvency II, if a company’s capital becomes lower than the minimum capital requirement, its licence is revoked by the supervisory authority, see Gatzert and Wesker [15]. Also, an insurance company supervised by the Swiss Financial Market Supervisory Authority (FINMA) can lose its license when its risk-based capital drops below the lowest threshold specified in the Swiss Solvency Test (SST), see FINMA Circ. 08/44 'SST’ [13]. Proceeds from liquidated assets are then paid to stakeholders. Hence, the policyholders also face early default risk of the issuing company accompanied with the early regulatory intervention.

Both surrender decision and early default intervention definitely have an direct impact on the fair value of participating policies since they change the policies’ payment streams. In the existing literature, most studies focus on only one of these two aspects. For example, Grosen and Jørgensen [17], Bacinello [2] and Andreatto and Corradin [1] study the fair valuation of participating life insurance contracts with embedded surrender options but have not counted in the early default risk triggered by the bad performance of the insurance company, while Grosen and Jørgensen [18], Jørgensen [20], Bernard et al. [5] and Chen and Suchanecki [8] take into account regulatory intervention in evaluating participating contracts, but leave out surrender risk. The only work that treats early default risk and surrender risk at the same time is Le Courtois and Nakagawa [10] who model surrender risk through a Cox process of an intensity which is correlated to the financial market but is independent of the company’s liquidation threshold. However, since the early termination of the insurance company imposed by the regulator reframes the payment structure
for the policyholders, which we consider as direct impacts, as a response the policyholders may change their surrender decision-making behaviors. This influence of the enforced early bankruptcy on the policyholders’ surrender behaviors can be considered as a 'by-product' of the regulatory intervention, which in turn affects the policies’ payments and correspondingly, the contracts’ value. In this paper, we closely analyze both the direct and indirect ('by-product') impacts of the early default risk on the pricing of the participating policies by endogenously modeling the policyholders’ surrender behaviors and uncovering the impacts of the early default regulation on the surrender behaviors. Besides, when regulatory rules change, insurance companies may react differently to the regulation by adopting different investment strategies, which again affects the contract values directly and indirectly through its influence on policyholders’ surrender behavior. We hence also study the impacts of the investment strategy on the fair value of the insurance contracts and how the insurance companies choose the investment strategies in face of the regulatory rules.

To describe the early default risk we adopt the regulatory framework in Grosen and Jørgensen [18], Jørgensen [20] and Bernard et al. [3], where liquidation is triggered as the insurance company’s asset value drops below a threshold. Concerning surrendering, in most literature, it is assumed that the policyholders are fully rational, which means they can terminate the contract at an optimal time point so that the contract value is maximized, see e.g., Grosen and Jørgensen [16][17], Andreatta and Corradin [1], Bacinello [2] [3], to just name a few. However, since there is not an active market to monitor the contract values, the surrender option is hardly exercised at the right time if a policyholder is not capable of evaluating the contract correctly. Also due to the lack of a trading market for the contracts, the policyholders, when in urgent liquidity needs, have to surrender their contracts at the insurance company and collect the surrender guarantees, which are usually lower than the fair contract values. Empirical evidences which confirm the so called emergency hypothesis are found e.g. in Kuo et. al. [22] and Kiesenbauer [21]. Given the limitations, it is more reasonable to consider policyholders as partly rational from a purely financial point of view. We adopt the approach of modeling policyholders’ partial rationality in Li and Szimayer [24]. They consider surrender as a randomized event assuming that arrival of the event follows a Poisson process with an intensity bounded from below and from above. The lower and upper bounds refer to the minimum surrender rate due to exogenous reasons and maximum surrender rate due to limited rationality respectively. The maximum contract values are then derived by choosing surrender intensities within the two bounds in the worst case scenario (from the perspective of the insurers). This approach corresponds to the spirit of Solvency II. CEIOPS\(^1\) has pointed out that it may be necessary to differentiate between

\(^1\)CEIOPS refers to the Committee of European Insurance and Occupational Pensions Supervisors. It
different insurance products for the purpose of the mass lapse stress. It is also pointed out by the CEIOPS that the lapse risk should be treated differently for different policyholders. For example, the risk of a mass lapse is substantially greater if the policyholders are institutional investors since they tend to be better informed and react more quickly. This indicates that the rationality level of the policyholders also plays a significant role in the analysis. Therefore, we will consider different bounded values of the surrender intensities and analyze how the influence of regulation rules on surrender behaviors differs with respect to policyholders’ rationality. Similar to Li and Szimayer [24] we derive a partial differential equation (PDE) to characterize the price of a participating policy. However, this PDE is only valid when the liquidation threshold has not been touched yet. Otherwise, the policy takes immediately the liquidation value. In this sense, we are solving a barrier option pricing problem. We apply the finite difference method proposed in Zvan et al. [25] and Zvan et al. [26] to solve this problem numerically.

The paper is organized as follows. In section 2 we model the insurance company and introduce the payoff structure of a participating policy. The early default regulatory framework is specified as well. Besides, both the financial market and the insurance market are modeled with respect to the stochastic processes of the underlying asset, the mortality risk intensity and the surrender risk intensity. In section 3 we derive the PDE for the price of the policy. In section 4 we analyze the effect of the regulatory framework and the investment strategy on surrender behaviors as well as on contract values. Section 5 concludes.

2 Model Framework

2.1 Company Overview

Inspired by the model framework in Briys and de Varenne [7], we consider a life insurance company which acquires an asset portfolio with the initial value $A_0$ at time $t_0$ financed by two agents, i.e., a policyholder and an equity holder. The policyholder pays $\alpha A_0$ with $\alpha \in (0, 1)$ to acquire the initial liability $L_0$, i.e., $L_0 = \alpha A_0$. The rest investment $E_0 \equiv (1 - \alpha)A_0$ is levied from the equity holder who has limited liability. The insurance company’s balance was replaced by the European Insurance and Occupational Pensions Authority (EIOPA) since 2011.

CEIOPS advises to take three lapse scenarios into consideration and choose the worst among them to calculate the capital charge for lapse risk, see CEIOPS [9] The advice of limiting to three scenarios is based on the complexity of situations in reality. The mass lapse event is one of the three scenarios considered during the calculation. In our simplified model, however, we are able to find the worst case scenario dynamically.

See CEIOPS [9].
sheet at time $t_0$ is shown in Table 1. The parameter $\alpha$ is called as a wealth distribution coefficient in Grosen and Jørgensen [18].

<table>
<thead>
<tr>
<th>Assets</th>
<th>Liabilities &amp; Equity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_0$</td>
<td>$L_0 \equiv \alpha A_0$</td>
</tr>
<tr>
<td></td>
<td>$E_0 \equiv (1 - \alpha)A_0$</td>
</tr>
</tbody>
</table>

Table 1: Insurance company’s balance sheet at $t_0$

It is assumed that the insurance company operates in a frictionless complete and arbitrage-free financial market over a time interval $[0, T]$, where time $T$ corresponds to the expiration date of the insurance contract. As the insurance contract expires at time $T$, the insurance company closes and its assets are liquidated and distributed to stakeholders.4

2.2 Participating Life Insurance Policy

By investing in the insurance company at time $t_0$, the policyholder signs a participating insurance contract which promises the policyholder a part of the insurance company’s profits in addition to the guaranteed minimum interest rate at maturity date $T$. If the policyholder dies before time $T$, the contract pays death benefits. Additionally, the policyholder can exercise the surrendere option embedded in the contract before maturity $T$ and collects surrender benefits from the insurance company. To summarize, the contract promises survival benefits, death benefits and surrender benefits, depending on which event happens first. In any event, the policyholder has a priority claim on the company’s assets and the equity holder receives what is left.

