The Valuation of the Catastrophe Equity Puts with Jump Risks

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Catastrophe Insurance Products

- **Catastrophe (CAT)**
  - An event resulting in the great loss
  - Man-made hazards and nature hazards
- **Man-made hazards** are arisen by human beings , such as
  - Wars, 911 Terrorist attack
- **Nature hazards** are natural disasters, such as
  - Earthquakes, Storms, Hurricanes
- **Definition** of Property Claim Service (PCS) index
  - A nature CAT is an event which causes in excess of 25 million US dollars in insured damages

Types of Insurance Instruments

- **Those that transfer CAT risk are**
  - Reinsurance
  - Exchange-traded derivatives
  - Swaps
  - CAT bonds
- **Those that provide contingent capital are**
  - Contingent surplus notes
  - CAT equity puts (we focus on Catastrophe equity puts)

Catastrophe Equity Puts (CatEPut)

(1) The reinsurance company (writer) issues the CatEPut to insurance company (buyer).

(2) If the actual CAT loss $L(T)-L(t_0)$ exceed the specified losses L and $S(T)$ is lower than K, the insurance company has the right to sell a specified amount of its stock to the writer at K

Insurance company

Reinsurance company

- The payoff of CatEPut

$$P(T)=\begin{cases} K-S(T), & \text{if } S(T)<K \text{ and } L(T)-L(t_0)>L, \\ 0, & \text{if } S(T)\geq K \text{ or } L(T)-L(t_0)\leq L, \end{cases} \tag{1}$$

Literatures

- Cox et al., (2004) propose the **pure Poisson process** to model the price per share of the insurance company's equity

$$S(t) = S(0)\exp\left\{\left[\mu(t) - \frac{1}{2}\sigma_S^2\right]t + \sigma_S W_S(t) - AN(t)\right\} \tag{2}$$

  - where S(0) is the initial price, $\{W_S(t): t>0\}$ a standard Brownian motion, and $\{N(t): t>0\}$ is a Poisson process with a constant parameter λ.
  - If a large claim occurs at time t, the price changes instantaneously from S_{t-} to $S_t = e^{-A} S_{t-}$.(A is called by jump size.)
- **The model includes two assumptions with**
  - The constant arrival rates of CAT
  - The constant Jump size.
    - The large losses are not equal when these hurricanes hit the US east coast.

Literatures

- Jaimungal and Wang (2006) propose the compound Poisson process to model the dynamics process of the stock price

$$S(t) = S(0)\exp\left\{\left[\mu(t) - \frac{1}{2}\sigma_S^2\right]t + \sigma_S W_S(t) - \alpha L(t)\right\} \qquad (3)$$

  - The loss process of the insured is assumed as the compound Poisson

$$L(t) = \sum_{n=1}^{N(t)} Y_n \qquad (4)$$

  - where {Yn, n=1,2,…} are i.i.d random variables representing the size of the i-th loss with p.d.f *f(y)*
  - α denotes the percentage drop in the share value price per unit of loss.
- Jaimungal and Wang (2006) extends the Cox's model (2004) for random jump sizes.
- The model includes two assumptions with
  - The constant arrival rates of CAT events
  - The constant percentage drops.

Poisson Process (PP)

- The arrival rates are equal in a Poisson process.
- However, the arrival rates are not equal in the United State during 1950 to 2004 in Figure 1 provided by ISO (Insurance Service Office).

