Optimal Capital Allocation: Mean-Variance Models

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Abstract
This paper studies capital allocation problem based on minimizing loss functions. Two capital allocation models based on the Mean-Variance principle are proposed. General formulas for optimal capital allocations for both models are derived according to quadratic distance measure. In particular, we discuss centrally symmetric distributions and gamma distributions. Some numerical examples are given to illustrate the results.

Mathematics Subject Classifications (2000): 60E15; 62N05; 62G30; 62D05

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1 Introduction and motivation

In this paper, we develop a new methodology for capital allocation. We believe this methodology better accounts for the nature of risk faced by an insurance firm, or a financial intermediary, in general. Let assume that a firm has a portfolio of risks $X_1, \ldots, X_n$. Assume that a company wishes to allocate the total capital $p = p_1 + \cdots + p_n$ to the corresponding risks. Dhaene et al. (2012) proposed a criteria, which is to set the capital amount $p_i$ to $X_i$ as close as possible as measured by some appropriate distance measures. More specifically, they proposed the following optimization problem to model the capital allocation problem:
allocation problem:
\[
\min_{p \in A} \sum_{i=1}^{n} v_i E \left[ \zeta_i D \left( \frac{X_i - p_i}{v_i} \right) \right]
\]  
(1.1)
for
\[
p \in A = \{ p \in \mathbb{R}^n : p_1 + \cdots + p_n = p \},
\]
where the \(v_i\) are nonnegative real numbers such that \(\sum_{i=1}^{n} v_i = 1\), and the \(\zeta_i\) are non-negative random variables such that \(E[\zeta_j] = 1\), and \(D\) is some suitable distance measurement function. A special case of Eq. (1.1) is the following optimization problem
\[
\min_{p \in A} \sum_{i=1}^{n} E \left[ (X_i - p_i)^2 \right].
\]  
(1.2)
As shown in Dhaene et al. (2012) that the solution to Eq. (1.2) is
\[
p_i = p n + \mu_i - \frac{1}{n} \sum_{j=1}^{n} \mu_j
\]
where \(\mu_j = E(X_j)\) for \(j = 1, \ldots, n\). This idea was further generalized in Xu and Hu (2012), where they defined the following loss function:
\[
L(p) = \sum_{i=1}^{n} D(X_i - p_i).
\]
They proposed the following optimization problem:
\[
\min_{p \in A} \mathbb{P}(L(p) \geq t), \quad \forall t \geq 0.
\]

In fact, the idea of minimizing the loss function has been discussed in the framework of premium calculation. For example, Laeven and Goovaerts (2004) used \(D(x) = \max\{x, 0\}\) as distance measure, and Zaks et al. (2006) used quadratic distance measure \(D(x) = x^2\). This topic was further pursued in Frostig et al. (2007), where they used the general convex distance measure.

However, most of the discussion on this topic in the literature has focused only on the magnitude of the loss function \(L\). In practice, we might also be interested in the variability of the loss function \(L\). The relevant idea has appeared in the premium calculation, see, for example, Valdez (2005) and Furman and Landsman (2006). Furman and Landsman (2006) used the tail variance risk (TVP) measure estimating the variability along the tails to compute the premium.

\[
TVP_q(X) = \text{TCE}_q(X) + \beta \text{TV}_q(X), \quad \beta \geq 0,
\]
where \(\text{TCE}_q\) means conditional tail expectation,
\[
\text{TCE}_q(X) = E(X|X > x_q), \quad \text{TV}_q(X) = \text{Var}(X|X > x_q),
\]
where \(x_q\) is \(q\)th quantile of \(X\) or Value-at-Risk (VaR). The idea of incorporating the variability with the mean might be traced back to the Mean-Variance framework; see, for example, Steinbach (2001) and Landsman (2010). The mean variance (MV) model uses the Mean-Variance risk measurement
\[
\text{MV}(X) = \mathbb{E}(X) + \beta \text{Var}(X), \quad \beta \geq 0,
\]
which is also known as the expected quadratic utility in finance literature.
Motivated by their work, in this paper we propose two new MV models to allocation capitals, which control both magnitude and variability of the loss function $L$ using the quadratic function as the distance measure. More specifically, we consider the following MV model:

$$P_1: \begin{cases} \min_{p \in A} \left\{ \alpha E \left[ \sum_{i=1}^{n} (X_i - p_i)^2 \right] + (1 - \alpha) \text{Var} \left( \sum_{i=1}^{n} (X_i - p_i)^2 \right) \right\}; \\ A = \{ p \in \mathbb{R}^n : p_1 + \cdots + p_n = p \}, \end{cases} \quad (1.3)$$

where $0 \leq \alpha \leq 1$. This optimization problem might be interpreted as $100\alpha\%$ compromise between the mean and variance. The second MV model follows the idea of traditional MV model:

$$P_2: \begin{cases} \min_{p \in A} \left\{ E \left[ \sum_{i=1}^{n} (X_i - p_i)^2 \right] + \beta \text{Var} \left( \sum_{i=1}^{n} (X_i - p_i)^2 \right) \right\}; \\ A = \{ p \in \mathbb{R}^n : p_1 + \cdots + p_n = p \}, \end{cases} \quad (1.4)$$

where $\beta > 0$. The optimization problems $P_1$ and $P_2$ are equivalent by setting $\beta = (1 - \alpha)/\alpha$. However, Models $P_1$ and $P_2$ have different interpretations.

