Dynamic Models of Catastrophic Risk Intermediation*

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Abstract

Sharp price spikes and large capacity swings would follow catastrophic shocks in the insurance industry. In this paper, we construct a series of catastrophic risk intermediation models to study whether and how catastrophe events can accelerate the insurance price changing process and the industrial organization developing process, focusing on the firm’s decision-making of underwritings and capital structures in costly external capital market. The first two-period cash flow model with one catastrophe event concludes that the insurance firm has optimal capital structures, and that the insurance firm with a good solvency position could obtain advantages in both underwriting capability and capital rising capacity. Such the good firm even may gain extra profit when it can take advantage of price spikes and the insured’ loyalty in post-catastrophe underwritings, and control the penalties of costly reinsurance rate and external capital cost after catastrophic shocks. Two dynamic models with stochastic catastrophic shocks are then developed in this paper. We find that the insurance cycle can be inspired by temporary catastrophic shocks according to the interaction of capital rising capacity and underwriting capability, and that the insurance cycle will be larger if the catastrophic risk is more volatile, and if the external capital market is harder, and if the solvency regulation is more relaxed. We also try to figure out why insurance firms have different performances during and after catastrophe events and predict such difference will become larger if the catastrophic risks are more volatile.

Keywords: catastrophic risk; insurance industry; capital structure; dynamic model
I Introduction

Capacity Constraint Theory and Risk Over Hang Theory suggest that sharp price spikes and large capacity swings would follow the capital shock due to the high external capital cost in the capital market. The mismatch of unexpected huge catastrophe losses and loss reserves funded by the relatively small amount of annual insurance premium tends to cause capital shortage in the whole insurance industry. Once such a catastrophe event occurs, the demand expansion and the supply reduction turn out to make the premium grow sharply and gradually moderate until the insurance industry recovers from capital replenishment. During this process, insurance firms expertizing in intermediating catastrophe risks may take advantage of price changes and benefit from clients’ loyalty in future underwritings, while other firms may encounter insolvency or bankruptcy problems resulting from capital insufficiency. One catastrophe event could act as an accelerated trigger, splitting insurance firms into high-quality ones and low-quality ones with respect to different underwriting efficiency and capital raising capability. Meanwhile, new investors would also enter the insurance market so as to pursue profits quite after the event. With current insurance firms categorized by catastrophe events in a succession and new comers continually entering into the game, changes in the insurance industry come out sequentially. Under the background of more frequent and sever catastrophe events nowadays, it is vital to research how the insurance firms response to such a new era of catastrophes. In this paper, we will build a series of cash flow models to simulate the insurance firm’s catastrophic risk intermediating process in costly external capital market,
focusing on the firm’s decision-making of its underwritings and capital structure. We try to analyze whether and how catastrophe events can accelerate the price changing process and the industrial organization developing process in both hard and soft insurance markets, and study why insurance firms have different post-catastrophe performances and what kind of firm can obtain an advantageous position in an insurance cycle.

In previous literature, many empirical papers support the capacity-constraint and risk overhang hypothesis in property insurance industry. Gron (1994) examines property–casualty insurance underwriting margins data to find that the short-run industry supply curve is upward sloping, and that variations in supply capacity have a significant, negative effect on movements in property-casualty profitability. Doherty, Lamm-Tennant, and Starks (2003) check the temporal and cross sectional variation in insurance company stock prices after 911, and the regression analysis demonstrates that firms suffering the lowest losses with less leverage would be able to exploit the post-loss hard market and this would be reflected in stock price performance. Grace, Klein and Kleindorfer (2004) estimates the demand for homeowners insurance and suggest that catastrophe demand is more price elastic than non-catastrophe demand, and that homeowners insurance demand is more price elastic in Florida than demand in New York. Grace and Klein (2009) examines homeowner insurance market developments, which indicates that Insurers have substantially raised insurance rates and reduced their exposures after the intense hurricane seasons of 2004 and 2005 as the availability of coverage has tightened, and shows that there has been substantial market restructuring in Florida but significantly less so in other
states. In the theoretical field, some models are built to study the relationship between shocks and capitalization. Froot, Scharfstein, and Stein (1993) develop a portfolio model of corporate risk management to analyze that capital-market imperfections can make risk-neutral firms be risk averse and be more risk averse if there is negative shock to internal capital. The portfolio model is extended by Gron and Winton (2001) for shocks in insurance industry, and they conclude that nonlife-insurance firms will reduce the willingness to engage in correlated business activities when the past risks cannot be easily diversified or hedged. Cummins and Danzon (1997) develop a risky debt model for an insurance firm to imply that insurance firms should have optimal capital structures because of the relationship between insurance demand and financial quality. This model is more general than the capacity constraint theory, which can predict an ambiguous relationship between price and a loss shock based on different assumptions about the effects of shocks on demand elasticity. In this paper we extend Cummins and Danzon (1997)’s model and develop a two-period cash flow model with one catastrophic event for an insurance firm to explore the firm’s optimal catastrophe risk intermediation strategy. It concludes that the insurance firm has optimal capital structure, which is determined by the tradeoff among marginal cost of reinsurance, marginal cost of external capital rising and marginal profit of underwritings. And it suggests that in the situation of tight capital supply and high insurance demand, the positive relation of catastrophic losses and insurance price can be observed. It also suggests that the insurance firm with a good solvency position could obtain advantages in both underwriting capability and capital
rising capacity. Such the firm even may gain extra profit when it can take advantage of price spikes and the insured’ loyalty after shock, and control the penalties of costly reinsurance rate and external capital cost in post-catastrophe underwritings. Moreover, it implies that the insurance cycle can be inspired in costly capital market by changes of the firm’s solvency position, which will be discussed further in the two dynamic models.