As the contract expires at maturity $T$, the policyholder receives a minimum guaranteed benefit, which is given by compounding the initial liability $L_0$ with a minimum guaranteed interest rate $g$, i.e., $L_T^g = L_0 e^{gT}$, and a bonus conditional on that the asset value generated by the contribution of the policyholder is enough to cover the minimum guaranteed benefit, i.e., $\alpha A_T \geq L_T^g$. Suppose $\delta$ is a participation rate in the asset surplus, the profits shared with the policyholder are $\delta(\alpha A_T - L_T^g)^+$. However, it may happen that at time $T$ when the company’s assets are liquidated, the assets’ value is lower than the value of the minimum guaranteed benefit. In this case, based on the assumptions that the policyholder has a priority claim on the company’s assets and the equity holder has limited liability, the policyholder collects

\footnote{For simplicity, we assume the company closes when the contract ends. It is not a strict assumption because it can be considered that assets raised from the policyholder and the equity holder are put in a separate fund, as the contract ends, the fund is closed and assets left in the fund are liquidated and distributed to the stakeholders.}
what is left, i.e., $A_T$, and the equity holder walks away with nothing in his hands. To sum up, when the contract survives until the maturity $T$ the policyholder receives survival benefits which take the form

$$
\Phi(A_T) = L_T^g + \delta[\alpha A_T - L_T^d]^+ - \left[ L_T^g - A_T \right]^+.
$$

(1)

The policyholder may die before the contract matures. We use $\tau_d$ to denote the death time of the policyholder aged $x$ at time $t_0$. At time $\tau_d < T$, the contract pays death benefits to the policyholder. We assume that death benefits have the same payment form as survival benefits, but with all the components evaluated at the death time $\tau_d$. We use $d$ and $\delta_d$ to denote the minimum guaranteed interest rate and the participate rate for calculating death benefits respectively. Then, the death payment has the following form at time $\tau_d$

$$
\Psi(\tau_d, A_{\tau_d}) = L_{\tau_d}^d + \delta_d[\alpha A_{\tau_d} - L_{\tau_d}^d]^+ - \left[ L_{\tau_d}^d - A_{\tau_d} \right]^+.
$$

(2)

Furthermore, by exercising surrender option embedded in the contract, the policyholder can terminate the contract before the expiration date $T$. We use $\tau_s$ to denote the surrender time. Once the surrender option is exercised, the company closes and its assets are liquidated and paid to the policyholder as specified in the contract but not more than the liquidated asset value. We consider the following surrender payment form for the policyholder

$$
S(\tau_s, A_{\tau_s}) = L_{\tau_s}^s - \left[ L_{\tau_s}^s - A_{\tau_s} \right]^+,
$$

(3)

where $L_{\tau_s}^s = (1 - \beta_{\tau_s})L_0 e^{s \tau_s}$ is the minimum surrender guarantee when the asset value suffices. Here, $s$ is the minimum guaranteed interest rate at surrender and $\beta_{\tau_s}$ is a penalty parameter which penalizes the policyholder for early terminating the contract and is assumed to be a deterministic decreasing function of the time. After the policyholder is paid off, the equity holder receives the rest of the value.

### 2.3 Early Default Mechanism

Now we introduce early default risk of insurance company into the model. We consider the presence of an external regulator who watches on the insurance company’s financial status over its operating time horizon. We abstract away from cumbersome bankruptcy rules and procedures applied to insurance companies in practice and assume the insurance company is on-going until either the external regulator intervenes before $T$ or the insurance contract expires at $T$. We adopt the regulatory mechanism introduced by Grosen and Jørgensen [18] and set up a default-triggering barrier based on the minimum survival
guarantee $B_t = \theta L_0 e^{\theta t}$, where $\theta$ is a default multiplier. Once the company’s asset value drops below the barrier before maturity $T$, the company is closed by the regulator and its assets are liquidated and distributed to the stakeholders. Accordingly, we define the early default time $\tau_b$ as the first time that the asset value drops below the barrier,

$$\tau_b = \inf \{t < T \mid A_t \leq B_t\}.$$

The default multiplier $\theta$ is set by the regulator, which actually reflects how intensively the regulator monitors the insurance company and how strongly the regulator intends to protect the policyholder. If the regulator believes that the insurance company is inclined to take the advantage of the policyholder by running a risky business or is not competent enough to manage its assets, the regulator may set a higher default multiplier to protect the policyholder. This implies that the insurance company must bear a higher early default risk. Otherwise, the regulator will set a lower default multiplier, which allows the insurance company to recover from its temporary bad performance. In our model, we restrict $\theta$ to be smaller than $1/\alpha$, which ensures $A_0 > B_0$ so that the insurance company does not default at the initial time $t_0$ when the contract is just issued to the policyholder. The restriction makes economic sense since in reality there would be no equity holder who would like to invest in a company that will be shut down by the government at the opening day.

We consider following early default benefits paid to the policyholder when the company is closed by the regulator

$$\Upsilon(\tau_b, A_{\tau_b}) = \min \{A_{\tau_b}, L_{\tau_b}^g\}.$$

The policyholder receives the lower payment value of the liquidated assets and the minimum survival guarantee accrued at the guarantee rate $g$ up to the early default date, i.e., $L_{\tau_b}^g = L_0 e^{g\tau_b}$. Accordingly, if the company has enough asset value to cover the minimum survival guarantee, the equity holder obtains what is left after paying off the policyholder; otherwise, the equity holder gets nothing.

### 2.4 Mathematical Formulation

In this section we model the financial market and the insurance market mathematically. We assume that the company invests its total initial assets in traded (risk-free and/or risky) assets on the financial market which is defined on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$. The risk-free interest rate, denoted as $r$, is assumed to be deterministic in time. Under the market probability measure $\mathbb{P}$, the company’s asset price process $A$ is assumed to be
governed by the following stochastic process

\[ dA_t = a(t, A_t)A_t dt + \sigma(t, A_t)A_t dW^P_t, \quad \forall t \in [0, T]. \]  

Here \( W^P \) is a standard Brownian motion under \( \mathbb{P} \). It generates the filtration \( \mathcal{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T} \) which reflects all the information available on the financial market. The functions \( a \) and \( \sigma > 0 \) refer to the expected rate of return and the volatility of the asset process respectively, and both are regular enough to guarantee the unique solution of (4).