The rest of this paper is organized as follows. In Section 2, we present our main results. Section 3 discusses some applications, which include elliptical distributions and gamma distributions. Some numerical examples are also given there. In Section 4, we present some discussions on the case of the absolute deviation distance measure.

## 2 Main results

The following lemma, called second-order sufficiency conditions (cf. Bertsekas, 1999, Proposition 3.2.1), will be used to prove our main result.

**Lemma 2.1** Assume that $f$ and $h$ are twice continuously differentiable, and let $L(p, \lambda) = f(p) + \lambda h(p)$.

If

$$\nabla_p L(p^*, \lambda^*) = 0, \quad \nabla_\lambda L(p^*, \lambda^*) = 0,$$

and for all $y \neq 0$ with $[\nabla_p h(p^*)]^T y = 0$,

$$y^T \nabla^2_{pp} L(p^*, \lambda^*) y > 0,$$

where $\nabla$ is the differential operator and $x^T$ is the transpose of $x$, then $p^*$ is a strict local minimum of $f$ subject to $h(p^*) = 0$.

Now, we are ready to present the following result.

**Theorem 2.2** If the covariance matrix $\Sigma$ of $(X_1, \ldots, X_n)$ is positive definite, then $p^* = (p_1^*, \ldots, p_n^*)$ is an optimal allocation solution to Problem $P_1$, given by

$$p_i^* = \frac{-\sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} \delta_l \sum_{j=1}^{n} a_{ij} \delta_j + \sum_{j=1}^{n} a_{ij} \delta_j}{\sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} \delta_l \sum_{j=1}^{n} a_{ij} \delta_j}, \quad i = 1, \ldots, n,$$