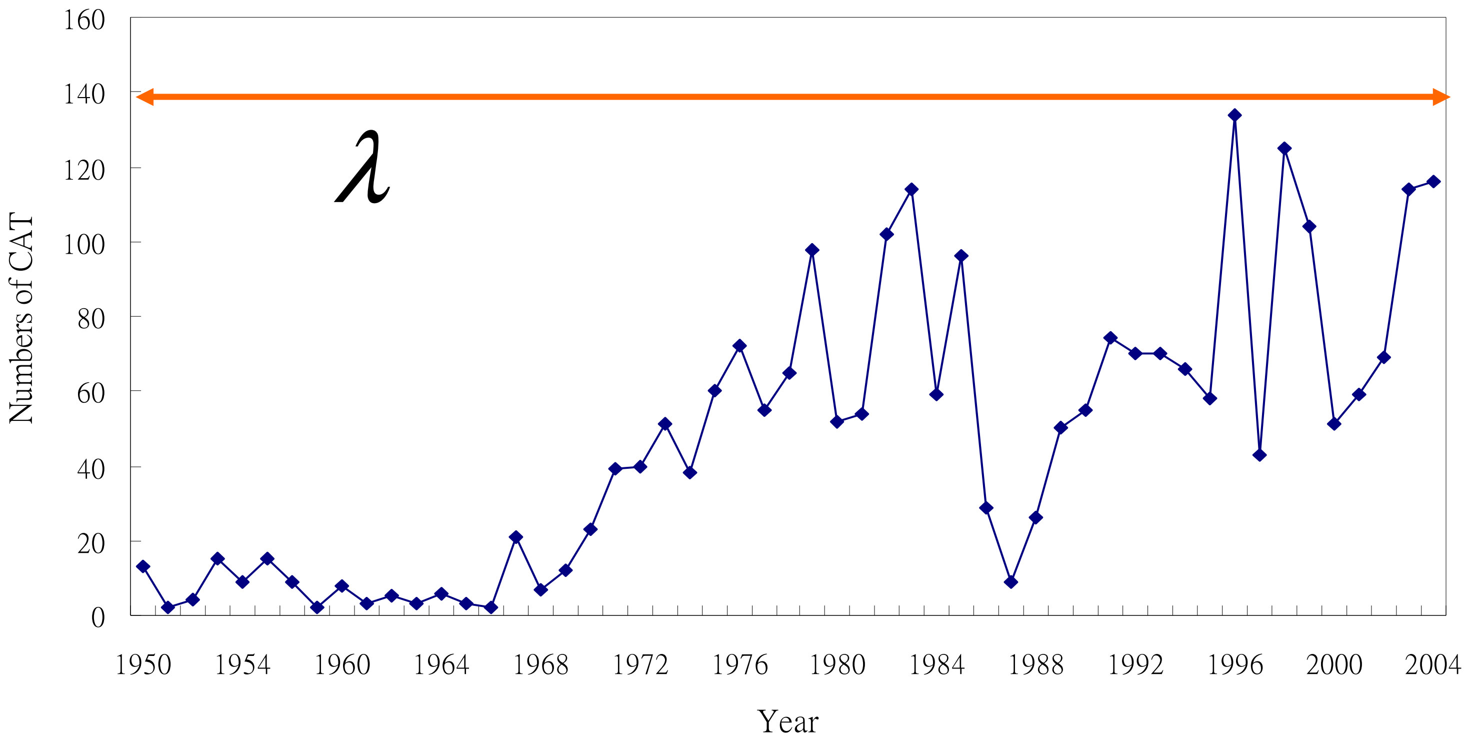


Figure 1: Number of CAT in the United State during 1950 to 2004

Markov Modulated Poisson Process (MMPP)

- More precisely, we could assume the numbers of CAT comes from two dependent arrival rates in MMPP.
- One is the small arrival rate, the other one is larger.
- The smaller jump rate stay for a long time, then the smaller jump rate change into the larger jump rate and it stays for a long time in Fig. 2.