Although few literatures research in catastrophic shocks in insurance industry by dynamic models, there are lots of papers studying technology shocks and financial shocks in the business cycle by dynamic models, which shed light to this paper. Kiyotaki and Moore (1997) construct a dynamic model in which lenders cannot force borrowers to repay their debts unless the debts are secured, and show that small and temporary shocks to technology or income distribution can generate large, persistent fluctuations in output and asset prices. Bernanke and Gertler (1989) develop a business cycle model in which the condition of borrowers’ balance sheet is a source of output dynamics, and show that shocks that affect net worth can initiate fluctuations. Szemely (2010) builds a dynamic firm financial model and show that changes in uncertainty of entrepreneurs and capital pricing quantitatively have large effects on optimal leverage and investment dynamics by collateral constraints. In this paper, we develop two dynamic cash models with catastrophic shocks within insurance firms to study the insurance firm’s catastrophic risk intermediation process. A dynamic cash flow model with temporary catastrophic risks for an insurance firm is constructed firstly. In this model, we find that temporary catastrophic shocks could play an important role on the interaction of capital rising capacity and
underwriting capability, and the effect will amplify and spread out to other time sectors. We solve the firm’s certainty-equivalent steady-state equilibrium by linear quadratic approximation and then simulate its decision path. We report the amplitude of fluctuations in the firm’s underwriting quantity, external capital raising amount and internal capital saving due to catastrophic risks, and we also report the correlations of them to show the insurance cycle within this insurance firm. We find that the insurance cycle will be larger if the catastrophic risk is more volatile, and if the external capital market is harder, and if the solvency regulation is relaxed. In the second dynamic model, we assume that a large number of heterogeneous insurance firms with different states coexist in a heterogeneous-agent economy, and the distribution of insurance firms in the model economy is cross sectional. In this model, each insurance firm will choose its own optimal decision-making of underwritings, internal capital saving and external capital raising strategy under stochastic catastrophic risks. According to the future simulation results in this model, we try to figure out why insurance firms have different performances during and after catastrophe events. We try to find the difference between good firms and bad firms in terms of capital rising capacity and underwriting technology for catastrophic risk intermediation, and predict such difference will become larger if the catastrophic risks are more volatile.

This paper is structured as follows. Section II develops a two-period cash flow model with one catastrophe event for an insurance firm in detail. Two dynamic cash flow models with catastrophic shocks are developed in Section III. Section IV shows initial simulation
results of the first dynamic model. Section V introduces the calibrations to do by the data and potential simulation results of the second dynamic model in future study. Section VI provides current conclusions and future empirical works.

II Two-Period Cash Flow Model with One Catastrophe Event for an insurance firm

We develop a two-period cash flow model with one catastrophic event for an insurance firm to explore the firm’s optimal catastrophe risk intermediation strategy, and find in which condition an insurance firm can benefit from catastrophe insurance underwritings. In this two-period model, the insurance firm originally has retained earnings $e_0$ as initial endowment, and one catastrophe event occurs during the first period. Figure 1 shows the time line in this model.

At the beginning of each of these two periods, the firm collects annual premium $\pi Q$ from the insured, in which $\pi$ is the insurance premium per coverage and $Q$ is the whole insurance coverage; and raises external capital $e$ from equity holders as revenue; and purchases reinsurance for its partial coverage $\beta Q$ with the reinsurance premium per
coverage C, where $\beta$ is the ratio of reinsurance coverage to the whole coverage. At the end of each period, the firm indemnifies the insured for covered losses $L$ in the last period and pays the required return of capital $R$ to equity holders; meanwhile, the firm can receive the reimbursement of $\beta L$ from the reinsurer and finally may make some profit in the balance sheet. The catastrophe event occurring during the first period probably causes different levels of losses: $L^i$ with probability of $P^i$ where $i = 1, 2, \ldots I$. Here we assume that the expected value of the loss amount is equal to the insured coverage, and $L^i < L^j$ if $i < j$.

Correspondingly, each economic variable in the second period would have different states with superscript $i$. Here we assume that the insurance firm is a price-taker without any old liability before the catastrophic shock and the premium per coverage $\pi$ is determined by the insurance market; however, after the catastrophic shock, the insurance industry changes the premium in the next period due to insurance supply reduction along with capital shortage and thus the firm would set up its own insurance rate to target its customers in different states $i$ in the second period, denoted by $\pi'$.

In this model, $b$ is defined as the ratio of assets to liabilities, which implies the firm’s solvency ability and impacts the insurance coverage $Q$, the return of capital $R$, and the reinsurance premium per coverage $C$; and $b^i$ denote the same ratio for each state $i$ in the second period. Similarly, $Q^i$, $Q^i$, $R^i$, $R^i$, $C^i$, $C^i$, $e^i$, $e^i$, $\beta^i$ and $\beta^i$ all denote the same economic variables as previous ones for each state in the second period. We also make some assumptions with regards to the following several functions: $Q(\pi, b)$ is a concave function with $Q_\pi < 0$, $Q_{\pi\pi} < 0$, $Q_b > 0$, $Q_{bb} < 0$ considering that the customer
tends to purchase more insurance product when the price is lower and the firm has lower insolvent probability; $C(b)$ is a convex function with $C_b < 0$, $C_{bb} > 0$ since the reinsurance cost will be more likely to grow up quickly if the firm becomes more insolvent; and $R(e, b)$ is a convex function with $R_e > 0$, $R_{ee} > 0$, $R_b < 0$, $R_{bb} > 0$ considering that the external capital cost will increase faster if the firm borrows a larger amount of capital and becomes more insolvent.