where

$$\delta_i = 4(1 - \alpha) \sum_{j=1}^{n} \sigma_{2,j,i} + 2\alpha \mu_i,$$

$$\sigma_{2,j,i} = \text{Cov}(X_j^2, X_i), \quad \mu_i = \mathbb{E}(X_i), \quad \text{and } (a_{ij})_{n \times n} \text{ is the inverse matrix of } A = 8(1 - \alpha)\Sigma + 2\alpha I_n, \text{ where } I_n \text{ is the identity matrix.}$$
Proof: The proof utilizes the methodology of Lagrange multipliers. Define
\[ f(p) = \alpha \mathbb{E} \left[ \sum_{i=1}^{n} (X_i - p_i)^2 \right] + (1 - \alpha) \text{Var} \left( \sum_{i=1}^{n} (X_i - p_i)^2 \right), \]
and
\[ h(p) = p - \sum_{i=1}^{n} p_i, \]
and
\[ L(p, \lambda) = f(p) + \lambda \left( p - \sum_{i=1}^{n} p_i \right). \]
Using Lemma 2.1, we set
\[ \frac{\partial L(p, \lambda)}{\partial p_i} = 0, \quad i = 1, \ldots, n, \quad (2.1) \]
and
\[ \frac{\partial L(p, \lambda)}{\partial \lambda} = 0. \quad (2.2) \]
Note that
\[ \text{Var} \left( \sum_{i=1}^{n} (X_i - p_i)^2 \right) = \sum_{i=1}^{n} \text{Var} \left( (X_i - p_i)^2 \right) + 2 \sum_{1 \leq i < j \leq n} \text{Cov} \left\{ (X_i - p_i)^2, (X_j - p_j)^2 \right\} \]
\[ = \text{Var} \left( (X_i - p_i)^2 \right) + 2 \sum_{j=2}^{n} \text{Cov} \left\{ (X_i - p_i)^2, (X_j - p_j)^2 \right\} + C_1 \]
\[ = 4p_i^2 \text{Var}(X_i) - 4p_i \text{Cov}(X_i^2, X_i) - 4p_i \sum_{j=2}^{n} \text{Cov} \left\{ X_i, (X_j - p_j)^2 \right\} + C_2, \]
where \( C_1 \) and \( C_2 \) are constants, which are not related to \( p_1 \). We have
\[ \frac{\partial L(p, \lambda)}{\partial p_1} = \alpha \left[ -2 \mathbb{E}(X_1) + 2p_1 + (1 - \alpha) \left( 8p_1 \text{Var}(X_1) - 4 \text{Cov}(X_1^2, X_1) - 4 \sum_{j=2}^{n} \text{Cov} \left\{ X_1, (X_j - p_j)^2 \right\} \right) \right] - \lambda \]
Here and henceforth, \( \mu_i = \mathbb{E}(X_i) \), \( \sigma_i^2 = \text{Var}(X_i) \), \( \sigma_{2,i} = \text{Cov}(X_i^2, X_i) \) and \( \sigma_{j,i} = \text{Cov}(X_j, X_i) \) for any \( i, j = 1, \ldots, n \). Similarly, for \( i = 2, \ldots, n \),
\[ -2 \alpha \mu_i + 2 \alpha p_i + (1 - \alpha) \left[ 8p_i \sigma_i^2 - 4 \sum_{j=1}^{n} \sigma_{2,j,i} + 8 \sum_{j \neq i}^{n} p_j \sigma_{j,i} \right] - \lambda = 0, \]
which can be written as the matrix form
\[ [8(1 - \alpha)\Sigma + 2\alpha I_n] \mathbf{p} = \mathbf{z}, \]
where \( \Sigma = (a_{ij})_{n \times n} \), \( p^T = (p_1, p_2, \ldots, p_n) \), and
\[
z^T = (\lambda + \delta_1, \lambda + \delta_2, \ldots, \lambda + \delta_n)
\]
with
\[
\delta_i = 4(1 - \alpha) \sum_{j=1}^{n} \sigma_{2,j,i} + 2\alpha \mu_i, \quad i = 1, \ldots, n.
\]
Since the covariance matrix \( \Sigma \) and \( I_n \) are positive definite, hence nonsingular, the unique solution is
\[
p = A^{-1} z,
\]
where \( A^{-1} = (a_{ij})_{n \times n} \) is the inverse matrix of
\[
A = 8(1 - \alpha) \Sigma + 2\alpha I_n,
\]
which is also symmetric. Hence, it follows that
\[
p_i = \frac{\lambda \sum_{j=1}^{n} a_{ij} + \sum_{j=1}^{n} a_{ij} \delta_j}{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}}.
\]
(2.3)
From Eq. (2.2),
\[
p_1 + p_2 + \cdots + p_n = p,
\]
it holds that
\[
\lambda = \frac{p - \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij} \delta_j}{\sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}}.
\]
(2.4)
Combining Eqs. (2.3) and (2.4), we have
\[
p_i^* = \frac{p - \sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} \delta_l}{\sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl}} \sum_{j=1}^{n} a_{ij} + \sum_{j=1}^{n} a_{ij} \delta_j, \quad i = 1, \ldots, n.
\]
Now, let us consider the Hessian of the Lagrangian. Note that
\[
\nabla_p^2 L(p^*, \lambda^*) = 8(1 - \alpha) \Sigma + 2\alpha I_n.
\]
We have for all \( y \neq 0 \) with \( \nabla_p h(p^*)^T y = 0 \), i.e., \( \sum_{i=1}^{n} y_i = 0 \),
\[
y^T \nabla_p^2 L(p^*, \lambda^*) y = 8(1 - \alpha) y^T \Sigma y + 2\alpha y^T y > 0,
\]
where the inequality follows from the fact that \( \Sigma \) is positive definite. Hence, the required result follows immediately.

Using the similar argument, we give the optimization solution to Problem P2.

**Theorem 2.3** If the covariance matrix \( \Sigma \) of \( (X_1, \ldots, X_n) \) is positive definite, then \( p^* = (p_1^*, \ldots, p_n^*) \) is an optimal allocation solution to Problem P2, given by
\[
p_i^* = \frac{p - \sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} \delta_l}{\sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl}} \sum_{j=1}^{n} a_{ij} + \sum_{j=1}^{n} a_{ij} \delta_j, \quad i = 1, \ldots, n,
\]
where
\[
\delta_i = 4\beta \sum_{j=1}^{n} \sigma_{2,j,i} + 2\mu_i,
\]
\( \sigma_{2,j,i} = \text{Cov}(X_j^2, X_i), \mu_i = \mathbb{E}(X_i) \), and \( (a_{ij})_{n \times n} \) is the inverse matrix of \( A = 8\beta \sum + 2I_n \).
As a direct corollary, we have the following result.

**Corollary 2.4** Let $X_1, \ldots, X_n$ be independent random variables. Then $p^* = (p_1^*, \ldots, p_n^*)$ is an optimal allocation solution to Problem P1, given by

$$p_i^* = \mu_i + \frac{1}{4(1-\alpha)\sigma_i^2 + \alpha} \left[ p - \sum_{j=1}^n \mu_j - \sum_{j=1}^n [2(1-\alpha)\gamma_j\sigma_j^2]/[4(1-\alpha)\sigma_j^2 + \alpha] \right] + 2(1-\alpha)\gamma_i\sigma_i^2,$$

where $\mu_j = \mathbb{E}(X_j)$, and $\gamma_j = \mathbb{E}(X_j - \mu_j)^3/\sigma_j^3$, the skewness of $X_j$.