As the payoff of the contract depends not only on the asset value itself but also on the occurrence of the death event or the surrender event, we enlarge the filtration \( \mathcal{F} \) to summarize all the information relevant to the contract valuation. The filtration \( \mathcal{F} \) is thus enlarged to \( \mathcal{G} = \mathcal{F} \vee \mathcal{H} \) where \( \mathcal{H} \) is generated jointly by the jump processes \( H_t = 1_{\{\tau_d \leq t\}} \) and \( J_t = 1_{\{\tau_s \leq t\}} \), i.e., the information about whether the policyholder dies before time \( t \) and whether he surrenders the contract before time \( t \). The probability space is enlarged correspondingly to \( (\Omega, \mathcal{G}, \mathbb{G}, \mathbb{P}) \). The hazard rate of the random time \( \tau_d \), also called mortality intensity, is denoted as \( \mu \) and is assumed to a deterministic function of time.\(^5\) Similarly, we call the hazard rate of the random time \( \tau_s \) the surrender intensity and denote it as \( \gamma \). The surrender intensity \( \gamma \) needs to be modeled here. Based on our arguments in the introduction, we take into account that the policyholder has partial rationality in surrendering the contract and follow the approach adopted in Li and Szimayer [24]. The policyholder may have his personal reasons which urge him to surrender the contract prematurely. Hence, the surrender intensity at least takes the value of \( \rho \). In addition, the policyholder will also update his financial market information to some extent, at the intensity of \( \bar{\rho} - \rho \), so as to make the decision about whether to surrender the contract or not. The more frequently he updates and/or analyzes financial information, i.e. the larger the increase in the intensity is, the more rational he is. In case that \( \bar{\rho} = \infty \), we are back in the setting where the policyholder may surrender the contract at any time when it is optimal to do so and a pure American-style contract should be priced by solving an optimal stopping problem. The decision will be made by comparing the continuation value of the contract and the value of surrender benefits, which are denoted as \( v(t, A) \) and \( S(t, A) \) respectively. Depending on which decision the policyholder makes, the endogenous surrender intensity is thus either 0

\(^5\)In the literature there are many discussions on stochastic mortality intensity which is more consistent with the reality, see e.g. Bacinello et al. [4], Biffis [6], Dahl [11], Dahl and Møller [12]. However, the stochastic feature of the mortality intensity does not have too much influence on the contract value, see Li and Szimayer [23]. Hence, we assume a deterministic mortality intensity for simplicity and focus more on the surrender risk and the early default risk.
or $\bar{\rho} - \rho$. To summarize, the surrender intensity takes the form of

$$
\gamma_t = \begin{cases} 
\rho, & \text{for } S(t, A) < v(t, A) \\
\bar{\rho}, & \text{for } S(t, A) \geq v(t, A)
\end{cases}
$$

which actually characterizes the surrender intensity in the worst case scenario from the perspective of the insurer and maximizes the contract value within the bound $(\rho, \bar{\rho})$, see Li and Szimayer [24] for more details.

Under the assumption that the financial market is complete and arbitrage-free for the insurance company, a unique risk-neutral probability measure $Q$ exists, see Harrison and Kreps [19], under which the contract can be fairly priced. Under the risk-neutral probability measure $Q$, the company’s asset process is described by

$$
dA_t = r(t) A_t dt + \sigma(t, A_t) A_t dW^Q_t, \quad \forall t \in [0, T],
$$

where $W^Q$ is a standard 1-dimensional Brownian motion under $Q$. Furthermore, we assume that mortality risk and surrender risk are diversifiable for the insurance company.\(^6\) Hence, the mortality intensity $\mu$ and surrender intensity $\gamma$ also represent $Q$-intensities under the risk-neutral probability measure $Q$, see Li and Szimayer [24] for a more detailed illustration of the difference between the $P$-intensities and the $Q$-intensities.

### 3 Contract Valuation

In this section we evaluate the contract by taking both the (early) default risk and the surrender risk into consideration. We are not able to find a closed-form pricing formula for the contract, since the surrender intensity can only be determined endogenously within our model. However, by applying the PDE approach, we can specify the surrender intensity and the contract value simultaneously. The contract value is thus not represented by a pricing formula but characterized by a PDE equation and solved numerically with the finite difference method. This approach is applied by Li and Szimayer [24] in a similar setting when they price unit-linked life insurance contracts with surrender guarantees. The crucial point in this paper is that after introducing the early default mechanism, the contract payoff to the policyholder is linked to the solvency of the company and has a barrier option property. Thus, in order to evaluate the contract, we need to distinguish between the region

\[^6\]Notice that the policyholder in our model represents a large pool of policyholders in reality.
option pricing. For \( A_t \leq B_t \) at time \( t \in (0, T) \), the insurance company must be liquidated and the policyholder only obtains \( Y(t, A_t) \). For \( A_t > B_t \), we represent the contract value at time \( t \leq \tau_d \wedge \tau_s < T \) as

\[
V_t = \begin{cases} 
  t < \tau_d \wedge \tau_s & V(t, A_t) + \begin{cases} 
  t = \tau_d, \tau_d < \tau_s & \Psi(t, A_{\tau_d}) + \begin{cases} 
    t = \tau_s, \tau_s < \tau_d & S(\tau_s, A_{\tau_s}) 
  \end{cases}, 
  \end{cases} 
\end{cases}
\]

Then on the set \( \{ t < \tau_d \wedge \tau_s \wedge \tau_b \wedge T \} \), we apply the balance law based on the no-arbitrage condition

\[
\frac{\partial v(t, A_t)}{\partial t} + r(t)v(t, A_t)dt = \mathbb{E}_Q[dV_t|G_t]. \tag{6}
\]

On the set \( \{ t < \tau_d \wedge \tau_s \wedge \tau_b \wedge T \} \), we compute the differential of \( V \) as

\[
dV_t = dv(t, A_t) + (\Psi(t, A_t) - v(t, A_t))dH_t + (S(t, A_t) - v(t, A_t))dJ_t, \text{ for } 0 \leq t < T, \tag{7}
\]

where \( H \) and \( J \) refer to the jump processes with the \( Q \)-intensities \( \mu \) and \( \gamma \) respectively. A jump in \( H \) or \( J \) leads to the change in the payment liability either at the amount of \( \Psi(t, A_t) - v(t, A_t) \) or \( S(t, A_t) - v(t, A_t) \). Taking the expected value of \( dV_t \), we obtain

\[
\frac{\partial v(t, A_t)}{\partial t} + r(t)v(t, A_t)dt = \mathbb{E}_Q[dv(t, A_t)|G_t] + (\Psi(t, A_t) - v(t, A_t))\mu(t)dt + (S(t, A_t) - v(t, A_t))\gamma(t)dt. \tag{9}
\]

By applying Ito’s Lemma to \( dv(t, A_t) \), we have

\[
\mathbb{E}_Q[dv(t, A_t)|G_t] = \mathbb{E}_Q[\mathcal{L}v(t, A_t)dt + \sigma(t, A_t)A_t\frac{\partial v(t, A_t)}{\partial A}dW_Q(t)|G_t] = \mathcal{L}v(t, A_t)dt,
\]

where \( \mathcal{L} \) is defined as

\[
\mathcal{L}f(t, A) = \frac{\partial f}{\partial t}(t, A) + r(t)A\frac{\partial f}{\partial A}(t, A) + \frac{1}{2}\sigma^2(t, A)A^2\frac{\partial^2 f}{\partial A^2}(t, A).
\]

Thus we summarize the pricing PDE with the following proposition.