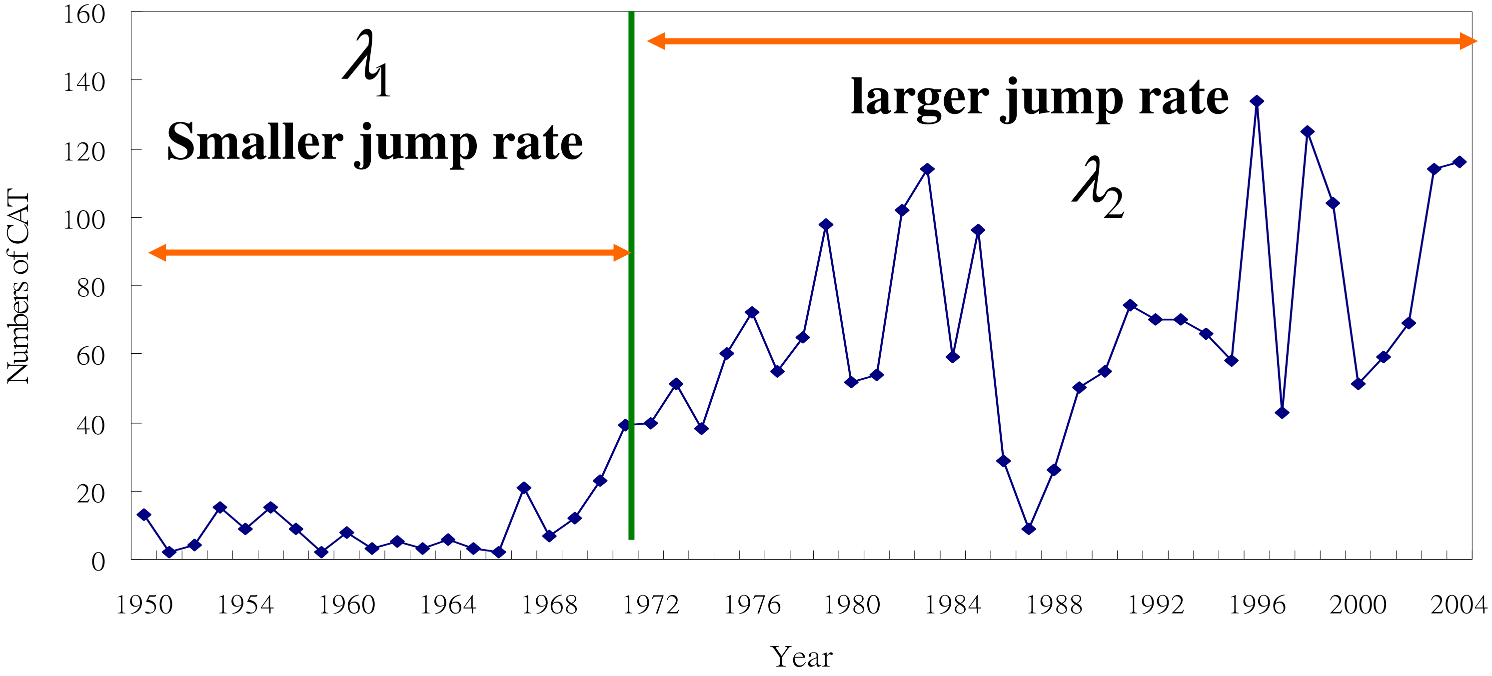


Figure 2: Number of CAT in the United State during 1950 to 2004

Doubly Stochastic Poisson Processes (DSPP)

- Generally, we assume the numbers of CAT come from a DSPP with deterministic rates $\lambda(t)$ or with stochastic rates.
- The arrival rates are deterministic in the United State during 1950 to 2004 in Figure 3 provided by ISO.