**Proof:** Since $X_1, \ldots, X_n$ are independent, we have

$$a_{ii} = \frac{1}{8(1-\alpha)\sigma_i^2 + 2\alpha}, \quad \delta_i = 4(1-\alpha)\sigma_{2,i,i} + 2\alpha\mu_i, \quad i = 1, \ldots, n,$

as $a_{ij} = \sigma_{2,j,i} = \sigma_{ij} = 0$ for $i \neq j$. Therefore,

$$p_i^* = \frac{p - \sum_{j=1}^n a_{jj}\delta_j}{\sum_{j=1}^n a_{jj}} a_{ii} + a_{ii}\delta_i = \frac{1}{4(1-\alpha)\sigma_i^2 + \alpha} \left[ p - \sum_{j=1}^n [2(1-\alpha)\sigma_{2,j,j} + \alpha\mu_j]/[4(1-\alpha)\sigma_j^2 + \alpha] \right] + 2(1-\alpha)\sigma_{2,i,i} + \alpha\mu_i.$$

Note that

$$\frac{\sigma_{2,j,j}}{\sigma_j^2} = \gamma_j \sigma_j + 2\mu_j.$$

Hence, the desired result follows. \qed

The following result is a direct consequence of Theorem 2.2, which discusses the exchangeable random variables.

**Corollary 2.5** Let $X_1, \ldots, X_n$ be exchangeable random variables. If $p^* = (p_1^*, \ldots, p_n^*)$ is an optimal allocation solution to Problem P1 (P2), then

$$p_1^* = \cdots = p_n^* = \frac{p}{n}.$$

In fact, if $X_1, \ldots, X_n$ are exchangeable random variables, we may have a general result. To present the result (Theorem 2.8), we need the following two lemmas. First, we recall the concepts of majorization and of Schur-convexity. Let $x_{(1)} \leq x_{(2)} \leq \cdots \leq x_{(n)}$ be the increasing arrangement of components of the vector $x = (x_1, \ldots, x_n)$. For vectors $x, y \in \mathbb{R}^n$, $x$ is said to be majorized by $y$, denoted by $x \preceq_m y$, if

$$\sum_{i=1}^j x_{(i)} \geq \sum_{i=1}^j y_{(i)} \text{ for } j = 1, \ldots, n-1,$$

and $\sum_{i=1}^n x_{(i)} = \sum_{i=1}^n y_{(i)}$. A real-valued function $\phi$ defined on a set $A \subseteq \mathbb{R}^n$ is said to be Schur-convex on $A$ if, for any $x, y \in A$,

$$x \preceq_m y \Rightarrow \phi(x) \geq \phi(y).$$

Lemma 2.6 (Marshall, et al., 2011) Let I ⊂ R be an open interval and let φ : I^n → R be continuously differentiable. Then φ is Schur-convex on I^n if and only if φ is symmetric on I^n, and

\[ (p_1 - p_2) \left[ \frac{\partial \phi(p)}{\partial p_1} - \frac{\partial \phi(p)}{\partial p_2} \right] \geq 0, \quad p \in I^n. \]

Lemma 2.7 Let X_1, …, X_n be exchangeable random variables. If

\[ (p_1, \ldots, p_n) \succeq_m (p_1^*, \ldots, p_n^*), \]

then

\[ \mathbb{E} \left( \sum_{i=1}^{n} (X_i - p_i)^2 \right) \geq \mathbb{E} \left( \sum_{i=1}^{n} (X_i - p_i^*)^2 \right), \]

and

\[ \text{Var} \left( \sum_{i=1}^{n} (X_i - p_i)^2 \right) \geq \text{Var} \left( \sum_{i=1}^{n} (X_i - p_i^*)^2 \right). \]

Proof: Because X_i’s are exchangeable, it is easy to verify that

\[ (p_1, \ldots, p_n) \succeq_m (p_1^*, \ldots, p_n^*) \implies \mathbb{E} \left( \sum_{i=1}^{n} (X_i - p_i)^2 \right) \geq \mathbb{E} \left( \sum_{i=1}^{n} (X_i - p_i^*)^2 \right). \]

Now, define

\[ f(p) = \text{Var} \left( \sum_{i=1}^{n} (X_i - p_i)^2 \right), \]

which is a symmetric function of p. Note that

\[ \frac{\partial f(p)}{\partial p_i} = 8p_i \text{Var}(X_1) - 4 \text{Cov}(X_i^2, X_1) - 4 \sum_{j=2}^{n} \text{Cov}(X_1, (X_j - p_j)^2). \]

Hence, it follows that

\[ \frac{\partial f(p)}{\partial p_1} - \frac{\partial f(p)}{\partial p_2} = 8(p_1 - p_2)\sigma^2 - 4 \left[ \text{Cov} \left\{ X_1, (X_2 - p_2)^2 \right\} - \text{Cov} \left\{ X_2, (X_1 - p_1)^2 \right\} \right] \]

\[ = 8(p_1 - p_2)\sigma^2 - 4 \left[ -2p_2 \text{Cov}(X_1, X_2) + 2p_1 \text{Cov}(X_2, X_1) \right] \]

\[ = 8(p_1 - p_2)(\sigma^2 - \sigma_{12}^2), \]

where \( \sigma^2 = \text{Var}(X_1) = \text{Var}(X_2) \), \( \sigma_{1,2} = \text{Cov}(X_1, X_2) \). Therefore,

\[ (p_1 - p_2) \left[ \frac{\partial h(p)}{\partial p_1} - \frac{\partial h(p)}{\partial p_2} \right] \geq 0, \]

where \( \sigma^2 \geq \sigma_{12} \) is guaranteed by the Cauchy-Schwarz inequality. Hence, the required result follows from Lemma 2.6.