Then insurance firm’s cash flow in the first period should be $\{e_0 + [\pi - C(b)\beta]Q(\pi, b) - (1-\beta)\sum_{i=1}^{l} P^i L^i r^{-i} + e - R(b, e)r^{-1}\}$, where $r$ is the discount rate in each period. In the second period, the cash flow in state $i$ should be $\{ [\pi^i - C_i(b^i)\beta^i - (1-\beta^i)r^{-i}] Q_i(\pi^i, b^i) + e^i - R_i(b^i, e^i)r^{-1}\}$. To maximize the profit within these two periods, the firm would choose the optimal amount of external capital $\{e, e^i\}$ and reinsurance ratio $\{\beta, \beta^i\}$ for each state $i$ in both of these two periods, and set up the optimal premium per coverage $\{\pi^i\}$ for each state $i$ in the second period. To avoid the insolvency problem at the end of the first period, we need to make sure that the cash flow in the state of the largest losses $L^i$ should be non-negative. The optimization problem for the insurance firm in this model is as follows,

Max $\{e_0 + [\pi - C(b)\beta] Q(\pi, b) - (1-\beta)\sum_{i=1}^{l} P^i L^i r^{-i} + e - R(b, e)r^{-1} + r^{-i}\sum_{i=1}^{l} P^i \{ [\pi^i - C_i(b^i)\beta^i - (1-\beta^i)r^{-i}] Q_i(\pi^i, b^i) + e^i - R_i(b^i, e^i)r^{-1}\}\}$

s.t. The objective maximization function above $\geq 0$

Here, we have

$$b = \frac{e_0 + (\pi - C\beta)Q + e - R r^{-1}}{(1-\beta)Q r^{-1}};$$
\[ b^i = \frac{[r(\pi - C)Q - (1-\beta)L^i] + [r(e^i + e) - R] + [(\pi^i - C^i\beta^i)Q^i + (e^i - R^i r^{-1})]}{(1-\beta^i)Q^i r^{-1}}; \]

\[ \sum_{i=1}^L p^i L^i = Q; \]

\[ \sum_{i=1}^L p^i = 1; \]

i = 1, 2… I.

We will discuss some results of the optimization problem above in two different capital markets: we want to look at the insurer’s choices in the costly capital market, but let us look at the free capital flow market at first.

I) Free capital flow market

When the capital can be flowing freely with no cost in the market, conditions (1) and (2) will be held that the marginal cost of reinsurance and external capital is the interest rate.

\[ C = C^i = r^i; \quad \text{(1)} \]

\[ R_e = R_e^i = r; \quad \text{(2)} \]

These two conditions imply that the insurance company could choose any reinsurance ratio of \( \beta^i \) between 0 and 1 and raise any feasible external capital \( e^i \) from equity holders. In other words, there is no need for the insurance firm to reserve fund to prepare for future loss payment. Based on First Order Conditions (FOCs) and Comparative Statics Analysis (CSA) of the optimization problem under these two conditions, we get the following equations:

\[ Q_b = C_b = R_b = Q_{b^i}^i = C_{b^i}^i = R_{b^i}^i = 0; \quad \text{(3)} \]

\[ E_{Q^i \pi^i} = -\frac{\pi^i}{\pi^i - r^{-1}}; \quad \text{(4)} \]

Equation (3) describes the fact that the solvency ratio of the insurance firm, \( b \), has no
impact on the insurance demand $Q$, the reinsurance cost $C$, or the equity cost $R$, because the firm can always raise revenue as high as she needs with no extra charge. Equation (4) is the price elasticity of insurance demand in each state during the second period, showing that the insurance demand would be relatively less elastic if the premium is increasing. It also implies that the second-period premium of the insurance firm would be determined by its specific price elasticity in each state, and has nothing to do with the previous loss payment. In all, in a free capital economy, the firm’s solvency position does not matter and there is no insurance cycle within the insurance firm. In such a situation, neither the capacity constraint theory nor the risk over hang theory has any effect at all.

II) Costly capital flow market

When the capital flows with some cost in the market, the conditions (1) and (2) should be changed into the inequities (5) and (6).

$$C, C^i > r^i; \quad (5)$$

$$R_e, R^i_e > r; \quad (6)$$

In this case, the insurance firm tends to choose optimal intermediation strategy to reserve fund to make preparation for future loss payment. From the FOCs and CAS of the optimization problem, we get the following results:

$$MP^i = T^i Q^i_{b^i} - \beta^i Q^i_{C^i b^i} - r^i R^i_{b^i} = -\frac{Q^i + T^i Q^i_{\pi^i}}{b^i_{\pi^i}} = \frac{(C^i - r - 1)Q^i}{b^i_{\pi^i}}; \quad (7)$$

$$\frac{d\pi^i}{dL^i} = \beta^i C^i_{b^i} Q^i_{\pi^i L^i} - (Q^i_{b^i} + T^i Q^i_{\pi^i b^i} + b^i_{\pi^i} MP^i_{b^i}) \frac{|SOC|b^i_{L^i}^{-1}|}{|SOC|}; \quad (8)$$
\[
\frac{de^i}{dL^i} = \frac{r^{-1}R^i_e b^i_e - b^i_e MP^i_b}{|SOC|*|b^{-1}_L|}; 
\]

(9)

\[
\frac{d\beta^i}{dL^i} = \frac{(c^i - r^{-1})Q^i_{\beta} + Q^i_c b^i_{\beta} - b^i_{\beta} MP^i_b}{|SOC|*|b^{-1}_L|}; 
\]

(10)

\[
E_{\pi^i} = -\frac{(Q^i + MP^i b^i_{\pi})\pi^i}{T^i Q^i}; 
\]

(11)

\[
\Delta^j - \Delta^i = (T^j Q^j - T^i Q^i) - [r^{-1}(R^j - R^i) - (e^j - e^i)] - (1 - \beta)(L^j - L^i) 
\]

(12)

Where \(T^i = \pi^j - C^i \beta^j - (1 - \beta^j)r^{-1}\) and \(i < j\) for all equations above.