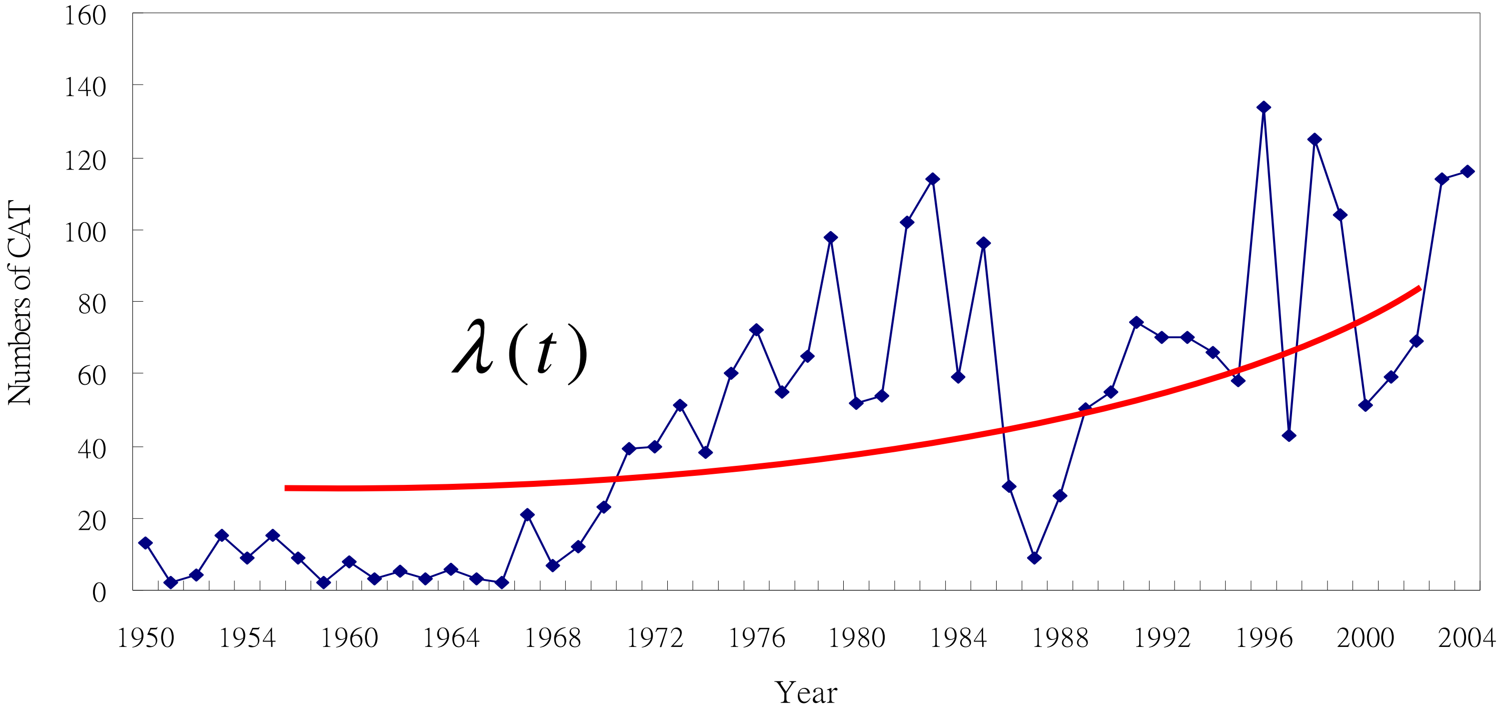


Figure 3: Number of CAT in the United State during 1950 to 2004

Marked Point Processes (MPP)

- More generally, we can assume the arrival rate follows a MPP by a sequence

$$\{(\tau_n, Y_n), n = 1, 2, \ldots\}.$$

- The τ_n interprets the times of potential CAT events and satisfies

$$\tau_1 < \tau_2 < \ldots \tau_n < \ldots, \sup_n \tau_n = \infty$$

- In this paper, we propose a general stochastic intensity (MPP) dependent on some environmental variable to model the arrival rates of CAT events.