Hence, from Lemma 2.7, we directly have the following result.

Theorem 2.8 If X_1, …, X_n are exchangeable random variables, then the smaller variability (in the sense of majorization) between capital allocations leads to the smaller loss in the MV models.
3 Applications and Examples

In this section, we discuss some specific multivariate distributions, and give some numerical examples.

3.1 Centrally symmetric distributions

A random vector \( \mathbf{X} = (X_1, \ldots, X_n) \) has a distribution centrally symmetric about \( \mu = (\mu_1, \ldots, \mu_n) \) if \( \mathbf{X} - \mu \) and \( \mu - \mathbf{X} \) have the same distribution. It includes many distribution families, one of which is elliptical distribution family. Elliptical distributions are generalizations of the multivariate normal distributions. The class of elliptical distributions contains many well-known distributions as special cases: multivariate normal, multivariate Cauchy, multivariate exponential, and multivariate t-distributions, etc. For more discussions, please refer to Fang et al. (1987), Furman and Landsman (2006) and references therein. Note that for centrally symmetric distributions, we have

\[
\sigma_{2,j;i} = E(X_j^2 X_i) - E(X_j^2) E(X_i)
\]

Since elliptical distributions are symmetric about the means, we have

\[
E[(X_j - \mu_j)^2(X_i - \mu_i)] = 0.
\]

That is,

\[
E(X_j^2 X_i) = E(X_j^2) \mu_i + 2 \mu_j E(X_j X_i) - 2 \mu_j^2 \mu_i
\]

Hence, we have

\[
\delta_{2,j;i} = 2 \mu_j \sigma_{j,i}.
\]

Now, we have the following result.

**Proposition 3.1** Assume \( X_1, \ldots, X_n \) have a distribution centrally symmetric about \( \mu = (\mu_1, \ldots, \mu_n) \). If the covariance matrix \( \Sigma \) of \( (X_1, \ldots, X_n) \) is positive definite, and \( \mathbf{p}^* = (p^*_1, \ldots, p^*_n) \) is an optimal allocation solution to Problem P1 (P2), then

\[
p^*_i = \frac{p - \sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl} \delta_l}{\sum_{k=1}^{n} \sum_{l=1}^{n} a_{kl}} \sum_{j=1}^{n} a_{ij} + \sum_{j=1}^{n} a_{ij} \delta_j,
\]

where \( \delta_l = 8(1-\alpha) \sum_{j=1}^{n} \mu_j \sigma_{j,i} + 2 \alpha \mu_i \) (\( \delta_l = 8\beta \sum_{j=1}^{n} \mu_j \sigma_{j,i} + 2 \mu_i \)), \( \mu_j = E(X_j) \), and \( \sigma_{j,i} = \text{Cov}(X_j, X_i) \).

Next, we present a numerical example to illustrate Proposition 3.1.

**Example 3.2** We use the data reported in Panjer (2002). In that example, an insurance company has 10 lines of business with risks represented by the random vector \( \mathbf{X} = (X_1, \ldots, X_{10}) \). Panjer (2002) reported the correlation matrix, and Valdez and Chernih (2003) reported the covariance matrix \( \Sigma \). The inverse of matrix \( \mathbf{A} = 8(1-\alpha) \Sigma + 2 \alpha \mathbf{I}_n \) can be computed by software R. It can also be verified that \( \Sigma \) is positive definite. The estimated mean vector is

\[
\mu = (25.69, 37.84, 0.85, 12.70, 0.15, 24.05, 14.41, 4.49, 4.39, 9.56).
\]
Now, we use Proposition 3.1 to compute the optimal capital allocations for Problem P1 (P2). Table 1 reports optimal allocation strategies for different $\alpha$’s with a total capital $p = 200$. Table 2 reports the case for different $\beta$’s for Problem P2. It might be observed that we have negative allocations which reflect the diversification benefit, in that these lines of business may reduce capital requirements for the company as a whole.

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<th>.1</th>
<th>2</th>
<th>.3</th>
<th>.4</th>
<th>.5</th>
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<td>-17.464</td>
<td>-6.564</td>
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<td>5.407</td>
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<td>111.949</td>
<td>103.745</td>
<td>95.468</td>
<td>86.719</td>
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<tr>
<td>$p_3^*$</td>
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<td>125.578</td>
<td>91.796</td>
<td>70.746</td>
<td>55.914</td>
<td>44.620</td>
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<td>88.926</td>
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<td>92.691</td>
<td>90.543</td>
<td>86.746</td>
</tr>
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<td>$p_7^*$</td>
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<td>30.753</td>
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</tr>
<tr>
<td>$p_8^*$</td>
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<td>-94.850</td>
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<td>-70.303</td>
<td>-60.548</td>
<td>-50.878</td>
</tr>
</tbody>
</table>

Table 1: Optimal capital allocations for Problem P1 with a total capital $p = 200$ and different $\alpha$ values.

