Asset Allocation and Consumption Rule in the Presence of Background Risk and Insurance Market *

Jin-Ruey Lu  
Department of Finance  
National Don Hwa University, Taiwan

Wen-chang Lin  
Department of Finance  
National Chung Cheng University, Taiwan

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Corresponding Author:  
Wen-chang Lin  
168 University Rd.,  
Min-Hsiung, Chia-Yi  
Taiwan 621  
Tel-phone: 886-5-2720411-34210,  
E-Mail: finwcl@ccu.edu.tw

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Abstract

The paper examines how background risk can affect the decisions of insurance, risky assets allocation and consumption rate simultaneously. Given the interdependence between background risk and asset risk, we show the agent can improve his utility if he can account for security risk, insurable and uninsurable background risk jointly in his decisions. We find that background risk and his risk attitude not only affect the ratio of risk free asset in his portfolio but also the relative holding of risky assets. One role of insurance is forming a hedging demand, thus can mitigate the impact from background risk to his portfolio. With insurance, his consumption path can be smoother. We conclude that our results can partly explain the well-known “equity premium puzzle” as well as “asset allocation puzzle” in the financial literature.

Keywords: Insurance Demand, Portfolio Choice, Consumption, Asset Allocation Puzzle, Background Risk
1. Introduction

Earlier studies in insurance demand (e.g., Arrow, 1963; Harris and Raviv, 1978; Holmström, 1979) mostly focus on single (pure) risk and using insurance as the only tool of risk reduction. Later studies (e.g., Gould, 1969; Mayers and Smith, 1983; Briys, 1986, 1988, Gollier, 1994 etc.) then view pure risk as one of the risks in many market risks; thereby, it is nature to derive the insurance decision which is not independent of other financial decisions. Dionne and Doherty (2000) considered several risks, including pure risk, investment risk, background risk and the risk from information asymmetry simultaneously, and pointed out that only insurance policy is not sufficient to manage risks efficiently. Other financial instruments should be jointly used with insurance. To date, many researchers have devoted to the study regarding the integrated decision-making of insurance and portfolio choice. For example, Mayers and Smith (1983) is one of the earliest papers to solve the optimal insurance demand for an agent who also invests his wealth in financial stock and bond in which insurance risk and financial assets are assumed to be correlated. Bryis (1986) later analyzed insurance demand along with investment and consumption decisions in the context of Merton's (1969) portfolio choice problem. Gollier (1994) investigated the strategy of bearing insurable risk and showed how precautionary saving can dominate insurance under certain conditions\(^1\). Meyer and Ormiston (1995) derived the optimal insurance in an aggregate financial portfolio framework. In their study, insurance quantity decision is considered as one of portfolio selection decisions which is used to reduce the quantity of risky asset.

Typically, risk can be classified into two types, i.e., the financial risk mostly caused by the uncertainty of securities return, and the non-financial insurable “background risk”, including pure loss, inflation, labor income risk, political turmoil and natural disasters etc. (see also Doherty and Schlesinger, 1983; Heaton and Lucas, 1997; 2000). Several recent studies (Heaton, Lucas, 1997; 2000; Viceira, 2001) have derived the optimal asset allocation and consumption rate for a representative agent by taking into account the effect of background risk. Many empirical evidences (e.g., Campbell et al., 1999; Campbell and Viceira, 2002; Jagannathan et al., 1998) found that background risk may be positively correlated with the securities investment risks. Under this assumption, Viceira (2001) showed that the increase of labor income risk

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\(^1\) Gollier (1994) showed that insurance demand vanishes in the long run if loading fee exceeds a critical value. However, insurance purchasing is a transitory strategy to protect capital accumulation.
may turn to reduce investor’s consumption and the holding of risky asset. However, not all background risk is uninsurable. A few types of background risk can be transferred through insurance policies in which insurance company can further pool the risk by writing a large numbers of policies and/or simply reduce the risk by reinsurance. For example, an agent can purchase unemployment insurance policy to reduce part of labor income risk. On the other hand, there are many other types of background risk which can be transferred neither by insurance nor by financial instruments, such as domestic political turmoil.

One of the most important financial decision making theories in finance is the joint analysis of portfolio choice and consumption rate which is pioneered by Samuelson (1969), Merton (1971, 1973) etc. and by using continuous-time dynamic programming technique. Their studies are extended by several researchers later to include other risks, such as insurance risk, real asset investment and background risk etc. For example, Svensson and Werner (1993) added a dynamic income process in Merton’s model by using the separation theorem. Unfortunately, after a great deal of effort has been made, several critical decisions (e.g., asset allocation and consumption problem) are still found to be inconsistent with empirical findings and remains as puzzles. For example, most studies predict that a rational investor will choose to allocate his wealth only on a risk free asset and a risky portfolio in which the weights of risk free asset and risky portfolio for an agent depend on his risk attitude. From the famous “mutual fund separation theorem”, it is predicted the relative proportion of risky assets in the risky portfolio will be invariant for everyone. However, this prediction is inconsistent with empirical findings. For instance, Bodie, Merton and Samuelson (1992) investigated the investor’s portfolio selection by accounting for the effect of non-tradable labor income. They found that younger workers tend to have a higher ratio of his wealth invested in the riskier asset than aged workers. Canner et al., (1997) called such inconsistency the “asset allocation puzzle”, which has received many research attentions recently. For example, Bajeux-Besnainou et al., (2003) showed that the risky assets allocation of an investor, who has a hyperbolic absolute risk averse utility, is largely affected by his risk preference, given that the risky asset price is not independent of stochastic interest rates.. Munk et al., (2004) studied the

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2 For instance, Canner et al., (1997) examined many market advices regarding portfolio choice, including cash, bonds, and stocks. They found that most advices are inconsistent with the mutual fund separation theorem.
optimal asset allocation strategy of an investor, who is with power utility function and invests in cash, bond and stock, respectively. Under the setting of mean-reverting stock returns and stochastic interest rate, they found that the optimal decision of portfolio selection is inconsistent with most investment advices by fund managers. Shalit and Yitzhaki (2003) also have similar conclusions by using the approach of stochastic dominance rule.

In both financial and insurance literature, the studies on the joint decisions of insurance and financial portfolio choice in which the associated risks (e.g., insurable and uninsurable background risk, securities risks) are not independent have been explored extensively. In addition to several earliest papers (e.g., Mayer and Smith, 1983; Doherty and Schlesinger, 1983) which already considered the interdependence between insurance and financial decisions, there are many later studies (e.g., Koo, 1998; Heaton and Lucas, 1997; 2000; Viceira, 2001) have addressed how background risk can affect securities investment. Among them, Heaton and Lucas (1997) found that consumption level is not only sensitive to the risk attitude but also labor income risk. They yet predict that the allocation of risky assets is under a considerable wide range. Koo (1998) found that optimal consumption as well as optimal holding ratio of stock will be smaller if market liquidity is constrained and in the presence of uninsurable income risk. Heaton and Lucas (2000) also found that investors may take diverse asset allocation strategies depend on different source and degree of background risk. Viceira (2001) founds that the stock holding ratio for the investors with work is larger than the investors already retired.