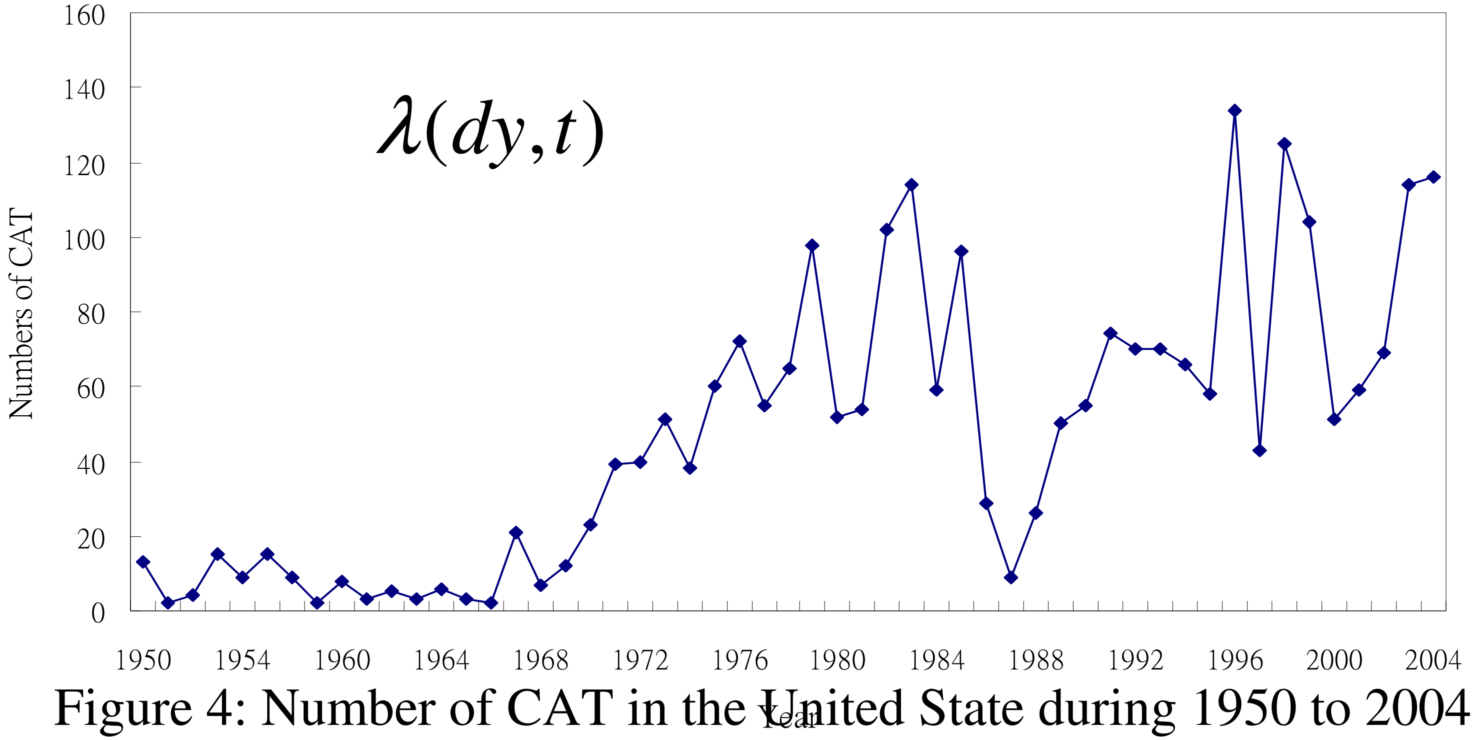


Figure 4: Number of CAT in the United State during 1950 to 2004

Motivation

- Jaimungal and Wang (2006) propose the stock process

$$S(t)=S(0)\exp\left\{\int_0^t \mu(s)ds-\frac{1}{2}\sigma_S^2 t+\sigma_S W_S(t)-\alpha L(t)\right\} \qquad (5)$$

- The loss-link-price process of the insured is assumed as follows

$$-\alpha L(t)=-\sum_{n=1}^{N(t)}\alpha Y_n \qquad (6)$$

- 1.Assume the arrival rate of CAT follows a Marked Point process

$$-\sum_{n=1}^{N(t)}\alpha Y_n \Rightarrow -\sum_{n=1}^{M(t)}\alpha Y_n \qquad (7)$$

- 2. Assume the general function $H(Y_n,\tau_n)>0$ with the percentage drop and loss distribution, because not all CAT events has same effect on stock price.

$$-\sum_{n=1}^{M(t)}\alpha Y_n \Rightarrow -\sum_{n=1}^{M(t)}H(Y_n,\tau_n) \qquad (8)$$

Jump Risk Models

- We propose the stock process with a Marked point process

$$S(t) = S(0)\exp\left\{\int_0^t \mu(s)ds - \frac{1}{2}\sigma_S^2 t + \sigma_S W_S(t) - L(t)\right\} \qquad (9)$$

$$\boxed{L(t) = \sum_{n=1}^{M(t)} H(Y_n, \tau_n)}$$

- Three special cases for Marked point processes are
  - Poisson Process (PP)

$$S(t) = S(0)\exp\left\{\int_0^t \mu(s)ds - \frac{1}{2}\sigma_S^2 t + \sigma_S W_S(t) - L_1(t)\right\} \qquad (10)$$

$$\boxed{L_1(t) = \sum_{n=1}^{N(t)} H(Y_n, \tau_n)}$$

  - Markov modulated Poisson process (MMPP)

$$S(t) = S(0)\exp\left\{\int_0^t \mu(s)ds - \frac{1}{2}\sigma_S^2 t + \sigma_S W_S(t) - L_2(t)\right\} \qquad (11)$$

$$\boxed{L_2(t) = \sum_{n=1}^{\Phi(t)} H(Y_n, \tau_n)}$$

  - Doubly stochastic Poisson process (DSPP)

$$S(t) = S(0)\exp\left\{\int_0^t \mu(s)ds - \frac{1}{2}\sigma_S^2 t + \sigma_S W_S(t) - L_3(t)\right\} \qquad (12)$$

$$\boxed{L_3(t) = \sum_{n=1}^{D(t)} H(Y_n, \tau_n)}$$

Some Assumptions

- Assume the interest rate follows Vasicek model

$$d\,r(t) = k\,[\theta - r(t)]\,dt + \sigma_r\, dW_r\,(t) \qquad (11)$$

- The interest rate and the stock price with Brownian motion are correlated.
- It is natural to assume that the interest rate process is stochastically independent of the loss measure.
- As Cox et al (2004), we also assume a liquid market for CatEPut exists, then standard derivative pricing theory implies an equivalent probability measure Q exists.