**Proposition 1.** For \( (t, A) \in [0, T) \times \mathbb{R}^+ \), the value of the participating policy described in section 2.2 is the solution of the partial differential equation

\[
\mathcal{L}v(t, A) + \mu(t)\Psi(t, A) + \gamma(t)S(t, A) - (r(t) + \mu(t) + \gamma(t))v(t, A) = 0, \tag{11}
\]

\[\text{Notice that in the region } A_t > B_t, \text{ there would not be early default after the instantaneous time period } dt \text{ since the asset process is assumed to be continuous in our model.}\]
subject to \( v(t, A) = \Upsilon(t, A) \) for \( A \leq B_t \), with

\[
\gamma_t = \begin{cases} 
\rho, & \text{for } S(t, A) < v(t, A), \\
\bar{\rho}, & \text{for } S(t, A) \geq v(t, A),
\end{cases}
\]

and termination condition \( v(T, A) = \Phi(A), \) for \( A \in \mathbb{R}^+ \).

The integral representation of the solution to the above pricing PDE is shown in Corollary 1 and proved in Appendix A. Due to the existence of an early default boundary, we have here a Dirichlet’s problem to solve.

**Corollary 1.** Suppose the surrender intensity \( \gamma \) is given. The value of the participating policy \( V \) can be represented on \( \{ t < \tau_s \land \tau_d \land \tau_b \land T \} \) by

\[
V_t = \mathbb{E} \left[ \int_t^{\tau_b \wedge T} e^{-\int_t^u (r(u)+\mu(u)+\gamma(u,A_u))du} (\mu(s)\Psi(s,A_s) + \gamma(s,A_s)S(s,A_s))ds \\
+ 1_{\{\tau_b \geq T\}} \Phi(A_T) e^{-\int_t^T (r(u)+\mu(u)+\gamma(u,A_u))du} + 1_{\{\tau_b < T\}} \Upsilon(\tau_b, A_{\tau_b}) e^{-\int_{\tau_b}^T (r(u)+\mu(u)+\gamma(u,A_u))du} \right] \bigg| \mathcal{F}_t
\]

In Li and Szimayer [24] they have proved formally the relationships between surrender intensities and unit-linked life insurance contracts values in a setting without the regulatory issue. We can prove in the similar way that the statement also holds within a given regulatory framework and under a given investment strategy. We illustrate it in Proposition 2 and the proof is provided in Appendix B.

**Proposition 2.** Suppose the early default mechanism is characterized by the default multiplier \( \theta \) and the insurance company’s investment strategy by \( \sigma \). Furthermore, suppose that \( v \) is the value function of the participating policy with the bounds of the surrender intensity being \( \rho \) and \( \bar{\rho} \), and that \( w \) is the value function of the contract with bounds \( \zeta \) and \( \bar{\zeta} \). Assume that \( \bar{\zeta} \leq \rho \) and \( \bar{\rho} \leq \bar{\zeta} \). Then we have \( w(t, A) \geq v(t, A) \), for \( (t, A) \in [0, \tau_b \land T] \times \mathbb{R}^+ \).

4 Numerical Analysis

In this section we adopt the finite difference method proposed by Zvan et al. [25] and Zvan et al. [26] to numerically solve the PDE with a continuously applied barrier (11) as stated in Proposition 1 and study the effects of the early default risk and the surrender risk on the fair valuation of the contract as well as on the insurance company’s investment strategies. The insurance company is set up with initial asset value \( A_0 = 100 \) and 85% of the asset value is acquired by the policyholder who buys the participating contract at
time $t_0$ as the initial liability, which means $\alpha = 0.85$. The contract matures in $T = 10$ years and promises the same participation rate $\delta = \delta_d = 0.9$ at maturity and at death. The risk-free interest rate is $r = 0.04$. The volatility of the company’s asset process is constant, i.e., $\sigma(t, A_t) = 0.2$. The volatility provides information about the riskiness of the insurance company’s investment strategy. A higher $\sigma$ indicates a higher riskiness of the investment strategy while a lower $\sigma$ implies a more conservative investment strategy.\footnote{On a market with only one risk-free and one risky asset, a higher asset volatility is achieved by investing more into the risky asset.} The minimum guaranteed interest rates at survival, at death and at surrender are $g = d = s = 0.02$. As for the mortality intensity, we follow Li and Szimayer [24] and assume that it follows a deterministic process $\mu(t) = A^\mu + Bc^x + t$ for the policyholder aged $x = 40$ at $t_0 = 0$ with $A^\mu = 5.0758 \times 10^{-4}$, $B = 3.9342 \times 10^{-5}$, $c = 1.1029$. Additionally, the penalty parameter takes the form

$$\beta_t = \begin{cases} 
0.05 & \text{for } t \leq 1, \\
0.04 & \text{for } 1 < t \leq 2, \\
0.02 & \text{for } 2 < t \leq 3, \\
0.01 & \text{for } 3 < t \leq 4, \\
0 & \text{for } t > 4.
\end{cases}$$

The parameters are summarized again in Table 2.

<table>
<thead>
<tr>
<th>Market Parameters</th>
<th>Contract Parameters</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_0$</td>
<td>$\alpha$</td>
</tr>
<tr>
<td>100</td>
<td>85%</td>
</tr>
<tr>
<td>$r$</td>
<td>$T$</td>
</tr>
<tr>
<td>0.04</td>
<td>10</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>$\delta, \delta_d$</td>
</tr>
<tr>
<td>0.2</td>
<td>0.9</td>
</tr>
<tr>
<td>$A^\mu$</td>
<td>$g, d, s$</td>
</tr>
<tr>
<td>$5.0758 \times 10^{-4}$</td>
<td>0.02</td>
</tr>
<tr>
<td>$B$</td>
<td>$c$</td>
</tr>
<tr>
<td>$3.9342 \times 10^{-5}$</td>
<td>1.1029</td>
</tr>
</tbody>
</table>

Table 2: Parameter specifications

The analysis in the following sections is conducted for a representative policyholder. Under the assumption that the pool of policyholders is large enough, the surrender intensity of a representative policyholder gives the indication at the portfolio level about the proportion of policyholders who will surrender the contracts. The implications for the large pool of policyholders will be summarized in section 5 to conclude the paper.
4.1 Effects of Regulatory Frameworks on Contract Valuation

In this section we analyze the effects of the early default risk on fair contract valuation. The magnitude of the early default risk depends on the strictness of the regulatory framework, which in our model is represented by the default multiplier $\theta$ that is specified by the regulator. It indicates how the regulator judges the insurance company’s ability to manage its assets. If the regulator is very confident about the expertise of the insurance company and about the financial market, it will tolerate the temporary poor performance of the company more and hence choose a lower default multiplier so that the company has the chance to recover. Otherwise, it will set a higher value to protect the policyholders from not being able to obtain the guaranteed benefits promised by the company. Although a lower (higher) default multiplier is less (more) effective to protect the policyholders from the downside development of the company, it gives the company more (less) chance to recover from the temporary bad performance and pay out more (less) to the policyholders when it does recover. Hence, the level of the default multiplier has great influence on the payoff of the contract and thus on the contract value.