Mehra and Prescott (1985) observed that real per capita consumption is more stable than the theoretical expectation in the U.S. during the period of 1889-1978 and referred it to be a puzzle. The puzzle is usually called the "equity premium puzzle" in finance. Mehra and Prescott (1985) found that mean return of stock is unexpectedly higher than the mean return of bond which can not be explained by consumption asset pricing model. The puzzle may be resolved in an economy with rational investors who have the characteristic of habit persistence (Constantinides, 1990). This is because it may lead their consumption path to be smoother when the time separability of preferences is relaxed. In this paper, our model including insurance risk will try to give the puzzle a possible solution. We will show that buying insurance can smooth the consumption path because the consumption process is likely to suffer from the unexpected impact from background risk.
In this study, we will derive the decisions not only optimal insurance, allocation between risk free asset and risky portfolio; more importantly, also derive the optimal allocation of risky assets. We will show a more rigorous condition for the existence of “mutual fund separation theorem” when there is background risk, and the condition that insurance decision may affect asset allocation of the agent. The major theoretic contribution in this paper can be summarized as follows. First, we show that how insurance decision can affect the asset allocation in the risky portfolio indirectly if we are allowed to hedge background risk through insurance market. Second, we develops a model considering background risk and introducing insurance market so that it may provide a possible solution of the well known “asset allocation puzzle” and the “equity premium puzzle” in the finance literature. The remainder of this paper is organized as follows. Section 2 presents an analysis framework of economic system. The main theoretic results about the individual’s optimal decisions in the presence of background risks are reported and discussed in Section 3. Section 4 presents a calibrated analysis of the results. Finally, Section 5 is concluding remarks.

2. The Economy

Follow the setting of continuous-time decision making similar to those developed by Merton (1971, 1973), Briys (1986, 1988), Gollier (1994), and many others. Without loss of generality and to acquire a more attractive theoretical result, we assume that the market is perfect\(^3\), and there is a representative insured/investor who wants to trade financial assets and purchase insurance policies to reduce his pure/background risk.

We assume that the risk free bond which has the following price process \((M(t)):\)

\[
\dot{M}(t) = rM(t)dt, \quad M(0) = 1
\]

(1)

where \(r \geq 0\) is the instantaneous risk free interest rate. In addition, we assume there are two risky assets, the first one is a dividend paying stock, and another is a default free risky bond. Both stock prices \((S(t))\) and bond prices \((B(t))\) (per share) follow geometric Brownian motions. That is, the return

\(^3\) I.e., there are no transactions costs, taxes, information requiring cost, or problems with indivisibilities of assets and liquidity, financial assets have no constraint on the short sales trading. Trading in assets takes place continuously in time.
dynamics of stock and risky bond can be described as follows, respectively.

\[ \hat{S}(t) = (r + \alpha_S)S(t) + \sigma_S S(t) \xi_S(t), \quad S(0) = S_0 \in (0, \infty) \]  

(2)

\[ \hat{B}(t) = (r + \alpha_B)B(t) + \sigma_B B(t) \xi_B(t), \quad B(0) = B_0 \in (0, \infty) \]  

(3)

where \( \alpha_S \geq 0 \) and \( \alpha_B \geq 0 \) are the instantaneous conditional expected percentage price change per unit time of stock and bond, respectively. In general, \( \alpha_S \geq \alpha_B \) since \( \sigma_S > \sigma_B \geq 0 \), which are the instantaneous return volatility of the stock and risky bond, respectively. Moreover, \( \alpha_S, \sigma_S, \alpha_B \) and \( \sigma_B \) are assumed to be constant over the time for simplicity which also implies the investment opportunity set is constant over time. \( \xi_S(t) \sim N(0, 1) \) and \( \xi_B(t) \sim N(0, 1) \) are the stochastic terms of asset price process. \( M_0, S_0 \) and \( B_0 \) are the initial price for the risk-free bond, stock and risky bond, respectively. From (2) and (3), the integral form of risky assets price processes are

\[ S(t) = S(0) + \int_0^t (r + \alpha_S)S(s)ds + \int_0^t \sigma_S S(s)dZ_S(s) \]  

(4)

\[ B(t) = B(0) + \int_0^t (r + \alpha_B)B(s)ds + \int_0^t \sigma_B B(s)dZ_B(s), \]  

(5)

respectively. \( dZ_S(t) = \xi_S(t)dt \) and \( dZ_B(t) = \xi_B(t)dt \) satisfy the filtered probability space \( \{\Omega, \Gamma, \{\Gamma(t)\}, P\} \), where \( \Omega \) is the possible state set, \( \Gamma(t) \) is the natural filtration of \( P \)-augment measures and \( \Gamma(s)^{\mathbb{F}} := \sigma\{Z(s); 0 \leq s \leq t\} \). Assume that the returns of stock and risky bond are instantaneously correlated with coefficient, \( \eta_S(t)dt = E(dZ_S(t)dZ_B(t)) \). The representative investor/insured consumes a representational good in order to maximize his utility. Let \( C(t) \) be the instantaneous quantity of consumption at time \( t \) which is a non-negative \( \Gamma(t) \)-measurable process. It also satisfies:

\[ C(t) \geq 0 \quad \text{and} \quad \int_0^t C(t)dt < \infty \quad \text{a.s.} \ \ t \geq 0 \]

Assume that the agent faces background risk, which is not tradable in the financial market. However, it is possible to transfer part of the risk to through insurance policy. Because background risk may come from a variety of resources, for simplicity, we assume that its dynamic can be characterized by a mixed geometric Brownian motion-Poisson jump process. Specifically, it
includes two stochastic terms, i.e., we use \( dZ_L(t) \) to describe the part which is smooth and continuous in time and state spaces, and let \( dQ(t) \) describing the part of discontinuous changes, which is characterized by a Poisson process with mean parameter \( \lambda \). According to several empirical evidences, (e.g., Campbell et al., 1999; Campbell and Viceira, 2002; Jagannathan et al., 1998), it is proper to assume the instantaneous correlation coefficients of background risk with respect to stock and risky bond \( \eta_{sl}(t)dt = E(dZ_s(t)dZ_L(t)) \) and \( \eta_{bl}(t)dt = E(dZ_b(t)dZ_L(t)) \), respectively, are nonzero.

Insurance premium is assumed to be constant over time. There is no moral hazard or adverse selection problem in the insurance market. In addition, assume that a fraction of individual’s wealth \( (k) \) suffers from background risk, i.e., the quantity of background risk exposure is \( Y(t) = kW(t) \). And, only a fraction of background risk \((1-\delta)\) can be transferred to insurer.