From Equation (7), we can find that the insurance firm has optimal capital structure in the costly capital flow market, and the capital structure is determined by the firm’s overall marginal profit with respect to its solvency ratio, denoted by \(MP^i\). Note that the marginal profit will be increased if the firm has a better solvency position in this model.

Specifically, the optimal \(e^i\) is the amount of external capital when the marginal cost of raising such capital is equal to \(MP^i\); the optimal \(\beta^i\) is the reinsurance ratio when the marginal cost of purchasing such reinsurance is equal to \(MP^i\); new premium \(\pi^i\) in the second period would be set up to make sure that the firm’s expected marginal profit of insurance underwritings is equal to \(MP^j\).

Equation (8) describes the effect of losses in the last period on the next-period insurance price, the sign of which is determined by \(Q^i_{\pi} b^i_{\pi}\) and \(b^i_{\pi}\). Firstly, we assume \(C^i_{b^i_{\pi}} = 0\) in order to check the sign in a simple way. If the insurance demand becomes more price elastic in response to a lower solvency ratio, with \(Q^i_{\pi} b^i_{\pi}\) and \(b^i_{\pi}\) being both positive, the effect will be negative. This situation can be plausible when people turn to buy other
available insurance products at the same cost from firms with higher solvency ability, or when people make use of other effective mechanisms to mitigate risks rather than purchase insurance. If $Q_{\pi, b, t}^i$ is negative, the insurance price elasticity of demand will be less in response to a lower solvency ratio. Then there is a chance that the relation between previous losses and future premium can be positive. This situation can be valid when there is supply shock in the insurance industry, and people cannot find any other effective risk solutions yet; the insurance firm can increase its own premium in a higher level and the insured insist on purchasing highly charged insurance products from firms with relatively higher solvency ability. This positive effect can be stronger in such circumstance that $b_{\pi, t}^i$ is also negative, implying that the effect of changing the solvency ratio on the demand changes dominates the effect of changing the premium on the demand changes. Therefore, in the extreme case with tight capital supply and high insurance demand in the insurance industry, the positive relation of shock losses and premium can be observed; and the solvency ability matters for the insurance firm to obtain an advantageous position in the insurance market. Then let us check this effect in an economy with the reinsurance market. The positive relation would be enlarged when $C_{b, t}^i$ is largely negative (costly) in the reinsurance market, which is consistent with the statement that price spikes after shock would be larger when the reinsurance rate is more sensitive to the firm’s solvency ratio during the period of a tight reinsurance market. Finally, we can find that the positively effect of losses on the next-period premium can
shrink when $Q_{b_i}^l$ is larger with the insured being more sensitive to the firm’s solvency ratio. It tells us that the price spike is rigidly limited for the insurance firm with quite a low solvency ratio after shock since many customers tend to quit. Therefore it is more likely for such a firm to encounter insolvency and even bankruptcy problem after a catastrophic event.

Equation (9) shows the effect of losses on the next-period external capital, whose sign is determined by $R_{e_i b_i}^l$ and $b_{e_i}^l$. If $R_{e_i b_i}^l$ is negative with the external capital cost being more sensitive to the capital amount in response to a lower solvency ratio, the relation between losses and external capital can be negative. In this case, the external capital market is too tight that the insurance firm tends to decrease its costly external capital rising amount, or make its solvency ratio as high as possible to attract equity holders with a relatively low external capital cost. If $R_{e_i b_i}^l$ is positive, with the external capital cost being less sensitive in response to a lower solvency ratio, the external capital market is not tight yet and the insurance firm may directly access more external capital to get rid of a higher loss.

Equation (10) provides the effect of losses on the next-period reinsurance ratio. It shows that the effect will be small if the marginal cost of reinsurance $C_{b_i}^l$ is largely negative with. It implies that the insurance firm would avoid reinsurance solution to transfer risks when the reinsurance market is tight.

Equation (11) is the price elasticity in the costly capital market. It shows that the insurance premium in the costly capital market is determined not only by its specific price
elasticity but also by its overall marginal profit and solvency position in each state. It also implies that the insurance cycle can be inspired in costly capital market by changes of the firm’s solvency position. If we let MP = 0 to make the solvency ratio useless, this equation will be the same as the equation (4) derived in a free capital market.

The difference of the firm’s overall profit between two states is shown by Equation (12). From the previous analysis, we can conclude that there is a chance for the insurance firm with a high solvency ratio to have larger expected profit with larger loss payment in the first period, denoted by \( \Delta^j - \Delta^i > 0 \). The part of \( (T^j Q^j - T^i Q^i) \) in this equation can be interpreted as underwriting premium spikes after a larger loss; and the part of \( [r^{-1}(R^j - R^i) - (e^j - e^i)] \) is extra external capital cost after a larger loss; and the part of \( (1 - \beta)(L^j - L^i) \) is the loss payment difference between two states. The possibility of positive profit difference between these two states can be increased when the highly solvent insurance firm can take advantage of price spikes and the insured’ loyalty after shock, and control the penalties of costly reinsurance rate and external capital cost in post-catastrophe underwritings. This is also the condition in which the insurance firm can benefit from catastrophic risk underwritings within these two periods.