Catastrophe Equity Put (CatEPut)

- The payoff of CatEPut

$$P(T) = 1_{\{L(T) > L + L(t_0)\}} \left(K - S(T)\right) 1_{\{S(T) < K\}} \qquad (13)$$

- Theorem 1: The pricing formula of CatEPut with the MPP

$$P_M(t;t_0) = E^{\mathbb{Q}} \left[1_{\{L(T)-L(t)>\tilde{L}\}} \left\{ K\, B(t,T) \mathbb{N}(-d_2^M) \right. \right. \qquad (14)$$

$$\left. \left. - S(t) \exp\left(\left[L(T) - L(t) - \int_t^T \int_0^\infty h(y,s) \lambda(dy,ds) \right] \right) \mathbb{N}(-d_1^M) \right\} \middle| F_t^L \right]$$

$$d_{1,2}^M = \frac{\ln\left(S(t)/K\, B(t,T)\right) \pm \frac{1}{2} \tilde{\sigma}^2(t,T) + \left(L(T) - L(t) - \int_t^T \int_0^\infty h(y,s) \lambda(dy,ds) \right)}{\tilde{\sigma}(t,T)}$$

$$\tilde{\sigma}^2(t,T) = \sigma_S^2 (T-t) + \frac{2k\rho\sigma_r\sigma_S + \sigma_r^2}{k^2} [(T-t) - U(t,T)] - \frac{\sigma_r^2}{2k} U^2(t,T) \qquad \tilde{L} = L + L(t_0) - L(t)$$

Catastrophe Equity Put (CatEPut)

- The formulas can be computed in MMPP and DSPP
- The price of the CatEPut with a DSPP

$$P_{DS}(t;t_0)=\int_0^{\infty}\sum_{m=1}^{\infty}\frac{\exp\left(-\int_t^T\lambda(ds)\right)\left(\int_t^T\lambda(ds)\right)^m}{m!}E^{\mathbb{Q}}\left[1_{\{L(T)-L(t)>\tilde{L}\}}\left\{K\,B(t,T)\mathbb{N}(-d_2^{DS})\right.\right.$$

$$\left.\left.-S(t)\exp\left(\left[L(T)-L(t)-\int_t^T\int_0^{\infty}h(y,s)\lambda(s)f(y)\,dy\,ds\right]\right)\mathbb{N}(-d_1^{DS})\right\}\middle|\Phi_{T-t}=m\middle|F_t^L\right]f(\lambda(s))\,d\lambda(s) \qquad (15)$$

$$d_{1,2}^{DS}=\frac{\ln\left(S(t)/K\,B(t,T)\right)\pm\frac{1}{2}\tilde{\sigma}^2(t,T)+\left[L(T)-L(t)-\int_t^T\int_0^{\infty}h(y,s)\,\lambda(s)\,f(y)\,dy\,ds\right]}{\tilde{\sigma}(t,T)}$$

Catastrophe Equity Put (CatEPut)

- The formulas can be computed in MMPP and DSPP
- The price of the CatEPut with a MMPP

$$P_{MM}(t;t_0)=\sum_{m=1}^{\infty}\mathrm{P}(\mathrm{m},\mathrm{T}\text{-}\mathrm{t})\,E^{\mathbb{Q}}\left[1_{\left\{L(T)-L(t)>\tilde{L}\right\}}\left\{K\,B(t,T)\mathbb{N}(-d_2^{MM})\right.\right. \tag{16}$$

$$\left.\left.-S(t)\exp\left(\left[L(T)-L(t)-\int_t^T\int_0^\infty h(y,s)\Lambda f(y)\,dy\,ds\right]\right)\mathbb{N}(-d_1^{MM})\right\}\middle|\Psi_{T\text{-}t}=m\middle|F_t^L\right]$$

$$d_{1,2}^{MM}=\frac{\ln\left(S(t)/K\,B(t,T)\right)\pm\frac{1}{2}\tilde{\sigma}^2(t,T)+\left[L(T)-L(t)-\int_t^T\int_0^\infty h(y,s)\Lambda f(y)\,dy\,ds\right]}{\tilde{\sigma}(t,T)}$$