Under the continuous-time setting, the investor/insured needs to simultaneously make the instantaneous quantity decisions of risk free asset \((w_M(t))\), stock \((w_S(t))\), risky bond \((w_B(t))\), insurance policy \((b(t))\) as well as consumption plan \((C(t))\). That is, every dollars of his wealth \((W(t))\) must allocate either in money account, risky asset, insurance policy or simply consume it. In sum, we can write the dynamic process of his wealth as:

\[
\begin{align*}
\frac{dW(t)}{W(0)} & = [rW(t) - C(t) + \alpha_S w_S W(t) + \alpha_B w_B W(t) - \lambda(1 + \pi)bY(t)]dt + \\
& + \sigma_S w_S dZ_S(t) + \sigma_B w_B dZ_B(t) + \\
& + (b - 1 - \delta)\sigma_Y Y(t)dZ_L(t) + (b - 1 - \delta)Y(t)dQ(t) \\
W(0) & \equiv W_0 \in (0, \infty)
\end{align*}
\]

where \( \pi \geq 0 \) is the insurance loading rate, i.e., the contract is known as actuarially fair if \( \pi = 0 \). \( b(t) \in [0, 1] \) is the proportion decision of insurable risk transferring to insurer to be made. I.e., the agent takes full insurance for the insurable background risk if \( b(t) = 1 \). Note that we need the following restriction about the insurable background risk to ensure the feasibility of insurance contract:

\[
1 - \frac{1}{k}(1 + \pi)^{-1} \geq \delta \geq 0
\]

\( \delta = 0 \) indicates that all background risk is insurable through insurance, and the upper limit rule outs the case of over insurance. \( \gamma \) is the risk averse coefficient of the insured.
Equation (6) says that the dynamic process of wealth is affected by two stochastic factors. One is from the investment of financial assets in which the investor/insured is allowed to trade them continuously in order to get optimal hedging effect. Another one is from non-financial risk; it consists of the continuous minor impacts and the large discontinuous impact. However, the existence of insurance market allows the agent to mitigate the non-financial risk. \((\lambda(1+\pi)bY(t))\) stand for the insurance premium he needs to pay.

3. Optimal Insurance, Portfolio Selection and Consumption

As most previous related studies, the agent is like to make decisions on insurance, portfolio allocation and consumption rules to maximize his expected discounted life time utility, i.e.,

\[
\max_{C,\nu_x,\nu_y,b} E_u \left[ \int_0^T e^{-\rho t} U(C(s),s)ds + e^{-\rho T} B(W(T),T) \right]
\]

subject to the wealth budget constraint (equation 6); where \(U(C(t),t)\) is the utility function of consumption at time \(t\) which is assumed to be strictly increasing and concave in \(C\). It also satisfies Inada (1963) conditions\(^4\). \(B(W(T), T)\) is the bequest function which is assumed to be concave in \(W\) and continuously differentiable in time \(t\). \(\rho\) is the associated discount rate for the utility and bequest functions. \(E,[|W(t),t|]\) is the conditional expectation operator conditional on wealth and time.

In the context of stochastic dynamic programming, i.e., the state equation is the wealth process; the control variables are insurance demand, stock investment, risky bond investment and consumption, respectively. The indirect utility of wealth function \(J(W(t),t)\) then can be written as:

\[
J(W(t),t) = \max_{C,\nu_x,\nu_y,b} E_0 \left[ \int_0^T e^{-\rho s} U(C(s),s)ds + e^{-\rho T} B(W(T),T) \right]
\]

The above indirect utility function is assumed to be second-order

\(^4\) That is, \(\lim_{t \to \infty} U_{ss}(C(t),t) = \lim_{t \to \infty} U_{ss}(C(t),t) = \infty\) and \(\lim_{t \to \infty} U_{ss}(C(t),t) = \lim_{t \to \infty} U_{ss}(C(t),t) = 0\).
differentiable, strictly increasing and concave in \( W \) and the time.

Applying the dynamic programming method,\(^5\) the optimality conditions of the problem is to solve the Hamilton-Jacobi-Bellman (HJB) equation:

\[
0 = \max_{C, w_s, w_b} \left( U(C(t)) - \rho J(W(t), t) + J_W(W(t), t)[rW(t) - C(t) + \alpha_s w_s W(t) + \alpha_b w_b W(t) - \lambda(1 + \pi) b Y(t)] + \lambda[J((b - \delta)Y(t), t) - J(Y(t), t)] + \frac{1}{2} J_{ww}(W(t), t)[\sigma_s^2 w_s^2 W^2(t) + \sigma_b^2 w_b^2 W^2(t) + (b - 1 - \delta)^2 \sigma_s^2 Y^2(t) + 2\sigma_s \sigma_b w_s w_b W^2(t)\eta_{sb} + 2\sigma_s \sigma_b (b - 1 - \delta) k W^2(t)\eta_{sl} + 2\sigma_b \sigma_b w_b (b - 1 - \delta) k W^2(t)\eta_{bl}] \right)
\]

where \( J_W = \partial J / \partial W \), \( J_{ww} = \partial^2 J / \partial W^2 \) are the first and second order partial derivatives with respect to wealth, respectively. The sufficient condition of obtaining unique interior maximum is \( J_{ww} < 0 \) in which \( J \) is strictly concave in \( W \). Subsequently, using the principle of Bellman optimality; we can solve the optimal decisions of an individual’s decisions. In the sequel, assume that background risk follows a pure Poisson jump process, as those assumed by Merton (1971, 1973) and Briys (1986), we have the following proposition.

**Proposition 1 (Markowitz-Merton-Briys)**

Given the price changes of the two risky assets are log-normally distributed, and background risk follows a Poisson jump process, then the optimal decisions of an individual’s consumption rules, risky asset allocation and insurance demand are:

\[
C^*(t) = \arg \max 0 = U_C(C(t), t) - J_W(W(t), t) \quad (9)
\]

\[
w^*_s(t) = \frac{G(W(t), t)}{1 - \eta_{sb}^2 W(t)} \left[ \frac{\alpha_s}{\sigma_s^2} - \frac{\alpha_s \eta_{sb}}{\sigma_s \sigma_b} \right] \quad (10)
\]

\[
w^*_b(t) = \frac{G(W(t), t)}{1 - \eta_{sb}^2 W(t)} \left[ \frac{\alpha_b}{\sigma_b^2} - \frac{\alpha_s \eta_{sb}}{\sigma_s \sigma_b} \right] \quad (11)
\]

respectively, where \( G(W,t) = -J_w / J_{ww} > 0 \).

**Proof.** Using the same procedures as in Kamien and Scheartz (1991)\(^6\), we can derive the above results.

In the above proposition, \( G(.) \) stands for the inverse of absolute risk aversion measure as the definition in Pratt (1964). Equation (9) ensures that the marginal utility of consumption equals to the marginal indirectly utility of wealth. The optimal stock holding and risky bond holding (equation 10 and 11) primary depend on two factors. The first one is the risk attitude. A higher value of \( G \) indicates a lower degree of risk averse, i.e., the agent will hold more risky securities. Another one is the mean-variance component of the risky assets, which is essentially the relative distance between the expected return/risk of stock and bond. Clearly, from equation (11), we see the agent will allocate more dollars in the bond if the risky bond has a good performance (in term of \( \alpha_b / \sigma_b \)). Given that background risk is completely insurable and there is no loading fee (\( \pi = 0 \)), the optimal quantity of insurance (equation 12) is full insurance, which is consistent with the findings by Briys (1986).