**III Dynamic Cash Flow Models with Catastrophic Shocks**

In this section, we construct two dynamic cash flow models in which catastrophic shocks affect both of underwriting profit in the firm and capital cost in the capital market. These two models are extensions of the previous two-period model, and emphasize on the
insurance cycle inspired by the firm’s changing solvency positions as a result of catastrophic risks. The first dynamic model is for a representative insurance firm with temporary catastrophic risks in an infinite time line, while the second one is developed for heterogeneous insurance firms with stochastic catastrophic risks.

I) Dynamic Model with Temporary Catastrophic Shocks for an Insurance Firm

In this model, we show that catastrophic shocks have a dual impact on the firm’s cash flows: not only are they factors of the firm’s operation income, but they also affect the firm’s capital raising capability. The dynamic interaction between the firm’s balance sheet and capital raising rationing turns out to be an amplifying transmission mechanism, by which the effects of temporary catastrophic shocks persistently spread to the following cash flow distributions. Figure II below summarizes all the positive and negative cash flows for the representative insurance firm from period $t$ to period $t+1$.

![Figure II](image)

To explain the model construction in detail, we take the cash flows during the period $t$ as an example. At the beginning of the period $t$, the insurance firm has retained earnings $K_t$ with a return rate of $r_t$. The firm then experiences a catastrophic shock with a loss ratio $\alpha_t$. The shock reduces the firm's cash flows, and the firm must raise capital to maintain its solvency. The raised capital is then invested with a return rate $r_{t+1}$, further exacerbating the effects of the initial shock.

This dynamic interaction continues over time, leading to a persistent spread of the effects of the catastrophic shocks.
accumulated by all previous operations, and it continues to be accumulated at a return rate of $r_t$ until the end of the period $t$. At the end of the period $t$, the firm collects the total premium of $\pi_t Q_{t+1}$ to underwrite one-period coverage policies for period $t+1$; meanwhile, the firm should pay for the total losses of $\alpha_t Q_t$ claimed during the period $t$, where $\alpha_t$ denotes the loss ratio (the ratio of coverage that incurs losses). In this model, we assume that the insurance firm expertizes in estimating regular losses it underwrites, with $\pi_t = E_t(\alpha_{t+1})$; however, the firm cannot predict the frequency and the severity of catastrophic shocks, which is expressed as a temporary positive volatility of $\alpha_t$. In other words, the insurance premium per coverage $\pi_t$ in this model is exogenously determined by the insurance industry with $\pi_t = E(\alpha_{t+1})$, and $E(\alpha_{t+1})$ is always fixed during a long time. At the end of period $t$, the insurance firm also needs to repay a gross return of $R_t$ for the capital $e_t$ raised from external capital market by the end of period $t-1$; meanwhile, the firm would raise new external capital $e_{t+1}$ with a promised gross return $R_{t+1}$. Here we assume that $R_t$ is a convex function with $R_e > 0$, $R_{ee} > 0$, $R_\alpha > 0$, $R_{\alpha\alpha} > 0$. This assumption follows a basic principal in the capital market that investors would increase (decrease) the capital cost when observing the fact that the firm incurs larger (smaller) losses and thus has a relatively worse (better) financial position. After paying out some dividend $D_t$ to the firm’s owners, the firm would gather $K_{t+1}$ to be the internal capital saving at the beginning of the period $t+1$, which will be accumulated at a return rate of $r_{t+1}$ until the end of period $t+1$. If we interpret the dividend payment in each period as the net profit after all the operations of each period, we can derive the cash flow equation (13)
at the end of the period $t$ as follows:

$$D_t = \pi_t Q_{t+1} + r_t k_t + e_{t+1} - \alpha_t Q_t - R_t(e_t, \alpha_t) - K_{t+1}$$  \hspace{1cm} (13)$$

From (13), we can find in this model that the insurance firm collects revenue from total premium $\pi_t Q_{t+1}$, newly raised external capital $e_{t+1}$ and beginning-of-period internal capital $K_t$; and the firm distributes the revenue into three categories: claimed loss payment $\alpha_t Q_t$, promised gross return $R_t(\alpha_t, e_t)$, and end-of-period saving internal capital $K_{t+1}$.

In this model, we assume that the insurance firm is subject to the solvency regulation, expressed by the Kenny Ratio $\eta_t$ (the ratio of premiums written to policyholders’ surplus). Such a ratio provides a measure of an insurer’s financial stability and solvency position.