### 3.2 Gamma distributions

Assume that $X_1, \ldots, X_n$ are independent gamma risks with $X_i \sim \Gamma(\alpha_i, \beta_i)$, where $\alpha_i, \beta_i > 0$ are shape and scale parameters, respectively, for $i = 1, \ldots, n$. The density of $X_i$ is given by

$$g_i(x) = \frac{\beta_i^{\alpha_i}}{\Gamma(\alpha_i)} x^{\alpha_i - 1} e^{-\beta_i x}, \quad x \geq 0.$$

Then, the skewness of $X_i$ is

$$\gamma_i = \frac{2}{\sqrt{\alpha_i}}.$$

The mean and variance of $X_i$ are $\mu_i = \alpha_i / \beta_i$, and $\sigma_i^2 = \alpha_i / \beta_i^2$, respectively. Hence, we may directly use Corollary 2.4 to compute capitals required for risks in Model P1.

Because the risks are usually dependent, we also consider a multivariate gamma distribution. There are a number of multivariate gamma distributions in the literature. In the following, we consider losses or
<table>
<thead>
<tr>
<th>β</th>
<th>.1</th>
<th>.2</th>
<th>.3</th>
<th>.4</th>
<th>.5</th>
<th>.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_1^* )</td>
<td>24.825</td>
<td>21.340</td>
<td>18.933</td>
<td>17.018</td>
<td>15.398</td>
<td>13.980</td>
</tr>
<tr>
<td>( p_2^* )</td>
<td>44.823</td>
<td>52.336</td>
<td>59.399</td>
<td>65.417</td>
<td>70.479</td>
<td>74.757</td>
</tr>
<tr>
<td>( p_3^* )</td>
<td>14.361</td>
<td>19.092</td>
<td>23.188</td>
<td>26.895</td>
<td>30.318</td>
<td>33.513</td>
</tr>
<tr>
<td>( p_4^* )</td>
<td>31.175</td>
<td>33.980</td>
<td>34.457</td>
<td>34.148</td>
<td>33.557</td>
<td>32.875</td>
</tr>
<tr>
<td>( p_5^* )</td>
<td>11.834</td>
<td>14.374</td>
<td>16.169</td>
<td>17.541</td>
<td>18.630</td>
<td>19.517</td>
</tr>
<tr>
<td>( p_6^* )</td>
<td>50.686</td>
<td>61.205</td>
<td>68.144</td>
<td>73.132</td>
<td>76.885</td>
<td>79.798</td>
</tr>
<tr>
<td>( p_7^* )</td>
<td>14.361</td>
<td>19.092</td>
<td>23.188</td>
<td>26.895</td>
<td>30.318</td>
<td>33.513</td>
</tr>
<tr>
<td>( p_8^* )</td>
<td>31.175</td>
<td>33.980</td>
<td>34.457</td>
<td>34.148</td>
<td>33.557</td>
<td>32.875</td>
</tr>
<tr>
<td>( p_9^* )</td>
<td>11.834</td>
<td>14.374</td>
<td>16.169</td>
<td>17.541</td>
<td>18.630</td>
<td>19.517</td>
</tr>
<tr>
<td>( p_{10}^* )</td>
<td>0.975</td>
<td>3.084</td>
<td>5.157</td>
<td>6.487</td>
<td>7.446</td>
<td>8.190</td>
</tr>
</tbody>
</table>

Table 2: Optimal capital allocations for Problem P2 with a total capital \( p = 200 \) and different \( \beta \) values.

risks which follow a multivariate gamma distribution introduced by Cheriyan (1941), and generalized by Mathai and Moschopoulos (1991). This distribution has been examined in Furman and Lansman (2005) for risk capital allocations. Let \( X_1, \ldots, X_n \) be independent gamma random variables with \( X_i \sim \Gamma(\alpha_i, \beta_i) \), \( i = 0, \ldots, n \). Denote

\[
Y_j = \beta_0 X_0 + X_j, \quad j = 1, \ldots, n.
\]

The joint distribution of the random vector \( \mathbf{Y}^\tau = (Y_1, \ldots, Y_n) \) is the multivariate gamma distribution defined in Mathai and Moschopoulos (1991). The moment generating function can be written as

\[
M_{\mathbf{X}}(\mathbf{t}) = \left( 1 - \sum_{j=1}^{n} \frac{t_j}{\beta_j} \right)^{-\alpha_0} \prod_{j=1}^{n} \left( 1 - \frac{t_j}{\beta_j} \right)^{-\alpha_j},
\]

where \( \mathbf{t} = (t_1, \ldots, t_n) \); see, for example, Furman and Lansman (2005).