It is possible to obtain the analytical solution of the above proposition if the form of both utility and bequest function exhibit are specified as isoelastic (i.e., constant relative risk aversion),

\[
U(x) = \begin{cases} 
\frac{x^\gamma}{\gamma}, & \text{if } \gamma \neq 0 \text{ if } x > 0 \\
\log(x), & \text{if } \gamma = 0 \\
-\infty, & \text{if } x \leq 0
\end{cases}
\]

where \( \gamma < 1 \). Hence, the indirectly utility function can be written as:

\[
J(x(t),t) = \frac{\mu^{\gamma^{-1}} x^\gamma}{\gamma}
\]

where \( \mu \) is the risk premium coefficient which is a function of \( \gamma \).

---


\(^7\) The parameter \( 1 - \gamma \) stands for the relative risk aversion measure as defined by Arrow-Pratt. Higher \( 1 - \gamma \) indicates more risk aversion. The sufficient condition for the utility function to be concave is \( 1 - \gamma > 0 \). The second part of utility function is the non-negative constraint of wealth process.
Corollary 1

If the marginal utility function of the agent exhibit iso-elastic, the explicit decisions are:

Optimal holding of riskless asset

\[
W^*_M(t) = 1 - \frac{1}{(1 - \eta_{SB}^2)(1 - \gamma)} \left[ \frac{\alpha_S}{\sigma_S^2} + \frac{\alpha_B}{\sigma_B^2} - \frac{(\alpha_S + \alpha_B)\eta_{SB}}{\sigma_S \sigma_B} \right]
\]  

(13)

Optimal relative holding of risky assets

\[
\frac{W^*_S(t)}{W^*_B(t)} = \frac{\frac{\alpha_S}{\sigma_S^2} - \frac{\alpha_B}{\sigma_B^2}}{\frac{\alpha_S}{\sigma_S} - \frac{\alpha_B}{\sigma_B}} \eta_{SB}
\]  

(14)

Optimal insurance

\[
b^*(t) = \frac{1}{k} \left( \frac{1}{1 + \pi} \right)^{\frac{-1}{\gamma}} + \delta
\]  

(15)

Optimal consumption rate

\[
C^*(t) = \frac{W(t)}{1 - \gamma} \left[ \rho - r \gamma + \lambda \left[ k^\gamma + \gamma k \delta (1 + \pi) - (1 - \gamma)(1 + \pi)^{\frac{-1}{\gamma}} \right] - \frac{\gamma}{2(1 - \eta_{SB}^2)(1 - \gamma)} \left( \frac{\alpha_S}{\sigma_S} - \frac{\alpha_B}{\sigma_B} \right)^2 \right]
\]  

(16)

Proof. See appendix 1.

The above corollary is consistent with traditional separation theorem\(^8\). The holding weight of risk free asset depends on his risk attitude and the performance of risky assets (equation 13). The increase of risky assets performance or the higher degree of risk averse reduces the holding of risk free asset. However, the relative holding of stock and risky bond remain unchanged even if the risk attitude and the wealth of the agent alter. That is, to maximize his (consumption) utility, the agent adjusts only the relative ratio between risk free asset and risky portfolio, as the prediction of Merton’s (1971, 1973). On the other hand, the optimal insurance depends on the loading rate

\(^8\) I.e., the separation of consumption plan, insurance demand and portfolio selection. This can be viewed as an integrated solutions of Merton (1971, 1973) and Briys (1986).
and his risk attitude (equation 15). Lower loading fee increases the demand of insurance. Finally, optimal consumption rate is positively related to current wealth level. The optimal consumption paths can be further portrayed as follows.

**Corollary 2**

*Under the same assumptions, then the optimal consumption path can be expressed as:*

\[
\frac{dC(t)}{C(t)} = \left[ \frac{r - \rho - \lambda(k^{\gamma} + \delta(k^{1+\pi}))}{1-\gamma} + \frac{2-\gamma}{2(1-\eta_{SB}^{2})(1-\gamma)} \left( \frac{\alpha_{S}}{\sigma_{S}} - \frac{\alpha_{B}}{\sigma_{B}} \right)^{2} \right] dt + \\
\left( \frac{1}{k(1+\pi)^{\frac{1}{1-\gamma}}} \right) dQ(t) + \frac{1}{(1-\eta_{SB}^{2})(1-\gamma)} \left[ \begin{array}{c}
\frac{\alpha_{S}}{\sigma_{S}} \\
\frac{\alpha_{B}}{\sigma_{B}} \\
-\eta_{SB} \\
1
\end{array} \right] dZ_{S}(t) + \\
\left[ \begin{array}{c}
-\eta_{SB} \\
1
\end{array} \right] dZ_{B}(t)
\]

\(17\)

**Proof.** See appendix 2

The result indicates that the increase of the probability of loss jump (i.e., background risk) lower consumption growth, which is also largely affected by the yields of risky assets. Moreover, we find that the volatility of optimal consumption paths is closely related to the risk of risky assets as well as background risk. The above optimal solutions can be achieved because of the formation of a hedging portfolio when the correlation matrix of the two risky assets is known.

Several studies (e.g., Mehra and Prescott, 1985; Constantinides, 1990) found that the actual consumption process is smoother than the prediction of the theoretic model used in most previous studies, which primarily consider only the financial investment, but ignore the effect of background risk as well as the existence of insurance market. By jointly considering the insurance market and the financial market, our model is allowed to examine how buying insurance can affect the consumption path. The relationship between the quantity of insurance purchase, i.e., full insurance, partial insurance and self-insurance, respectively, and consumption path can be summarized as follows.
Table 1. Insurance coverage and consumption process

<table>
<thead>
<tr>
<th>Insurance coverage</th>
<th>Expected consumption growth rate</th>
<th>Expected consumption volatility rate</th>
</tr>
</thead>
<tbody>
<tr>
<td>Full insurance</td>
<td>( m - \lambda (1 + \pi) k )</td>
<td>( \delta k \times 100% )</td>
</tr>
<tr>
<td>Optimal insurance</td>
<td>( m - \lambda(1+\pi)^{1-\gamma} - \lambda \delta k(1+\pi) )</td>
<td>( (k - (1+\pi)^{1-\gamma}) \times 100% )</td>
</tr>
<tr>
<td>Self-insurance</td>
<td>( m )</td>
<td>( (1+\delta)k \times 100% )</td>
</tr>
</tbody>
</table>

where

\[
m = \frac{r - \rho - \lambda(k' + \delta(1+\pi)k)}{1-\gamma} + \frac{2-\gamma}{2(1-\eta_s)(1-\gamma)} \left( \frac{\alpha_s - \alpha_B}{\sigma_s - \sigma_B} \right)^2 + \lambda k(1+\pi) \left[ \delta + \frac{1}{k} (1+\pi)^{1-\gamma} \right]
\]

is the associated parameter of consumption growth. Table 1 shows that the agent must bear entire impact of background risk to his consumption path if insurance is not available. Conversely, when full insurance is undertaken and there is a positive premium loading, all insurable background risk is eliminated but the expected consumption growth decelerates. Finally, if partial insurance is allowed, the optimal quantity of insurance is the one balancing the growth and the volatility of consumption path because it can attain the optimal lifetime utility. This, essentially, uses the same spirit as Gollier (1994). However, we find the above theoretical prediction may give a good explanation to the puzzle from actual consumption data.