The regulation turns out to be stricter if a lower Kenny Ratio is required. We then have the following capital rising constraint (14),

$$\frac{\pi_t Q_{t+1}}{(K_t + e_{t+1})} \leq \eta_t$$  \hspace{1cm} (14)$$

In the literature, the firm’s owners are always assumed to be risk averse towards the firm’s dividend payments all over the infinite time line. Here we assume a concave utility function of the dividend payments $U(D_t)$ for the firm’s owners. The firm’s owners will choose the optimal strategies of dividend payment $D_t$, underwriting insurance coverage $Q_{t+1}$, newly raised external capital $e_{t+1}$, and internal capital saving $K_{t+1}$ in each period to maximize its expected utility of dividend payments all over the time. So the optimization problem in this model can be built by (15) subject to (13) and (14), where $\beta$ is the discount rate.

$$\max E[\sum_{t=0}^{\infty} \beta^t U(D_t)]$$  \hspace{1cm} (15)$$
where $\{ Q_{t+1}, e_{t+1}, K_{t+1}, D_t \} \in \text{Arg} \{ \text{Max} E[\sum_{t=0}^{\infty} \beta^t U(D_t)] \}$

To know the dynamic transmission mechanism in this model, let us check the steady state solutions $\{ Q^*, e^*, K^*, D^* \}$ at first. We claim that, in the steady state, the insurance firm has an incentive to make the capital rising constraint (14) binding into (16). Equation (17) is then derived from cash flow equation (13) with binding constraint (16).

\[ \eta(K^*+e^*) = \pi Q^* \] (16)

\[ [R(e^*, \alpha^*) - (2- \tau) e^*] = [\pi - \alpha^* + (\tau-1) \pi / \eta] Q^* - D^* \] (17)

Equation (17) shows that the firm’s capital rising capacity and underwriting profit are mutually dependent in the steady state. The firm tends to raise external capital $e^*$ to expand its underwriting coverage $Q^*$ until the capital raising cost $[R(e^*, \alpha^*) - (2- \tau) e^*]$ is covered by the difference between the underwriting expansion profit $[(\pi - \alpha^*) + (\tau-1) \pi / \eta] Q^*$ and the steady state dividend payment $D^*$. If there is a positive shock to $\alpha$, the available fund from underwriting profit to raise external capital is decreasing, and such decreasing capital will decline further with a higher capital cost $R$ along with the shock; the largely decreased external capital raising capacity will reduce the future underwriting expansion in turn. It implies that small catastrophic shocks can generate large, persistent fluctuations in underwriting profit and capital raising capability. Note that, from Equation (17), the firm will raise more external capital with a higher rate of return on invested assets $\tau$ or a lower Kenny Ratio $\eta$, illustrating that the firm tends to raise more capital if it has higher investment returns or if it needs to satisfy a stricter solvency regulation.

Then we make the firm at the initial of time $t$ be in the steady state, and check the effect of
temporary catastrophic shocks on the cash flows in the current and the following periods. From (13) and (16), we get the dynamic motion of the newly raised equity as follows,

\[(r_t-1) e_{t+1} = \left[ \pi \left( 1 + \frac{\eta_t}{\eta_{t+1}} \right) Q_{t+1} - \frac{\eta_{t+1}}{\eta_{t+1}} Q_{t+2} - \alpha_t Q_t - D_t \right] - R_t(\alpha_t) + e_{t+2} \quad (18)\]

Similarly as we find from Equation (17), a positive volatility of \(\alpha_t\) can negatively impact the underwriting profit, which will reduce the fund available to raise new capital \(e_{t+1}\).

Moreover, with the external capital cost \(R_t\) increased by \(\alpha_t\), \(e_{t+1}\) shrinks more deeply, which in turn limits the future underwriting quantity \(Q_{t+1}\). Therefore, the insurance firm’s current catastrophic shocks will affect its future cash flows according to mutual dependence between capital raising capability and underwriting profit. Note that, from Equation (18), a lower \(\eta_t\) (a current stricter solvency regulation) leads to more capital rising during the period \(t\) since the firm needs to keep a good solvency position to expand its underwriting capacity; while a lower \(\eta_{t+1}\) (a future stricter solvency regulation) leads to less capital rising during the period \(t\) in order to avoid repaying a higher gross return of equity in the next period.

In this dynamic model, the insurance cycle within one insurance firm can be caused by temporary catastrophic shocks, according to dynamic interaction between its underwriting profit and external capital rising capacity. Such dynamic interaction between the firm’s balance sheet and capital rising rationing turns out to be an amplifying transmission mechanism, by which the effects of temporary catastrophic shocks persistently spread to the following cash flows. The amplifying effect will be larger (smaller) if the shock is more (less) volatile, and if the capital regulation constraint is more relaxed (stricter), and
if the capital market is more (less) sensitive to shocks, and if the firm relies on external capital more (less) heavily.

The initial results by linear quadratic approximation of this model will be discussed in section IV. The calibrations to do for this model will be introduced in section V.

II) Dynamic Model with Stochastic Catastrophic Shocks for Heterogeneous Firms

A dynamic model of heterogeneous insurance firms with stochastic catastrophic shocks is developed in this subsection. Both of the time line and the assumptions in this model are the same as the previous one, but it is built in a heterogeneous-agent economy instead. In this model economy, every firm receives stochastic catastrophic shocks $\alpha_t$, which generate underwriting insurance premium distribution, internal capital saving distribution and external capital raising distribution.

In this model, the insurance firms are heterogeneous with respect to the individual state vector $s = \{\alpha, Q, K, e\}$. We assume that the loss ratio $\alpha$ follows the markov process, which will be calibrated by the catastrophic loss data. The underwriting insurance coverage $Q$, the beginning-of-period internal capital $K$, and the external capital amount $e$ all change in accord with the firm’s optimal decisions in last period. Here we use $\alpha', Q', K', e'$ to denote the corresponding state variables in the next period.