From the moment generating function, we can easily obtain

\[
E(Y_j) = \frac{\alpha_0 + \alpha_j}{\beta_j}, \quad \text{Var}(Y_j) = \frac{\alpha_0 + \alpha_j}{\beta_j^2}.
\]

We may also compute

\[
\text{Cov}(Y_i, Y_j) = \frac{\alpha_0}{\beta_i \beta_j}, \quad i \neq j,
\]

10
\[
\text{Cov}(Y_k^2, Y_i) = 2\frac{\alpha_0(\alpha_0 + \alpha_k + 1)}{\beta_k^2\beta_i}, \quad k \neq i,
\]

and
\[
\text{Cov}(Y_k^2, Y_k) = 2\frac{(\alpha_0 + \alpha_k)^2(\alpha_0 + \alpha_k + 1)}{\beta_k^3}.
\]

Hence, we may compute \( \delta_i = 4(1 - \alpha)\sum_{j=1}^n \sigma_{2,j,i} + 2\alpha \mu_i \) in Theorem 2.2 as following
\[
\delta_i = \frac{8(1 - \alpha)\alpha_0}{\beta_i} \sum_{j \neq i} \frac{\alpha_0 + \alpha_j + 1}{\beta_j^2} + \frac{8(1 - \alpha)(\alpha_0 + \alpha_i)^2}{\beta_i^3}(\alpha_0 + \alpha_i + 1) + 2\alpha \frac{\alpha_0 + \alpha_i}{\beta_i}.
\]

Similarly, we can compute \( \delta_i = 4\beta \sum_{j=1}^n \sigma_{2,j,i} + 2\mu_i \) in Theorem 2.3.

Next, we present a numerical example for illustration.

**Example 3.3** Assume that a company has three business lines \( Y_1, Y_2, Y_3 \), which follow a multivariate gamma distribution defined in *Mathai and Moschopoulos* (1991), with underlying random variables \( (X_0, X_1, X_2, X_3) \). Assume that \( (\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (1, 2, 3, 5) \), and \( (\beta_0, \beta_1, \beta_2, \beta_3) = (.3, .1, .2, .4) \). Hence, the covariance matrix is
\[
\Sigma = \begin{pmatrix}
300 & 50 & 25 \\
50 & 100 & 12.5 \\
25 & 12.5 & 37.5
\end{pmatrix}.
\]

It can be verified that \( \Sigma \) is positive definite. The mean vector is \( (\mu_1, \mu_2, \mu_3) = (30, 20, 15) \). Now, assume the total capital is \( p = 70 \). According to Theorem 2.2, we list the capital allocations for different \( \alpha \)'s in Table 3. For model P2, we report the results for different \( \beta \)'s in Table 4. It is observed that the variability has a significant effect on the capital allocations.

<table>
<thead>
<tr>
<th>( \alpha )</th>
<th>( p_1^* )</th>
<th>( p_2^* )</th>
<th>( p_3^* )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5.405</td>
<td>51.654</td>
<td>12.941</td>
</tr>
<tr>
<td>.1</td>
<td>5.458</td>
<td>51.643</td>
<td>12.899</td>
</tr>
<tr>
<td>.2</td>
<td>5.525</td>
<td>51.629</td>
<td>12.846</td>
</tr>
<tr>
<td>.3</td>
<td>5.610</td>
<td>51.612</td>
<td>12.778</td>
</tr>
<tr>
<td>.4</td>
<td>5.724</td>
<td>51.588</td>
<td>12.688</td>
</tr>
<tr>
<td>.5</td>
<td>5.882</td>
<td>51.556</td>
<td>12.563</td>
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<tr>
<td>.6</td>
<td>6.118</td>
<td>51.506</td>
<td>12.376</td>
</tr>
<tr>
<td>.7</td>
<td>6.507</td>
<td>51.423</td>
<td>12.070</td>
</tr>
<tr>
<td>.8</td>
<td>7.273</td>
<td>51.256</td>
<td>11.472</td>
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<tr>
<td>.9</td>
<td>9.468</td>
<td>50.745</td>
<td>9.787</td>
</tr>
<tr>
<td>1</td>
<td>31.667</td>
<td>21.667</td>
<td>16.667</td>
</tr>
</tbody>
</table>

Table 3: Optimal capital allocations for different \( \alpha \)'s in Problem P1 with a total capital \( p = 70 \).
<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$p^*_1$</th>
<th>$p^*_2$</th>
<th>$p^*_3$</th>
</tr>
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<tbody>
<tr>
<td>.1</td>
<td>9.889</td>
<td>50.641</td>
<td>9.469</td>
</tr>
<tr>
<td>.2</td>
<td>7.724</td>
<td>51.155</td>
<td>11.122</td>
</tr>
<tr>
<td>.3</td>
<td>6.968</td>
<td>51.323</td>
<td>11.709</td>
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<tr>
<td>.4</td>
<td>6.584</td>
<td>51.406</td>
<td>12.009</td>
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<td>.5</td>
<td>6.352</td>
<td>51.456</td>
<td>12.192</td>
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<tr>
<td>.6</td>
<td>6.196</td>
<td>51.489</td>
<td>12.315</td>
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<td>.7</td>
<td>6.084</td>
<td>51.513</td>
<td>12.403</td>
</tr>
<tr>
<td>.8</td>
<td>6.000</td>
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</tr>
<tr>
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<td>5.882</td>
<td>51.556</td>
<td>12.563</td>
</tr>
<tr>
<td>2</td>
<td>5.644</td>
<td>51.606</td>
<td>12.751</td>
</tr>
</tbody>
</table>

Table 4: Optimal capital allocations for different $\beta$’s in Problem P2 with a total capital $p = 70$.