Catastrophe Equity Put (CatEPut)

- If $\lambda_1 = \lambda_2 = \ldots \lambda_I = \lambda$ (or $\lambda_t = \lambda$), and $L(t) = \alpha \sum_{n=1}^{N_t} Y_n$, the equation (15) and (16) reduce to the pricing formula of Jaimungal and Wang (2006)

$$
\begin{aligned}
P_{PP}(t;t_0) = \sum_{m=1}^{\infty} \frac{\exp\left(-\lambda(T-t)\right)\left(\lambda(T-t)\right)^m}{m!} E^{\mathbb{Q}}\Bigg[1_{\{L(T)-L(t)>\tilde{L}\}} \Big\{ & K\,B(t,T)\mathbb{N}(-d_2^{PP}) \\
& -S(t)\exp\left[\alpha\left((L(T)-L(t)) - \lambda\kappa(T-t)\right)\right]\mathbb{N}(-d_1^{PP}) \Big\} \Bigg| \Psi_{T-t} = m \Bigg| F_t^L \Bigg]
\end{aligned}
\qquad (17)
$$

$$
d_{1,2}^{PP} = \frac{\ln\left(S(t)/K\,B(t,T)\right) \pm \frac{1}{2}\tilde{\sigma}^2(t,T) + \alpha\left[L(T) - L(t) - \lambda\kappa(T-t)\right]}{\tilde{\sigma}(t,T)}
$$

Catastrophe Equity Put (CatEPut)

- If $\lambda_1=\lambda_2=\dots\lambda_I=\lambda$ (or $\lambda_t=\lambda$), $L(t)=\alpha 1_{\{N(t)>n\}}$, and the interest rate is deterministic, the equation (15) and (16) reduce to the result of Cox et al. (2004)

$$P_N(t;t_0)=\sum_{m=n}^{\infty}\frac{e^{-\lambda(T-t)}\left[\lambda(T-t)\right]^m}{m!}\left[Ke^{-r(T-t)}\mathbb{N}(-d_2^{PP})-S(t)\exp\left[-\alpha\left(m-\lambda k(T-t)\right)\right]\mathbb{N}(-d_1^{PP})\right] \tag{18}$$

$$d_{1,2}^{PP}=\frac{\ln\left(S(t)/K\right)+(r\pm\frac{1}{2}\sigma_S^2)(T-t)-\alpha\left[m-\lambda\kappa(T-t)\right]}{\sigma_S\sqrt{(T-t)}}$$

$$\kappa=\lambda(1-e^{-\alpha})$$

Numerical and Empirical Experiment

- In our numerical and empirical experiment, some parameters are given, and other parameters are estimated with the data.
- Then the true CatEput value is compared with the price of these jump risk models. Which jump risk model has the smallest error?
- **Some parameters are given** by Jaimungal and Wang (2006)
  - Initial stock price: S(0)=25
  - Volatility in equity: $\sigma_S = 0.2$
  - Initial interest rate: r(0)=0.02
  - Parameters in the interest rate: $k = 0.3$ $\theta = 5\%$ $\rho = -0.1$ $\sigma_r = 15\%$
  - Exercise price: K=80
  - Trigger level: L=0.25
  - Option term: T=4

Numerical and Empirical Experiment

- **Data source comes from** ISO
  - We use the number of hurricane in the PCS index during 1950 to 2004
- **Parameters are estimated** for the arrival rate of the different models
  - PP : The constant arrival rate is 3.93 overall years
  - MMPP :
    - The smaller arrival rate is 1.095 before 1970 and the larger arrival rate is 5.676 after 1970.
    - The transition probability is 0.045 (0.01) from state 1 (2) with smaller arrival rate to state 2 (1) with larger arrival rate
  - DSPP
    - Assume the arrival rate is deterministic as follows

$$\lambda(t) = 1.63 \times 10^{-6} t^4 \qquad (19)$$

  - Jump size : A lognormal distribution is estimated with location parameter=0.012 , scale parameter= 0.037.