The above analysis is carried out under the assumption that the dynamic behavior of background risk follows a pure Poisson process. This is plausible if background risk is mostly due to some unexpected events. In this case, the above analysis can be directly compared with the results by Merton (1969) and Briys (1988). However, there are other types of background risk which is more appropriate to be described using the continuous-time framework, such as the risk due to inflation. Subsequently, the risk may be also correlated with the risky assets. For example, the income from the commission of a life insurance agent may be correlated with interest rate. In the sequel, we use a mixed geometric Brownian motion and Poisson jump process to describe background risk.
Proposition 2

Given the price changes of the two risky assets are log-normally distributed, and background risk follows a mixed geometric Brownian motion and pure Poisson jump process, then the agent’s optimal decisions of consumption rules, risky asset allocation and insurance are:

Optimal holding of risk free asset:

\[
\begin{aligned}
w^*_M &= 1 - \frac{1}{(1 - \eta^2_{SB})(1 - \gamma)} \left[ \frac{\alpha_S}{\sigma^2_S} + \frac{\alpha_B}{\sigma^2_B} - \frac{(\alpha_S + \alpha_B)\eta_{SB}}{\sigma_S \sigma_R} \right] + \\
&\quad \left( b^* - 1 - \delta \right) k \frac{\eta_{SL} - \eta_{BL} \eta_{SB}}{\sigma_S} + \frac{\eta_{BL} - \eta_{SL} \eta_{SB}}{\sigma_B}
\end{aligned}
\]  
(22)

Optimal relative holding of risky assets:

\[
\begin{aligned}
\frac{w^*_S}{w^*_B} &= \frac{1}{(1 - \eta^2_{SB})(1 - \gamma)} \left( \frac{\alpha_S}{\sigma^2_S} - \frac{\alpha_B \eta_{SB}}{\sigma_S \sigma_R} \right) - \frac{\sigma_L (b^* - 1 - \delta) k}{\sigma_S (1 - \eta^2_{SB})} \left( \eta_{SL} - \eta_{BL} \eta_{SB} \right) \\
&\quad \frac{1}{(1 - \eta^2_{SB})(1 - \gamma)} \left( \frac{\alpha_B}{\sigma^2_B} - \frac{\alpha_S \eta_{SB}}{\sigma_S \sigma_R} \right) - \frac{\sigma_L (b^* - 1 - \delta) k}{\sigma_B (1 - \eta^2_{SB})} \left( \eta_{BL} - \eta_{SL} \eta_{SB} \right)
\end{aligned}
\]  
(23)

Optimal insurance coverage must satisfy the following nonlinear equation:

\[
\lambda k (b^* - \delta)^{1 - (1 - \gamma)} = \left( b^* - 1 - \delta \right) \sigma^2_L \left(1 - \frac{(\eta_{SL} - \eta_{BL})^2}{1 - \eta^2_{SB}} \right) - \\
\frac{\sigma^2_L}{1 - \eta^2_{SB}} \left[ \frac{\alpha_S}{\sigma_S} (\eta_{SL} - \eta_{SB} \eta_{BL}) + \frac{\alpha_B}{\sigma_B} (\eta_{BL} - \eta_{SB} \eta_{SL}) \right] - \lambda (1 + \pi) = 0
\]  
(24)

respectively.

Proof. Appendix 3

In proposition 2, it is important to note that, the optimal holding of risk free asset is not only affected the performance of risky asset (in term of \(\alpha / \sigma\)) but also need to account for the effect of background risk (equation 22) even if full insurance is taken (i.e., no insurable background risk). This is because, in our sense, the transferring of background risk through insurance company is incomplete; thereby, the leftovers risk still affects the decision of asset allocation. With optimal insurance, the relative holding of cash (risk free asset) should be negatively correlated with background risk when both stock
and bond are positively correlated with background risk. That is, the agent will form a hedging demand with respect to the risky assets to reduce the impact from background risk. On the contrary, if the risky assets correlate negatively with background risk, the relative holding of cash increases along with background risk. The agent will allocate more his wealth in risk free asset rather than in the risky assets.

On the other hand, the relative holdings of risky assets in his risky portfolio will depend on his risk attitude, insurance coverage and non-financial investment risk (equation 23). Surprisingly, with insurance, we find that the mutual fund separation theorem suggested by Merton (1971, 1973) is not supported in our case. Specifically, the higher degree of risk averse, the relative holding of stock/bond may increase or decrease depends on the relative performance of the risky assets. If the agent increases his insurance coverage, the effect of risk attitude and background risk to the allocation of the risky assets will reduces. Ultimately, the allocation of the risky assets is independent with his risk attitude if full insurance is purchased even if all background risk is insurable.

The non-linear equation (24) shows that there is only one variable left to be solved from which the optimal insurance can be obtained. Compared to equation (15), it is easy to see that the insurance decision is affected by premium loading, risk attitude, jump risks, the volatilities of the risky assets and background risk as well the correlations among them; however, it remains independent of the consumption decision. While the jump in the background risk does not affect the portfolio selection directly and only the stochastic part does so that it is possible for the agent to arrange a hedge operation against the risk from the securities investment and background risk.

Again, we investigate how insurance coverage affects consumption paths in the following corollary when background risk follows a mixed geometric Brownian motion and Poisson jump process.

**Corollary 3**

*Given the price changes of the two risky assets are log-normally distributed, and background risk follows a mixed geometric Brownian motion and pure Poisson jump process, then the optimal consumption paths is given by:*
\[
\frac{dC(t)}{C(t)} = \left[ r - \mu - \lambda (1 + \pi) k b^* + \frac{1}{(1 - \eta^2_{SB})(1 - \gamma)} \left( \frac{\alpha_S}{\sigma_S} - \frac{\alpha_B}{\sigma_B} \right)^2 \right. \\
\left. - \frac{(b^* - 1 - \delta) k \sigma_L}{(1 - \eta^2_{SB})} \left( \frac{\eta_{SL} - \eta_{BL} \eta_{SB}}{\sigma_S} + (\eta_{BL} - \eta_{SL} \eta_{SB}) \frac{\alpha_B}{\sigma_B} \right) \right] dt + \\
(b^* - 1 - \delta) k dQ(t) + (b^* - 1 - \delta) k \sigma_L dZ_L(t) + \\
\left\{ \frac{1}{(1 - \eta^2_{SB})(1 - \gamma)} \left( \frac{\alpha_S}{\sigma_S} - \frac{\alpha_B}{\sigma_B} \right) - \frac{(b^* - 1 - \delta) \sigma_L k}{(1 - \eta^2_{SB})} \left( \eta_{SL} - \eta_{BL} \eta_{SB} \right) \right\} dZ_S(t) + \\
\left\{ \frac{1}{(1 - \eta^2_{SB})(1 - \gamma)} \left( \frac{\alpha_B}{\sigma_B} - \frac{\alpha_S}{\sigma_S} \right) - \frac{(b^* - 1 - \delta) \sigma_L k}{(1 - \eta^2_{SB})} \left( \eta_{BL} - \eta_{SL} \eta_{SB} \right) \right\} dZ_B(t) 
\]

**Proof.** First obtain the coefficient of risk premium \( \mu \) using similar procedures as corollary 1. Then follow the same steps as in corollary 2, we have the above result given that \( C(t) = \mu W(t) \).