Furthermore, the policyholder takes into account the impacts of the protection from the regulator on the payments of his contract and adjusts his surrender behavior accordingly, which indirectly influences the contract value. Intuitively, the policyholder makes his surrender decision not only based on benefits that are promised by the insurance company but also on the ability of the insurance company to meet its promise. The early default mechanism insures the ability of the company to meet its promise by imposing a limit on the asset value of the insurance company. A higher default multiplier indicates that the policyholder has to worry less about the second issue because they are better protected and will surrender the contract only when the surrender benefits are very attractive. On the contrary, if the default multiplier is set lower so that the policyholders would not be protected completely, the policyholders must take the default risk of the insurance company seriously into account when implementing their surrender strategies. In this case, the policyholders may be willing to surrender the contract earlier to avoid losing too much of their initial investment.

In Table 3 we present the contract values for different default multipliers $\theta$ and different rationality levels represented by $(\rho, \bar{\rho})$. In the second column are the contract values in the case when there is no early default mechanism. From the third to the fifth column are the contract values with different levels of regulatory strength which are represented by the different values of the default multiplier $\theta$. For example, $\theta = 0.7$ means that the regulator does not allow the insurance company’s asset value to drop below 70% of the
minimum guarantee. $\theta = 1.1$ indicates that the regulator is more conservative and requires the company’s asset value to lie above 110% of the minimum guarantee. Comparing the contract values in columns 2-4 where the early default regulation is first introduced, then strengthened, the contract value increases gradually for all the types of policyholders. Introducing the early termination rule protects the policyholder from the downside risk and increasing the default multiplier enlarges the protection level. An interesting feature is that the effects of early termination regulation not only depend on the default multiplier $\theta$ but also on the rationality level of the policyholder. For example, a policyholder with $(\rho, \bar{\rho}) = (0, 0)$ never surrenders her contract. This is equivalent to a European-type contract which does not allow early termination.\(^9\) We can see that in the case when the policyholder is not competent enough to adopt a rational surrender strategy, an effective early termination regulation may help the policyholder improve his position. However, the benefits from the protection become smaller as the policyholder becomes financially more rational. The default multiplier does not play a significant role when the policyholder is able to exercise the surrender option optimally, i.e. $(\rho, \bar{\rho}) = (0, \infty)$.\(^10\) Since the fully rational policyholder can find the optimal surrender strategy anyway, he does not need the protection of the regulator. However, the positive effect of the strengthening of the regulation disappears as an over-regulation rule is carried out. As the default multiplier increases from 0.9 to 1.1, the contract value decreases in some cases, e.g., when $(\rho, \bar{\rho}) = (0, \cdot)$, $(\rho, \bar{\rho}) = (0.03, \infty)$, among which we can even observe the disadvantage of introducing the early default regulation. For the policyholder with $(\rho, \bar{\rho}) = (0, \infty)$ and $(\rho, \bar{\rho}) = (0.03, \infty)$, the contract becomes even lower than when there is no early default risk. As we have mentioned in Section 2.4 that the policyholder with $\bar{\rho} = \infty$ may surrender the contract at any time when it is optimal to do so, irrespective of exogenous reasons, he is able to protect himself from the downside risk. However, enforcing a termination regulation with a very large default multiplier stops him from obtaining more benefits in the favorable development of the insurance company, which actually lowers the contract’s value. Additionally, if we take a look at the column of the contract values for $\rho = 0$ and $\theta = 1.1$, the contracts have the same value. Since when the default multiplier is so high that the benefits that the policyholder obtains at the liquidation of the insurance company are higher than the surrender benefits, surrendering the contract becomes unattractive, which means there would also be no endogenous reasons to surrender the contracts prematurely. Therefore, if the policyholder does not surrender his contract for exogenous reasons ($\rho = 0$), the contract value stays at the same level as the European-style contract value 89.6619, no matter how financially rational the policyholder is.

\(^9\)However, the policyholder would be better off if he terminates the contract and collects the surrender guarantee when the value of the contract drops down. Notice that the contract value increases when $\bar{\rho} > 0$.

\(^10\)The contract values are all around 92 for $(\rho, \bar{\rho}) = (0, \infty)$ and $\theta = \{0, 0.7, 0.9\}$.
We have discussed in section 1 that due to personal reasons, the policyholder surrender his contract even though he knows the value of surrender guarantee offered by the insurance company is lower than the value of his contract. If the policyholder has to liquidate his contract before maturity, which means exogenous surrender does exist, i.e., \( \rho > 0 \), the contract’s fair value should be lower than when no exogenous surrender exists, \( \rho = 0 \). The decrease in contract value for a given \( \bar{\rho} \) measures the premium that the insurance company should not have charged due to personal non-avoidable liquidity reasons. We call this premium the liquidity premium. In Figure 1 we present the liquidity premia under different

![Figure 1: Liquidity premia as a function of the exogenous surrender intensity \( \rho \in [0, 0.3] \) and the default multiplier \( \theta \in [0, 1.1] \) when \( \bar{\rho} = \infty \).](image-url)
regulatory rules for different exogenous surrender intensities when $\rho = \infty$. We observe the following trends. First, within the same regulatory framework, the liquidity premium becomes larger as the exogenous surrender intensity increases. As the non-rational exogenous surrender intensity increases, the insurance company needs to compensate the policyholder more in terms of lowering contract value in order to make the contract more attractive to the policyholder. Second, the liquidity premium increases faster at a lower $\theta$-level while more slowly at a higher $\theta$-level, until the exogenous surrender intensity $\rho$ also becomes quite large. This indicates that the value of the liquidity premium is more sensitive to the policyholder’s exogenous surrender intensity level $\rho$ at a lower $\theta$-level, where the protection from the regulator is low and it is more necessary for the regulator to urge the insurance company to assess the exogenous surrender rates more correctly. On the contrary, as the intervention by the regulator is enhanced, the probability that the insurance company is closed increases. Liquidation may happen before the policyholder exercises the surrender option due to exogenous reasons. Since the policyholders is not penalized at the liquidation, he may receive more than the surrender guarantees they may otherwise obtain from surrendering the contract. Suppose that the insurance company is asked to keep the liquidity premium at a similar level so that the disadvantage to the policyholders who surrender the contracts due to exogenous reasons is kept at a comparable level, the company is then justified to penalize premature surrender more severely if the policyholders are already well protected by the regulator. However, the penalty should be lower if the protection level is low.