4 Discussion

It is of interest to consider the absolute deviation distance measure. For example, one may consider the following MV model:

$$P3: \begin{cases} \min_{p \in A} \{ \mathbb{E} [\sum_{i=1}^{n} |X_i - p_i|] + \beta \text{Var} (\sum_{i=1}^{n} |X_i - p_i|) \}; \\ A = \{ p \in \mathbb{R}^n : p_1 + \ldots + p_n = p \}, \end{cases}$$

where $\beta \geq 0$. Generally, we have to solve a nonlinear optimization problem for P3. There is no a simple closed form solution in contrast to Problem P1 or P2. Assume that $X_i$ has a distribution $F_{X_i}$, $i = 1, \ldots, n$. For the special case of $\beta = 0$, it has been shown in Dhaene et al. (2012) that the optimal capital allocations are

$$p^*_i = F_{X_i}^{-1}(\alpha) (F_U(p)), \quad i = 1, \ldots, n,$$

where $S^\alpha = \sum_{i=1}^{n} F_{X_i}^{-1}(\alpha)(U)$, $U$ is a uniform random variable on $(0,1)$, and $F_{X_i}^{-1}(\alpha)$ is $\alpha$-mixed inverse distribution function defined as

$$F_{X_i}^{-1}(\alpha)(s) = \alpha F_{X_i}^{-1}(s) + (1 - \alpha) F_{X_i}^{-1+}(s), \quad s \in (0,1), \ \alpha \in (0,1),$$

where $F_{X_i}^{-1}(s) = \inf \{ x \in \mathbb{R} | F_{X_i}(x) \geq s \}$ and $F_{X_i}^{-1+}(s) = \sup \{ x \in \mathbb{R} | F_{X_i}(x) \leq s \}$.

In the following, we present an example to illustrate the allocations based on the absolute deviation distance measure. Assume that $X_1, \ldots, X_n$ are independent and identically distributed risks. Intuitively, we should allocate even capital to each risk regardless the underlying distribution, which is confirmed by Theorem 2.8. However, this might not be true for the case of the absolute deviation distance measure. Note that

$$\text{Var} \left( \sum_{i=1}^{n} |X_i - p_i| \right) = \sum_{i=1}^{n} \text{Var}(|X_i - p_i|) = \sum_{i=1}^{n} [\mathbb{E} (X_i^2) + p_i^2 - 2p_i \mathbb{E} (X_i) - \mathbb{E}^2(|X_i - p_i|)].$$
Therefore,
\[
\frac{\partial \text{Var} \left( \sum_{i=1}^{n} |X_i - p_i| \right)}{\partial p_i} = 2 \left[ p_i - \mathbb{E}(X_i) \right] - 2 \mathbb{E}(|X_i - p_i|)(1 - 2 \bar{F}(p_i)),
\]
where \( \bar{F}(p_i) = 1 - F(p_i) \). Hence, we have
\[
(p_i - p_j) \left( \frac{\partial \text{Var} \left( \sum_{i=1}^{n} |X_i - p_i| \right)}{\partial p_i} - \frac{\partial \text{Var} \left( \sum_{i=1}^{n} |X_j - p_j| \right)}{\partial p_j} \right) = 2(p_i - p_j) \left[ p_i - (1 - 2 \bar{F}(p_i)) \mathbb{E}(|X_i - p_i|) - p_j + (1 - 2 \bar{F}(p_j)) \mathbb{E}(|X_j - p_j|) \right].
\]
If \( \ell(p) = p - (1 - 2 \bar{F}(p)) \mathbb{E}(|X_i - p|) \) is an increasing function of \( p \), it follows from Lemma 2.6 that
\[
(p_1, \ldots, p_n) \succeq_m (p_1^*, \ldots, p_n^*) \implies \text{Var} \left( \sum_{i=1}^{n} |X_i - p_i| \right) \geq \text{Var} \left( \sum_{i=1}^{n} |X_i - p_i^*| \right).
\]
Therefore, we have a similar conclusion to Theorem 2.8 by using absolute deviation distance measure.
Unfortunately, \( \ell(p) \) depends on the value of \( p \). For example, assume \( X_1, \ldots, X_n \) are standard exponential random variables. We plot \( \ell(p) \) in Figure 1. It is seen that \( \ell(p) \) is not a monotone function of \( p \).

![Figure 1: \( \ell(p) \) function.](image)

This means even for independent and identical risks, the capitals should not be always distributed evenly according to the absolute deviation distance measure! More research for capital allocations using the absolute deviation distance measure in MV models is needed.

**References**