By integrating the financial market and the insurance market, the corollary shows that the consumption path is principally affected by; (1). the size of the background risk which can be transferred to the insurer, and (2). the hedging function for securities investment and uninsurable background risk in the financial market. We see the consumption path becomes smoother if the agent buys insurance. Moreover, the risk from securities investment decreases because of the hedge demand for the insurance against the financial assets risk when part of background risk can be transferred to insurance company.

**4. Numerical Analysis**

In this section, we carry out a calibration analysis by giving some hypothetical values of parameters close to the real market\(^9\). We first examine the comparative static of the optimal insurance, the relative holding of financial assets and consumption rate with respect to various parameters, respectively. Subsequently, we investigate, to what extent; buying insurance can affect the allocation of risky assets as well as consumption path. The initial setting of parameter values are given as follows.

\(^9\) The numerical examples are illustrated given that the background risk follows a mixed geometric Brownian motion and pure Poisson process.
Table 2. Hypothetical Values of Parameters

<table>
<thead>
<tr>
<th>Risk Attitude</th>
<th>Financial Assets</th>
<th>Insurance Policy</th>
</tr>
</thead>
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<tr>
<td>$\gamma$</td>
<td>$\alpha_S$</td>
<td>$\delta$</td>
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<td>0.1</td>
<td>0.3</td>
</tr>
<tr>
<td>$\rho$</td>
<td>$\alpha_B$</td>
<td>$k$</td>
</tr>
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<td>0.06</td>
<td>0.7</td>
</tr>
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<td>$\sigma_L$</td>
<td>$\pi$</td>
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<td>0.2</td>
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<tr>
<td>$\sigma_B$</td>
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<tr>
<td>0.15</td>
<td>$\sigma_L$</td>
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</tr>
<tr>
<td>$r$</td>
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<td>$\eta_{SL}$</td>
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</tr>
<tr>
<td>$\eta_{BL}$</td>
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<td></td>
</tr>
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</table>

4.1 The Insurance Decision

Figure 1 shows the optimal insurance coverage by different loading rates as well as the frequencies of jump arrival. Interestingly, we find that the optimal insurance is relatively inelastic to the growth of loading rate (from 0% to 60%) while the jump frequency of background risk has a considerable effect on optimal insurance.

Figure 1. The Optimal Insurance by Various Premium loading and Frequency of Loss Jump
Figure 2. The Optimal Insurance by Various correlation Coefficients between Financial Risk and Background Risk

As shown in previous studies, we recognize that the optimal insurance will be not the same as the decisions made unless financial risk is not independent with background risk. The fact that lower correlation between the two risks increase the insurance demand (Figure 2) implies that, insurance can be used to transfer background risk only if the correlation between the two risks is low. However, because insurance policy usually comes with considerable loading fee, a more efficient way of improving his utility is to form a hedging component in his decisions when the correlation between the two risks is not too low.

Figure 2 also shows that, the optimal decision is buying full insurance when the correlations are negative. Full insurance is the constraint commonly employed in insurance contract to eradicate possible moral hazard and adverse selection problems in the insurance market. Without such constraint, the optimal decision will be over full insurance because the agent not only uses insurance to transfer insurable background risk but also obtain higher diversification effect of financial risk.
Figure 3. Optimal Insurance by Various Sizes of Background Risk and Financial Risk

Figure 3 illustrates the effect of background risk \( (\sigma_L) \) on optimal insurance. The optimal insurance coverage is positively related to the volatility of stock, which means the agent can purchase more insurance to eliminate insurable risk so that total risk is also reduced. That is, there is a hedge demand embedded in insurance policy to against financial risk. It is essentially the same spirit of integrated risk management (IRM).

4.2 The Portfolio Choice

Table 4 and 5 show how the holding of stock and the relative holding of stock and bond can be affected by the background risk and insurance price.
Table 4. The Optimal Holding of Stock ($w_s$) by Various Premium loading and Loss Jump Frequency

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\pi = 0.0$</th>
<th>$\pi = 0.1$</th>
<th>$\pi = 0.2$</th>
<th>$\pi = 0.3$</th>
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<td>1.2373</td>
<td>1.2373</td>
<td>1.2373</td>
<td>1.2373</td>
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<td>1.2521</td>
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Table 5. Optimal Holding Ratio of Stock to Bond ($w_s / w_b$) by Various Premium Loading ($\pi$) and Jump Frequency ($\lambda$)

<table>
<thead>
<tr>
<th>$\lambda$</th>
<th>$\pi = 0.0$</th>
<th>$\pi = 0.1$</th>
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<th>$\pi = 0.3$</th>
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<td>2.5961</td>
</tr>
<tr>
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<td>2.5868</td>
<td>2.5881</td>
<td>2.5890</td>
<td>2.5905</td>
<td>2.5917</td>
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</table>

The optimal holding of stock, *ceteris paribus*, is positively related to the jump frequency of background risk because the agent likes to mitigate the impact from loss events (see Table 4) through financial market. On the other hand, it is negatively correlated with premium loading because the agent likes to hold more cash in order to pay insurance premium. Nevertheless, the
effect of increasing loading fee to stock holding is relatively minor. Moreover, optimal relative holding ratio of stock to bond \((w_s / w_b)\) is negatively correlated with the jump frequency of background risk (see Table 5). In contrast to the increase of stock holding as in Table 4, the agent decreases his holding ratio of stock with respect to bond when the probability of loss events increases. While the holding of stock and bond both increase, we note that the bond increases more than the stock holding because the return/risk ratio of bond is lower. Still, both assets can provide the hedging function against background risk. Again, we find that the increase of insurance premium has only minor effect to optimal relative holding ratio of stock to bond.