Numerical and Empirical Experiment

- The four global fit measures for the difference between true value and model value with MMPP, DSPP, and PP.
- Average percentage error (APE)

$$APE = \frac{1}{E(P_{\mathrm{R}})} \sum_{n=1}^{N} \frac{\left|P_{\mathrm{R}} - P_{\mathrm{T}}\right|}{N} \qquad (21)$$

- Average absolute error (RMSE)

$$AAE = \sum_{n=1}^{N} \frac{\left|P_{\mathrm{R}} - P_{\mathrm{T}}\right|}{N} \qquad (22)$$

- Average relative percentage error (ARPE)

$$ARPE = \frac{1}{N} \sum_{n=1}^{N} \frac{\left|P_{\mathrm{R}} - P_{\mathrm{T}}\right|}{P_{\mathrm{R}}} \qquad (23)$$

- Relative measure square error (RMSE)

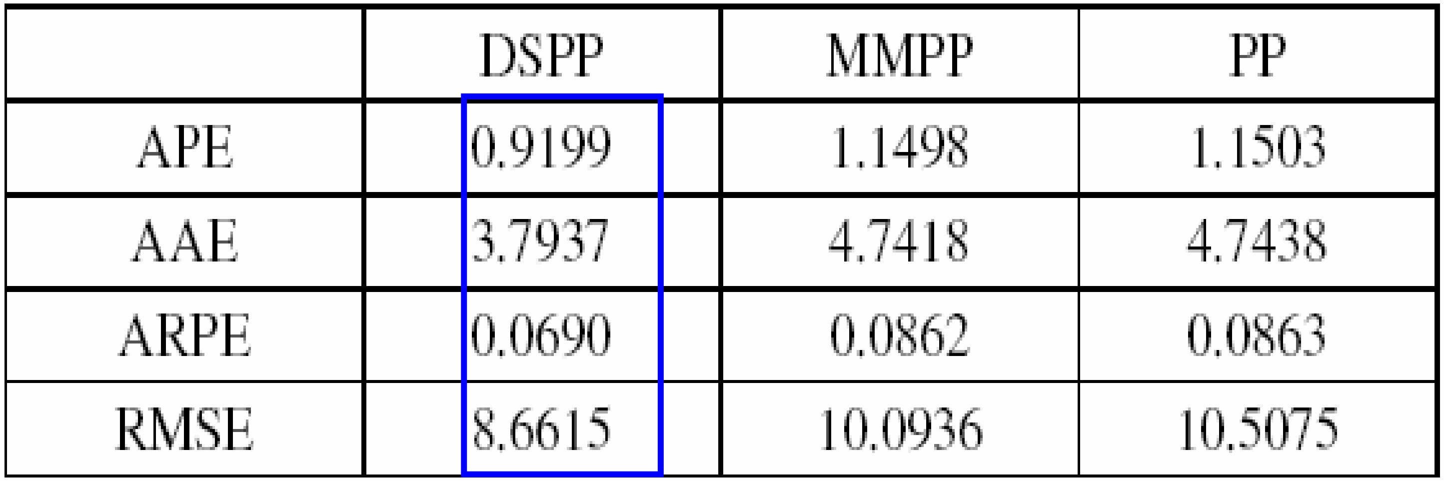
$$RMSE = \sqrt{\sum_{n=1}^{N} \frac{(P_{\mathrm{R}} - P_{\mathrm{T}})^2}{N}} \qquad (24)$$

- where P_R denotes the CatEPut value under the real number of CAT,
  P_T presents the CatEPut value under the model,
  $E(P_R)$ is the mean of the CatEPut value P_R .

Results of Experiment

- Table 1 shows four measurement errors of the CaTEPut value under different jump risk models, including DSPP, MMPP, and PP.
- Under DSPP, the four measurement errors are smallest than other risk models.

Table 1: Global fit error measurements



	DSPP	MMPP	PP
APE	0.9199	1.1498	1.1503
AAE	3.7937	4.7418	4.7438
ARPE	0.0690	0.0862	0.0863
RMSE	8.6615	10.0936	10.5075

Conclusions

- Model
  - We extend Jaimungal and Wang's (2006) model by stochastic intensity and stochastic loss-link-price effect to value the catastrophe equity puts under a stochastic interest rate model.
- Numerical and empirical experiments
  - Based on the numerical and empirical experiments, the value of the catastrophe equity put under the DSPP model is closed to the true value of the catastrophe equity put.

Future Researches

- Early excise features
  - Because CatEPut are quite illiquid, the early exercise features will be considered.
- Some other structured risk management products
  - This model can be applied to other structured risk management products such as double-trigger products.

The End

Thanks for your listening