Similar to the above discussion on the impacts of exogenous surrender intensity on the contract value, endogenous surrender intensity also influences the contract value. Since the policyholder has limited information on financial market situation and/or limited knowledge to evaluate the contract on his own, i.e., $\rho < \infty$, he may fail to surrender the contract when he should do so. The contract value decreases for given $\rho$ as the upper bound surrender intensity $\bar{\rho}$ changes from infinitely large value to a not enough high value. This decrease in the contract value measures the premium that the insurance company should not have charged due to limited information and/or limited evaluation ability, which we name as rationality premium. In Figure 2 we plot the rationality premia as a function of the upper bound surrender intensity $\bar{\rho}$ and the default multiplier $\theta$ given $\rho = 0$. We observe that the rationality premium increases with the increase of $\bar{\rho}$ at a given protection level $\theta$ settled by the regulator. It is natural to observe that the rationality premium decreases with $\rho$, compare Proposition 2. Furthermore, for given rationality level $(\rho, \bar{\rho})$, the rationality premium decreases when the intervention level from the regulator is enhanced. Intuitively, after the early default mechanism is introduced and even enhanced, the policyholder has fewer sur-
render strategies at disposal to carry out which lower his gains from a rational surrender action. Therefore, the insurance company should charge lower rationality premium. On the other hand, the intervention by the regulator helps the policyholder close his contract prematurely without bringing penalties to him. Such an intervention protects the policyholder at the moment when he should exercise the surrender option but does not do it for various reasons. Hence, the contract value may also increase. The overall effect depends on the power of these two aspects and is reflected by the final contract value presented e.g. in Table 3. We also see that the variation of the rationality premium is more obvious when the early default threshold is lower indicating the importance of assessing the endogenous surrender intensity correctly in an non-strict regulation environment.

![Figure 2: Rationality premia as a function of the upper bound surrender intensity $\bar{\rho} \in [0.3, 30]$ and the default multiplier $\theta \in [0, 1.1]$.](image)

4.2 Effects of Insurance Company’s Investment Strategies

In this section we focus on the influence of insurance company’s investment strategies on contract valuation. Furthermore, we analyze the insurance company’s risk-shifting incentives within the two regulatory frameworks. The investment strategy is represented by the volatility $\sigma$ of the underlying asset $A$. The higher the volatility $\sigma$ is, the higher is the risk that the insurance company has entered into. In Table 4 we display the contract values for different values of $\sigma$. The default multiplier $\theta$ is set to be 0.9.\footnote{We have also studied the cases with $\theta = 0.7$ and 1.1. However, we have not found any qualitative difference in the effect of the volatility $\sigma$ and hence do not present all the results here.}
Table 4: Contract values for different investment strategies represented by $\sigma$ and different rationality levels represented by $(\bar{\rho}, \tilde{\rho})$, $\bar{\theta} = 0.9$

<table>
<thead>
<tr>
<th></th>
<th>$\sigma = 0.1$</th>
<th>$\sigma = 0.2$</th>
<th>$\sigma = 0.3$</th>
<th>$\sigma = 0.1$</th>
<th>$\sigma = 0.2$</th>
<th>$\sigma = 0.3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>(0, 0)</td>
<td>85.3380</td>
<td>85.6141</td>
<td>84.7199</td>
<td>86.4204</td>
<td>90.4937</td>
<td>92.4513</td>
</tr>
<tr>
<td>(0, 0.03)</td>
<td>85.5737</td>
<td>86.0368</td>
<td>85.2578</td>
<td>86.5152</td>
<td>90.5088</td>
<td>92.4547</td>
</tr>
<tr>
<td>(0, 0.3)</td>
<td>86.7156</td>
<td>88.1531</td>
<td>87.9902</td>
<td>87.0566</td>
<td>90.6270</td>
<td>92.4836</td>
</tr>
<tr>
<td>(0, $\infty$)</td>
<td>88.3422</td>
<td>92.0546</td>
<td>93.3676</td>
<td>88.3424</td>
<td>92.0628</td>
<td>93.4082</td>
</tr>
<tr>
<td>(0.03, 0.03)</td>
<td>82.8209</td>
<td>81.8567</td>
<td>79.7188</td>
<td>83.7463</td>
<td>86.6744</td>
<td>88.1438</td>
</tr>
<tr>
<td>(0.03, 0.3)</td>
<td>84.0278</td>
<td>84.2656</td>
<td>83.0419</td>
<td>84.3402</td>
<td>86.8343</td>
<td>88.1934</td>
</tr>
<tr>
<td>(0.03, $\infty$)</td>
<td>85.5405</td>
<td>88.5391</td>
<td>89.6150</td>
<td>85.5407</td>
<td>88.5436</td>
<td>89.6439</td>
</tr>
<tr>
<td>(0.3, 0.3)</td>
<td>78.2582</td>
<td>75.4561</td>
<td>71.5565</td>
<td>78.4962</td>
<td>78.0482</td>
<td>77.8317</td>
</tr>
<tr>
<td>(0.3, $\infty$)</td>
<td>80.7500</td>
<td>80.7500</td>
<td>80.7500</td>
<td>80.7500</td>
<td>80.7500</td>
<td>80.7500</td>
</tr>
</tbody>
</table>

The influences of the investment strategies are different in different regulatory frameworks. For the no early default case, we observe three tendencies, which depend on the policyholder’s rationality. When $(\rho, \bar{\rho}) = (0, \infty), (0.03, \infty)$, the policyholder is considered to be very financially rational because the contract will be terminated immediately when it is optimal to do so. The increase of the underlying asset risk also implies the potentially higher rate of return, which a rational policyholder can exactly capture. Hence, the contract value increases with the asset risk. For $(\rho, \bar{\rho}) = (0, 0), (0, 0.03), (0, 0.3)$ and $(0.03, 0.3)$, which indicates either low exogenous surrender intensity or relatively but not enough high endogenous surrender intensity, we observe firstly the increase and then the decrease in the contract value. When the underlying asset risk increases but still stays at a lower level, the downside risk is still limited and the optimal surrender intensity during the life time of the contract stays anyway at a lower level. However, the chance in the participation of the favorable development of the asset value increases. Hence, overall the contract value increases slightly when $\sigma$ increases from 0.1 to 0.2. When the asset risk further increases, the downside risk could be so high that it is necessary to check more frequently whether to surrender the contract or not. A lower rational surrender intensity in this case would then lead to a lower contract value. When $(\rho, \bar{\rho}) = (0.03, 0.03)$ or $(0.3, 0.3)$, the endogenous surrender intensity is zero and the policyholder surrenders his contract only for exogenous reasons that is not related to the contract value at all. A higher asset risk requires a more rapid and correct response to the changing market situation. When the policyholder is not willing to do so, the contract value for the policyholder will decrease with the increase of the volatility $\sigma$.

As the early default mechanism is implemented by the regulator, the contract value
increases as the volatility $\sigma$ increases in most cases, except when the probability that the policyholder surrender the contract due to exogenous reasons is relatively large, i.e., $\rho = 0.3$. From Table 4 we observe that the contract value decreases and stays constant with $\sigma$ when $(\rho, \tilde{\rho}) = (0.3, 0.3)$ and $(\rho, \tilde{\rho}) = (0.3, \infty)$ respectively. In these cases, the policyholder surrenders the contract even when the asset value develops, which deprives him of the chance of participating in the asset appreciation. Since the policyholder is protected by the regulator through the early default barrier, the potential downside risk is limited while the potential participation in the favorable asset performance is still possible. As long as the policyholder is not so urgent in cashing out of his contract, he can benefits more from the regulator’s protection as the riskiness of the investment strategy increases and his contract value increases accordingly.