Table 6. Optimal Relative Holding Ratio of Stock to Bond \((w_s / w_b)\)
by Various Correlation Coefficients between Risky Assets and Background Risk \((\eta_{SL} \text{ and } \eta_{BL})\)

<table>
<thead>
<tr>
<th>(\eta_{SL})</th>
<th>-0.2</th>
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<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
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<td>1.2540</td>
<td>1.2285</td>
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<td>1.1799</td>
<td>1.1567</td>
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<td>1.2750</td>
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</table>

Table 6 shows that the relative holding of risky assets \((w_s / w_b)\) positively correlates with the correlation coefficient between stock and background risk \((\eta_{SL})\), but negatively correlates with the correlation between bond and background risk \((\eta_{BL})\). Moreover, we also find that various degrees of correlation between assets and background risk have insignificant effect on the holding of stock.
Table 7. Optimal Relative Holding Ratio of Stock to Bond \( \left( \frac{w_s}{w_b} \right) \) by Various Volatilities of Assets

<table>
<thead>
<tr>
<th>( \sigma_s )</th>
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<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
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<td>0.0164</td>
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<td>0.0160</td>
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</tbody>
</table>

Table 7 shows how the relative holding ratio of stock to bond \( \left( \frac{w_s}{w_b} \right) \) is a decreasing function of the volatility of stock, given the volatility of background risk unchanged. The relative holding ratio also positively correlates with the volatility of background risk \( \left( \sigma_L \right) \) which indicates the agent needs higher holding ratio of stock to form a hedge portfolio when the backgrounds risk increases. The optimal holding ratio of stock needs to take into account not only for its risk but also the hedging function of background risk.

4.3 The Effect of Insurance Demand and Risk Attitude to Portfolio Choice

As we mentioned in previous section, the use of insurance contract for background risk transferring can also affect the allocation of financial assets. Table 8 shows how the optimal allocations of stock, bond and cash are affected by insurance coverage and risk attitude, respectively. Panel A and B in Table 8 illustrate two scenarios in which the insurable ratio of background risk \( \left( \delta \right) \) equals to 0.3 and 0, respectively. The holding ratios of the two risky assets increase significantly as the coefficient of risk aversion \( \left( \gamma \right) \) increases; on the contrary, the holding of cash decreases. The consequence that asset allocation is closely related with the agent’s risk attitude is consistent with standard portfolio choice theory.
Table 8. Optimal Assets Allocation by Various Insurance Quantity and Risk Attitude ($\gamma$)

<table>
<thead>
<tr>
<th></th>
<th>Panel A : $\delta = 0.3$</th>
<th>Panel B : $\delta = 0$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$w_S$</td>
<td>$w_B$</td>
</tr>
<tr>
<td>$b=0.00$</td>
<td>$\gamma = -0.5$</td>
<td>1.4792</td>
</tr>
<tr>
<td></td>
<td>$\gamma = 0.0$</td>
<td>2.1538</td>
</tr>
<tr>
<td></td>
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<td>4.1776</td>
</tr>
<tr>
<td>$b=0.33$</td>
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<td>1.4459</td>
</tr>
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<td></td>
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</tr>
<tr>
<td></td>
<td>$\gamma = 0.5$</td>
<td>4.1443</td>
</tr>
<tr>
<td>$b=0.66$</td>
<td>$\gamma = -0.5$</td>
<td>1.4125</td>
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<td>2.0871</td>
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<tr>
<td></td>
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<td>4.1110</td>
</tr>
<tr>
<td>$b=1.00$</td>
<td>$\gamma = -0.5$</td>
<td>1.3792</td>
</tr>
<tr>
<td></td>
<td>$\gamma = 0.0$</td>
<td>2.0538</td>
</tr>
<tr>
<td></td>
<td>$\gamma = 0.5$</td>
<td>4.0776</td>
</tr>
</tbody>
</table>

More interestingly, if background risk is only partially insurable (Panel A), we find that the agent will allocate more stock than bond. The relative ratio ($w_S/w_B$) is positively correlated with the degree of risk aversion; i.e., the risk preference of the agent can affect his allocation of risky assets. However, the relative ratios (stock to bond) become more inelastic to the agent’s risk attitude as insurance coverage increases. This means the existence of insurance market can reduce the variation of risky assets holding ratio for various market investors, which are with different risk preferences. Ultimately, when that background risk is completely insurable by the insurance market (i.e., $\delta = 0$, Panel B) and full insurance is mandatory, the holding ratios of stock to bond would be completely inelastic to the agent’s risk preference. This says that, in addition to the investment opportunity set, the financial decision does not depend on risk preference as well as background risk either. That is, the “mutual fund separation theorem” holds only if the agent can transfer all background risk through insurance policies.

From Table 8, we also find that the holding ratios of stock and bond are both negatively related with insurance coverage because of the less need of hedging demand for the risky assets against background risk. Another interesting case is that if the agent does not purchase any insurance (or
insurance is not available), then the need of the hedge demand is largest in order to mitigate the adverse events from capital market as well as insurance pure loss.

4.4 The Effect of Assets Performance to Portfolio Choice

In the study of intertemporal CAPM model, Merton (1973) showed that the holding ratio of risky assets is also closely related to the moment characteristics of asset returns and their correlation coefficients. Clearly, the performance (in term of the expected return to volatility, or risk premium, $\frac{\alpha_s}{\sigma_s}$) of risky asset can be an important influential factor to the holding ratio, figure 4 shows the resulting optimal relative holding ratio of stock to bond under various degree of risk aversion and under two scenarios, high expected return ($\alpha_s = 10\%$) and low expected return ($\alpha_s = 7\%$), respectively. We find that the agent allocate more money in stock than in bond when risk premium is high, and the relative ratio is negatively related with the degree of risk aversion. In contrast, when the expected return of the stock is low, the agent holds less stock than bond, and the holding ratio is positively related with the degree of risk aversion. Figure 4 also shows that the portfolio choice of the agent heavily relies on the relative performance of the risky assets. The effect of the degree of risk aversion on the relative holding ratios of stock to bond is found to be lower if more insurance coverage is purchased. Similarly, if background risk is completely insurable (i.e., $\delta = 0$), Figure 5 shows that the relative holding ratio of stock to bond does not depend on the degree of risk aversion when full insurance is undertaken.
\[ \alpha_S = 10\% \quad \alpha_S = 7\% \]

Figure 4. Risky Assets Allocation by Various Degrees of Risk Aversion When Background Risk is Partially Insurable \((\delta = 0.3)\)

\[ \alpha_S = 10\% \quad \alpha_S = 7\% \]

Figure 5. Risky Assets Allocation by Various Degrees of Risk Aversion When Background Risk is Completely Insurable \((\delta = 0)\)
4.5 The Effect of the Volatility of Background risk on Portfolio Choice

Figure 6 shows how the background risk can affect the relative holding ratios of stock to bond. By setting the magnitude of background risk from 0% to 80%, we find the agent reduces his stock holding rather than the bond, i.e., a rational agent will move more his wealth to the bond in order to reduce the risk load by the increasing background risk. However, if he purchases full insurance meanwhile, the effect of increasing background risk would be smaller.