The same as the previous model, the firm’s fund sources for each period are underwriting premium, newly raised external capital, and internal capital, and the fund is distributed in each period into three categories: loss payment, equity return payment, and saving internal capital. Every firm chooses its optimal decision path for each specific fund source.
and fund usage to maximize its expected utility of dividend payment over the infinite time line. Let \( V(s) \) be value function of the firm in individual state \( s \), and \( P(\alpha' | \alpha) \) be the state transition probability function of loss ratio. The optimization problem for the firm should be as follows,

\[
V(s) = \text{Max} \left[ U(D) + \beta \sum_{\alpha'} P(\alpha' | \alpha) V(s') \right]
\]

a) Control variables’ constraints

\[
D \geq 0; \\
Q' > 0;
\]

b) Cash flow constraint

\[
D = \pi Q' + rK - \alpha Q - K' + e' - R(e, \alpha);
\]

c) Capital rising constraint

\[
\pi Q' \leq \eta (K + e');
\]

d) The law of motion of state variables

\[
s' = \{\alpha', Q', K', e'\}.
\]

For the aggregation, we construct distribution function for firms in different states. The firm’s states can include loss ratio, underwriting coverage, internal capital saving and external capital amount. Let \( x(s) \) be the firm’s aggregation probability density function, and let \( X(s) \) be the corresponding cumulative distribution function. The firm aggregation for each period is normalized to unity, shown as follows,

\[
\int_{\alpha^* Q^* K^* E} dX(s) = 1
\]

The law of motion of the firm’s aggregation distribution is as follows,
\[ x(s') = \int_{\alpha^*Q^*K^*E} 1 \{Q' = Q'(s); K' = K'(s); e' = e'(s)\} * P(\alpha'|\alpha) \ dX(s) \]

Then aggregated underwriting insurance coverage, internal capital saving, and external capital rising amount for each period are as follows,

\[ Q' = \int_{\alpha^*Q^*K^*E} Q(s) \ dX(s) \]
\[ K' = \int_{\alpha^*Q^*K^*E} K(s) \ dX(s) \]
\[ e' = \int_{\alpha^*Q^*K^*E} e(s) \ dX(s) \]

Based on the model construction above, we can quantitatively derive the optimal decisions of dividend payment, underwriting coverage, external capital rising, and internal capital saving among heterogeneous insurance firms with stochastic catastrophic risks. The potential simulation results of this heterogeneous-agent model in future study will be introduced in section V.

**IV Initial Simulation Results of the First Dynamic Model**

In this section, we calibrate the first dynamic model based on economic literature, and the initial simulation results are provided. We will discuss calibrations to do by using real data in section V.

I) Initial Calibrations

a) Utility function and gross return function

Consistent with the business cycle literature, we assume that the utility function of dividend payments for the firm’s owners is a CRRA function as follows,
\[ U(D) = \frac{D^{1-\gamma} - 1}{1-\gamma} \]

As we discussed before, we assume that the gross return function in external capital market is a convex function as follows.

\[ R(e, \alpha) = r_e \alpha^{\theta_e} e^{\theta_e} \]

Perimeter \( r_e \) is the gross return rate, and perimeters \( \theta_e \) and \( \theta_e \) are elasticity of catastrophic shock and external capital rising amount in the gross return function. In this section, we simply refer to economic literature to get these parameters. We will discuss how to calibrate them by real data in section V.

b) State Transition Probability Function

With catastrophic shocks for each individual state, the firms receive different levels of catastrophic loss shocks \( \alpha \). The motions of catastrophic loss shocks follow Markov Chains. The stochastic process of \( \alpha \) is,

\[ \ln(\alpha') = \rho \ln(\alpha) + \varepsilon' \text{ where } \varepsilon' \sim N(0, \sigma^2) \]

In this section, we just cite parameters of stochastic technology shocks in economic literature to be our parameters \( \rho \) and \( \sigma \).

c) Economic parameters

Table 1 is a list of main economic parameters for the benchmark economy in these two models. These parameters’ values are all consistent with economic literature.
### Table 1

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Notations</th>
<th>Values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Time discount factor</td>
<td>$\beta$</td>
<td>0.94</td>
</tr>
<tr>
<td>Kenny ratio</td>
<td>$\eta$</td>
<td>2.00</td>
</tr>
<tr>
<td>Coefficient of relative risk aversion</td>
<td>$\gamma$</td>
<td>1.50</td>
</tr>
<tr>
<td>Gross return rate of investment</td>
<td>$r$</td>
<td>1.06</td>
</tr>
<tr>
<td>Gross return rate of external capital</td>
<td>$r_e$</td>
<td>1.20</td>
</tr>
<tr>
<td>Catastrophic shock Elasticity</td>
<td>$\theta_\alpha$</td>
<td>1.05</td>
</tr>
<tr>
<td>External Capital Elasticity</td>
<td>$\theta_e$</td>
<td>1.05</td>
</tr>
<tr>
<td>Autocorrelation of $\ln(\alpha)$</td>
<td>$\rho$</td>
<td>0.80</td>
</tr>
<tr>
<td>Standard deviation of $\varepsilon'$</td>
<td>$\sigma$</td>
<td>0.003</td>
</tr>
<tr>
<td>The number of insurance quantity nodes</td>
<td>$i_{max}$</td>
<td>5</td>
</tr>
<tr>
<td>The number of internal capital nodes</td>
<td>$k_{max}$</td>
<td>5</td>
</tr>
<tr>
<td>The number of external capital nodes</td>
<td>$j_{max}$</td>
<td>5</td>
</tr>
<tr>
<td>The number of catastrophic shock nodes</td>
<td>$m_{max}$</td>
<td>5</td>
</tr>
</tbody>
</table>

II) Linear Quadratic Approximation for the first dynamic model

a) The relation between catastrophic risk and underwriting quantity

The figure III below tells us that the underwriting insurance coverage in the benchmark economy is negatively correlated with catastrophic risk. The insurance firm reacts after each catastrophic risk: the insurance firm tends to decrease its underwriting coverage to
avoid potential large losses when they observe an occurrence of a high catastrophic risk.