Similar to what we did in section 4.1, we present in Figure 3 the liquidity premia under different investment strategies (adopted by the insurance company) in both a world without the intervention of the regulator, see Figure 3 (a), and a world with the early default mechanism, see Figure 3 (b). We see that, for given rationality level $(\rho, \tilde{\rho})$, liquidity premium increases with the volatility of the underlying asset in both cases. As the investment risk of the insurance company increases with a larger volatility $\sigma$, the probability that the policyholder sells back his contract due to exogenous reasons to the insurance company which has been experiencing financial difficulties becomes higher. This indicates that increasing the riskiness of the investment generally does harm to the policyholder who is likely to cash out of his contract when personal difficulties occur. Hence, the contract value decreases more in order to attract policyholders as the insurance company’s investment risk increases, which happens in both cases with and without the early default regulation.

We present the rationality premia depending on investment strategies and endogenous surrender intensities in Figure 4. We see that, given the same endogeneous surrender intensity $\tilde{\rho}$, the rationality premium increases monotonically with the riskiness of the investment strategy $\sigma$ when there is no early default mechanism. The rationality premium is much higher when the endogenous surrender intensity $\tilde{\rho}$ is low. Unlike a policyholder who can track the performance of the insurance company and act optimally to maximize their benefits, a partially rational policyholder faces the risk of mistakenly holding a contract whose value is lower than the value of the surrender guarantee. The risk increases as the insurance company invests more in the risky asset and the company’s asset value becomes more volatile and is reflected by the increasing rationality premium with respect to $\sigma$. However, when the early default mechanism exists, the rationality premium first increases and then decreases with $\sigma$. The decreasing effect can be explained by the protection from the early default mechanism, which as a remedy for the policyholder’s ‘insufficient’ surrendering leads
Figure 3: Liquidity premia as a function of the exogenous surrender intensity $\bar{\rho} \in [0, 0.3]$ and the volatility $\sigma \in [0.05, 0.5]$ when $\bar{\rho} = \infty$.

Figure 4: Rationality premia as a function of the upper bound surrender intensity $\bar{\rho} \in [0.3, 30]$ and the volatility $\sigma \in [0.05, 0.5]$ when $\rho = 0$. 
the insurance company to lower the rationality premium.

Due to the influence of the policyholder’s rationality and the regulatory framework on contract value, the insurance company’s investment strategy is also affected by these two factors. We assume that the insurance company performs in the interest of the equity holder. Since the contract value can be regarded as the market value of the insurance company’s liabilities when the insurance company is ongoing, the purpose of the company, maximizing the residual value for the equity holder, is thus to minimize the value of the policyholder’s policy. From Table 4 we can infer which investment strategy the insurance company tends to adopt. If there is no early default rule and the rationality level of the policyholders is very high, the insurance company will prefer to take low risk investment. This gives us two implications. First, if the policyholder is rational enough to surrender their policies, the regulator does not need to interfere into the insurance company in order for the company to avoid too risky investment. Second, looking back into the history, insurance companies have not always taken conservative strategies. Although there are many reasons for them not to do so. For example, the market interest rate was too low in the past and the insurance company has to invest more riskily to achieve higher excess return so as to meet their payment obligations. Another aspect that we can infer from our study is that the insurance company has actually assumed that the policyholders will not always act optimally. Considering this, it is then inappropriate to price surrender option as a pure American-style option as it is often assumed in the literature, since the policy tends to be overpriced under this assumption which is unfair for the policyholders. It should be noticed that we are not conducting fair contract analysis in this paper. Actually, the contract parameter should be chosen in such a way that the contract value is exactly equal to the initial investment of the policyholder. If the insurance company chooses the fair contract parameters by assuming a high rationality level and leading us to think that it will adopt a low risk investment strategy under this assumption, it actually has the incentive to increase the riskiness of the investment strategy afterwards. This problem will be avoided when the early default regulation is introduced. We can read out from Table 4 that the insurance company will always prefer the low risk investment. Of course, if the fair contract parameters are chosen under the assumption of the full rationality of the policyholders, the true value to them is actually lower when the surrender behavior deviates from the full rationality.
5 Conclusion

In this paper we have studied the impacts of early default risk and surrender risk on participating life insurance policies. Early default is triggered if regulator requires the insurance company to be liquidated once its asset value touches a prespecified threshold. Surrender risk is represented by a surrender intensity which is bounded from below and from above and accounts for the limited rationality of a representative policyholder in making surrender decisions. The lower bound refers to the policyholder’s surrender intensity for exogenous reasons while the upper bound is achieved if the surrender value is higher than the active contract value. Since early default risk affects the contract’s payment structure, it influences the policyholder’s surrender behavior. We have derived the pricing PDE equation which characterizes the contract value and solved it numerically with the finite difference method. Based on the numerical examples, we have analyzed the influence of early default risk, given that the insurance company’s investment strategy is known, on the policyholder’s surrender behavior and consequently on contract valuation. Furthermore, we have analyzed the influence of insurance company’s investment strategy on the policyholder’s surrender behaviors as well as on contract values given the regulatory rule prescribed by the regulator.

The analysis for a representative policyholder can be transferred to a large pool of policyholders. Many implications can be drawn from our analysis. First, if policyholders are able to surrender their participating policies optimally, it is not necessary for the regulator to set a regulatory rule to monitor the insurance company. The insurance company is actually monitored by the policyholders themselves. However, since policyholders are mostly not completely rational to make financially optimal decisions, an early default regulation protects these policyholders so long as the regulation is not too strict. Over-regulation is also disadvantageous to irrational policyholders. Second, enhancing the regulation exerts a negative effect on the rationality premium for those policyholders who would have more freedom to construct their own optimal stopping strategy if there were fewer regulatory constraints. Since the effects of the regulation are different for the policyholders with different surrender reasons and the contract value that we are looking for is the average contract value for all the policyholders, which is denoted as the contract value for a representative policyholder in our paper, the overall effect on the contract value could be positive. In particular, we have observed that the early default regulation has protected the whole contract value when the insurance company adopts a more risky investment strategy, even though the rationality premium has decreased due to increase in the riskiness of the investment strategy. Third, without the introduction of the regulatory framework, we are not clear about the effect of the investment strategy on contract value. We find that the equity holder prefers to adopt a less risky investment strategy if the policyholders are able
to surrender the contracts optimally. Since the equity holder knows that policyholders are most of the time not financially rational enough and there are always exogenous reasons for them to surrender the contracts prematurely, he actually tends to invest more riskily. However, when the early default barrier is settled, an increase in the riskiness of the investment strategy will generally have a positive effect on the contract value from the perspective of the policyholders. The equity holder will then have the incentive to reduce the riskiness of their investment, which is independent of the rationality level of the policyholders. This result is consistent with the goal of the regulator.

Appendix A: Proof of Corollary 1

Proof of Corollary 1. We follow the proof of Theorem 2.1 in Freidlin [14] for a similar Dirichlet problem.