Figure 6. Risky Assets allocation by Various Sizes of Background Risk

4.6 The Consumption Process

In previous sections, we have shown the relationship between the consumption process and the associated determinants in the insurance market and the financial markets. In this section, we will use Monte Carlo simulation to portray the resulting optimal consumption process, and it is simulated under three scenarios, i.e., full insurance, optimal insurance and no insurance, respectively. With full insurance, a relatively smoother consumption process is generated because all insurable background risk has been transferred. The remaining volatility is due to the residual risk from
risky assets as well as the uninsurable background risk. The consumption process with optimal (partial) insurance fluctuates more volatile because part of insurable background risk is not transferred. Finally, with no insurance, the consumption process consists of several large jumps due to the uninsured jump risk. The simulation confirms that the consumption process is smoother if the agent buys insurance. This is also consistent with numerous empirical findings, which show the actual consumption process is much smoother than the prediction by the theoretical model considering only financial market.

Figure 7. Simulated Consumption Process

5. Conclusion

The paper investigates how financial risk and uninsurable background risk can jointly affect the decisions of assets allocation, insurance demand and consumption for a representative agent. Since all decisions can affect his expected utility, they should be made simultaneously. By integrating both
financial market and insurance market, we show that, if some background risk can be transferred through insurance contract while the remaining risk may be used to form a hedging demand against financial risk. One of the key findings in this study is that background risk not only affects portfolio choice but also the relative holding of the risky assets. Thereby, we conclude that the well-known “mutual fund separation theorem” in the finance literature holds only if all background risk is insurable in the insurance market and full insurance is taken. Moreover, we indicate that insurance can mitigate the impacts from background risk to his portfolio choice through forming a hedging demand. Finally, insurance can also smooth the consumption path which is largely affected by background risk. As a result, buying insurance not only reduces insurable background risk but also smooth the dynamic of risky assets choice as well as consumption rate. We believe that our findings can provide a possible explanation to the well-known “equity premium puzzle” and “asset allocation puzzle” in the financial literature.

References


University.


Appendix 1

Proof of Corollary 1

Substitute the utility function by the iso-elastic marginal utility in proposition 1, where the holding of risk free asset is \( w_m = 1 - w_s - w_b \).

Thereby, through equation (12) we have \((1 + \pi)(\mu W)^{-1} = (\mu(b - \delta)Y)^{-1}\), and the optimal insurance can be obtained directly. Also substitute equations (10), (11) and (15) into HJB equations. Under the assumption of iso-elastic marginal utility, we have

\[
0 = \left(\frac{\mu W}{\gamma} - \rho \frac{\mu^{-1} W}{\gamma} + (\mu W)^{-1} \right) r W(t) - \mu W + \frac{W}{(1 - \eta_{SB}^2)(1 - \gamma)} \left( \frac{\sigma_s^2}{\sigma_s^2} + \frac{\alpha_b^2}{\sigma_b^2} - \frac{2\alpha_s\alpha_b\eta_{SB}}{\sigma_s\sigma_b} \right) - \\
\lambda(1 + \pi) \left( \frac{1}{k} (1 + \pi)^{-\gamma} + \delta \right) Y(t) + \lambda \left( \frac{\mu^{-1} Y}{\gamma} \right) \left( \frac{1}{k^\gamma} (1 + \pi)^{-\gamma} - 1 \right) - \\
\frac{1}{2} \left( 1 - \eta_{SB}^2 \right)^2 \left( 1 - \gamma \right)^2 \left[ \frac{2\alpha_s^2 \eta_{SB}^2}{\sigma_s^2} - \frac{2\alpha_b^2 \eta_{SB}^2}{\sigma_b^2} + \frac{2\alpha_s \alpha_b \eta_{SB}^3}{\sigma_s \sigma_b} - \frac{2\alpha_b^2 \eta_{SB}^3}{\sigma_b^2} + \frac{2\alpha_s \alpha_b \eta_{SB}^3}{\sigma_s \sigma_b} \right]
\]

the risk premium coefficient is:

\[
\mu = \frac{1}{1 - \gamma} \left[ \rho - r \gamma + \lambda \left( k^\gamma + \gamma \delta (1 + \pi) - (1 - \gamma)(1 + \pi)^{-\gamma} \right) - \frac{\gamma}{2(1 - \eta_{SB}^2)(1 - \gamma)} \left( \frac{\alpha_s}{\sigma_s} - \frac{\alpha_b}{\sigma_b} \right)^2 \right]
\]

Take it into the optimality condition for the consumption rules, i.e.,

\[ C = U_C^{-1}(J_w(W(t), t) = \mu W(t), \text{ equation (16) follows.} \]

Appendix 2

Proof of Corollary 2

Using the fact that \( C(t) = \mu W(t) \), taking logarithm on both sides, then differentiates the equation with respective to consumption and wealth, respectively, we obtain the following path.
\[
\frac{dC(t)}{C(t)} = \frac{dW(t)}{W(t)} = \left[ r - \frac{1}{1-\gamma} \left( \rho - r\gamma + \lambda \left( k' + \gamma \kappa \delta (1+\pi) - (1+\pi)^{\frac{1}{1-\gamma}} (1-\gamma) \right) \right) \right.
\]
\[
\quad + \frac{\gamma}{2(1-\eta^2_B)(1-\gamma)} \left( \frac{\alpha_s}{\sigma_s} - \frac{\alpha_B}{\sigma_B} \right)^2 dt + \left( \frac{1}{k} (1+\pi)^{\frac{1}{1-\gamma}} - 1 \right) dQ(t) + \\
\quad - \frac{1}{1-\eta^2_B}(1-\gamma) \left( \frac{\alpha_s}{\sigma_s} - \frac{\alpha_B}{\sigma_B} \right) dZ_s(t) + \frac{1}{(1-\eta^2_B)(1-\gamma)} \left( \frac{\alpha_B}{\sigma_B} - \frac{\alpha_s \eta_B}{\sigma_s} \right) dZ_b(t)
\]

Rearrange the above equation, we have the result.

**Appendix 3**

Proof of proposition 2

Using the principle of Bellmen optimality, we differentiate \( J \) in equation (8) with respect to stock holding, bond holding and insurance quantity, respectively, then we have the following first-order conditions:

\[
0 = J_w(W(t), t) \alpha_s + J_{ww}(W(t), t) \left( \sigma_s^2 + w_s W(t) \sigma_s \eta_B + (b - 1 - \delta) Y(t) \sigma_B \eta_{BL} \right)
\]

\[
0 = J_w(W(t), t) \alpha_B + J_{ww}(W(t), t) \left( \sigma_B^2 + w_B W(t) \sigma_B \eta_B + (b - 1 - \delta) Y(t) \sigma_s \eta_{SL} \right)
\]

\[
0 = -\lambda (1+\pi) J_w(W(t), t) + \lambda J_{ww}(W(t), t) \left( (b - \delta) Y(t), t \right) + J_{www}(W(t), t) \left( w_s Y(t) \sigma_s \eta_{SL} + w_B W(t) \sigma_B \eta_{BL} + (b - 1 - \delta) W(t) \sigma_s^2 \right)
\]

Solving the above equations and substitute \( J \) for the explicit indirectly utility function, then the optimal decisions can be obtained. The non-linearly equation for optimal insurance can be derived by simply taking stock holding and bond holding into HJB equation.