Define
\[

g(t, A) := \mu(t)\Psi(t, A) + \gamma(t, A)S(t, A),
\]
\[
c(t, A) := r(t) + \mu(t) + \gamma(t, A),
\]
\[
Y^A_t := -\int_0^t c(s, A_s)\,ds,
\]
\[
U^A_t := v(t, A_t)e^{Y^A_t}.
\]

According to the Ito Lemma, we obtain, for all \(t < s < \tau_b \wedge T\) where \(t < \tau_s \wedge \tau_b \wedge T\), the stochastic differential equation of \(U^A_s\) as

\[
dU^A_s = \frac{\partial U^A_s}{\partial s} \, ds + \frac{\partial U^A_s}{\partial A_s} \, dA_s + \frac{1}{2} \frac{\partial^2 U^A_s}{\partial A_s^2} \, dA_s^2 \\
= \left[ \frac{\partial v}{\partial s}(s, A_s)e^{Y^A_s} - v(s, A_s)e^{Y^A_s}c(s, A_s) \right] \, ds + \frac{\partial v}{\partial A_s}(s, A_s)e^{Y^A_s}(r(s)A_s \, ds + \sigma(s, A_s)A_s \, dW_s) \\
+ \frac{1}{2} \frac{\partial^2 v}{\partial A_s^2}(s, A_s)e^{Y^A_s}\sigma^2(s, A_s)A_s^2 \, ds \\
= e^{Y^A_s}(Lv(s, A_s) - c(s, A_s)v(s, A_s))ds + e^{Y^A_s} \frac{\partial v}{\partial A_s}(s, A_s)\sigma(s, A_s)A_s \, dW_s \\
= e^{Y^A_s}(-g(s, A_s))ds + e^{Y^A_s} \frac{\partial v}{\partial A_s}(s, A_s)\sigma(s, A_s)A_s \, dW_s
\]

The last equation follows from equation 11 in Proposition 1.

Integrating both sides of the above equation from \(t\) to \(\tau_b \wedge T\) and take the conditional
expectation on both sides. Under the assumption that
\[
\mathbb{E} \left[ \int_0^T \| e^{Y_A} \frac{\partial v}{\partial A_s} \sigma(s, A_s) A_s \|^2 ds \right] < \infty
\]
which ensures
\[
\mathbb{E}_Q \left[ \int_t^T e^{Y_A} \frac{\partial v}{\partial A_s} (s, A_s) A_s dW_s \bigg| \mathcal{F}_t \right] = 0,
\]
we obtain
\[
\mathbb{E}_Q[U_{\tau_b \wedge T}|\mathcal{F}_t] = U_t + \mathbb{E}_Q \left[ \int_t^{\tau_b \wedge T} e^{Y_A} g(s, A_s) ds \bigg| \mathcal{F}_t \right],
\]
and thus
\[
v(t, A_t) = \mathbb{E}_Q \left[ \int_t^{\tau_b \wedge T} e^{Y_A - Y_{\tau_b}} g(s, A_s) ds + e^{Y_A - Y_{\tau_b} v(t, A_t)} \bigg| \mathcal{F}_t \right].
\]

Since
\[
e^{Y_{\tau_b} - Y_{\tau_b}} v(t, A_t) = 1_{\{\tau_b < T\}} e^{Y_{\tau_b}} v(t, A_t) + 1_{\{\tau_b \geq T\}} e^{Y_T} v(T, A_T)
\]
\[
= 1_{\{\tau_b < T\}} e^{Y_{\tau_b}} \gamma(t, A_t) + 1_{\{\tau_b \geq T\}} e^{Y_T} \Phi(A_T),
\]
where the last equation results from the boundary conditions in Proposition 1, we obtain, by substituting \(g(\cdot, \cdot), c(\cdot, \cdot),\) and \(Y\) with their original forms,
\[
V_t = \mathbb{E} \left[ \int_t^{\tau_b \wedge T} e^{-\int_t^s (r(u) + \mu(u) + \gamma(u,A_s)) du} \mu(s) \Psi(s, A_s) + \gamma(s, A_s) S(s, A_s) ds \right.
\]
\[
+ 1_{\{\tau_b \geq T\}} \Phi(A_T) e^{-\int_{\tau_b}^T (r(u) + \mu(u) + \gamma(u,A_u)) du} + \left. 1_{\{\tau_b < T\}} \Upsilon(t, A_{\tau_b}) e^{-\int_{\tau_b}^T (r(u) + \mu(u) + \gamma(u,A_u)) du} \right| \mathcal{F}_t
\]

Corollary 1 is therefore proved. \(\square\)

**Appendix B: Proof of Proposition 2**

**Proof of Proposition 2.** The function \(v\) is the solution of the PDE (11) with terminal condition \(v(T, A) = \Phi(A)\), boundary condition \(v(t, A) = \Upsilon(t, A)\) for \(A \leq B_t\), and bounds \(\underline{\rho}\) and \(\bar{\rho}\). The function \(w\) is the solution of the same PDE (11) with identical terminal condition \(w(T, A) = \Phi(A)\) and boundary condition \(w(t, A) = \Upsilon(t, A)\) for \(A \leq B_t\) but different bounds \(\underline{\zeta}\) and \(\bar{\zeta}\). Assume that \(\underline{\zeta} \leq \underline{\rho}\) and \(\bar{\rho} \leq \bar{\zeta}\). Now define \(z = w - v\). It follows directly that
\( z(T, A) = w(T, A) - v(T, A) = \Phi(A) - \Phi(A) = 0 \) and \( Z(t, A) = \Upsilon(t, A) - \Upsilon(t, A) = 0 \) for \( A \leq B_t \). To obtain the dynamics of \( z \) take the difference of the PDEs describing \( w \) and \( v \), i.e.:

\[
0 = \mathcal{L}w(t, A) + \mu(t)\Psi(t, A) + \gamma^w(t, A)S(t, A) - (r(t) + \mu(t) + \gamma^w(t, A))w(t, A) \\
- (\mathcal{L}v(t, A) + \mu(t)\Psi(t, A) + \gamma^v(t, A)S(t, A) - (r(t) + \mu(t) + \gamma^v(t, A))v(t, A)) \\
= \mathcal{L}z(t, A) + (\gamma^w(t, A) - \gamma^v(t, A))(S(t, A) - w(t, A)) - (r(t) + \mu(t) + \gamma^v(t, A))z(t, A),
\]

where \( \gamma^v \) and \( \gamma^w \), respectively, are given by (2.4) using the appropriate bounds. Similar to the proof of Proposition 1, we obtain the stochastic representation of \( z \) as follows

\[
z(t, s) = \mathbb{E}^{t,A}_Q \left[ \int_t^{u\wedge T} e^{-\int_t^u (r(x)+\mu(x)+\gamma^w(x,A_u))dx} (\gamma^w(u, A_u) - \gamma^v(u, A_u))(L(u, A_u) - w(u, A_u))du \right],
\]

where \( \mathbb{E}^{t,s}_Q \) denotes the expectation conditioned on \( A_t = A \). From the definition of \( \gamma^w \) in (2.4) and the assumption \( \bar{\zeta} \geq \bar{\rho} \) we see that if \( (S - w) \geq 0 \) we have \( \gamma^w = \bar{\zeta} \geq \bar{\rho} \geq \gamma^v \) and thus \( (\gamma^w - \gamma^v) \geq 0 \). On the other hand, if \( (S - w) < 0 \) then \( \gamma^w = \bar{\zeta} \). By assumption we have \( \bar{\zeta} \leq \rho \) and thus \( \gamma^w \leq \rho \leq \gamma^v \), or, \( (\gamma^w - \gamma^v) \leq 0 \). Thus, we see that the integrand in the above equation is nonnegative and therefore \( z \geq 0 \). Since \( z = w - v \) we obtain \( w \geq v \). □

## References


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