![Figure III](image)

**Figure III**

b) The relation between external capital and catastrophic risk

In the benchmark economy with costly external capital market, Figure IV shows that the external capital rising amount is negatively correlated with catastrophic risk. The external capital tends to shrink due to higher capital cost along with an occurrence of catastrophe events. Figure III and IV support the capacity constraint theory that supply capacity is reduced in the insurance industry due to capital shortage after catastrophic risks.

![Figure IV](image)

**Figure IV**

If we reduce the external capital gross return rate $r_e$ in benchmark economy by 10%, and
also reduce the elasticity $\theta_e$ and $\theta_\alpha$ by 4% respectively, the external capital turns out to be positively correlated with catastrophic risk with smaller fluctuations. It is shown in Figure V. In such situation with a soft capital market, the insurance firm is more likely to resort to external capital market in order to expand its underwriting capacity and reserve more for future catastrophic losses.

![Figure V](image)

c) The insurance cycle amplitude

The left hand side of Table 2 shows the amplitudes of each variable’s fluctuation in the insurance cycle of the benchmark economy, while the right one shows the corresponding amplitudes in an economy with a higher $\sigma$ up to 0.005.

<table>
<thead>
<tr>
<th>st.dev. (%)</th>
<th>st.dev. (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Underwriting Liability</td>
<td>3.095</td>
</tr>
<tr>
<td>Coverage q</td>
<td>3.112</td>
</tr>
<tr>
<td>Dividend d</td>
<td>0.007</td>
</tr>
<tr>
<td>Internal Capital k</td>
<td>0.007</td>
</tr>
<tr>
<td>External Capital ex</td>
<td>2.997</td>
</tr>
<tr>
<td>Ex Marginal Cost $y$</td>
<td>0.252</td>
</tr>
<tr>
<td>Catastrophic Risk alpha</td>
<td>0.380</td>
</tr>
</tbody>
</table>
From Table 2, we can find that the effect of the insurance cycle resulting from more volatile catastrophic shocks is larger.

The left hand side of Table 3 shows the corresponding amplitudes in an economy with $\theta_\alpha$ increasing to 1.1. It illustrates that the insurance cycle effect is stronger when the capital market is more sensitive towards catastrophic events. The right one shows the corresponding amplitudes in an economy with a higher $\eta$ up to 2.5, and it illustrates that the insurance cycle effect is enlarged if the solvency regulation is more relaxed.

<table>
<thead>
<tr>
<th></th>
<th>st.dev.(%)</th>
<th></th>
<th>st.dev.(%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Underwriting Liability</td>
<td>3.217</td>
<td>Underwriting Liability</td>
<td>3.100</td>
</tr>
<tr>
<td>Coverage q</td>
<td>3.211</td>
<td>Coverage q</td>
<td>3.128</td>
</tr>
<tr>
<td>Dividend d</td>
<td>0.007</td>
<td>Dividend d</td>
<td>0.007</td>
</tr>
<tr>
<td>Internal Capital k</td>
<td>0.007</td>
<td>Internal Capital k</td>
<td>0.007</td>
</tr>
<tr>
<td>External Capital ex</td>
<td>3.021</td>
<td>External Capital ex</td>
<td>3.012</td>
</tr>
<tr>
<td>Ex Marginal Cost y</td>
<td>0.297</td>
<td>Ex Marginal Cost y</td>
<td>0.282</td>
</tr>
<tr>
<td>Catstrophic Risk alpha</td>
<td>0.372</td>
<td>Catstrophic Risk alpha</td>
<td>0.371</td>
</tr>
</tbody>
</table>

These compared results support the statement derived in our models that the insurance cycle will be larger if the loss ratio is more volatile, and if the external capital market is tighter, and if the regulation is more relaxed.

V Calibrations and Simulations to Do

I) Calibrates to do in these two dynamic models

We will calibrate parameters in these two dynamic models by real data in future, which will be generally collected from A.M. Best, NAIC Annual Statement, and Sigma, Bloomberg Professional and Insurance Information Institute Websites.

For the gross return rates of both internal capital and external capital in the insurance
industry, we will check the A.M. Best for aggregate data. For perimeters $\theta_a$ and $\theta_e$ in external capital gross return function, we need to do regressions of loss ratios and equity rising amount on the return of equity to get relevant empirical coefficients based on the data from A.M. Best and NAIC Annual Statement.

The State transition probability function for stochastic catastrophic shocks will be estimated by AR1 process of loss ratios in both of industry and firm levels. We will use loss ratio data from NAIC Annual Statement to fit the lognormal distribution for insurance firms. Five nodes of persistent catastrophic shocks as well as unconditional distribution of these nodes will be given by simulating the median of loss ratios in both of industry and individual levels, and the Markov Transition Matrix will be calculated by using bivariate normal distribution function.

II) Heterogeneous-Agent simulations to do

According to future simulations in heterogeneous insurance firm model, we try to analyze that the good firm with higher internal capital saving and better underwriting technology is able to raise relatively more external capital to expand its underwriting capacity; while the bad firm with lower internal capital saving and worse underwriting technology tends to raise relatively lower external capital and its profit shrinks due to its limited underwriting capacity. These statements will be shown by figures of simulation results. We intend to suggest bad firms tend to quit from related lines and face the challenge of higher reinsurance rate and further capital shortage, which are more likely to go bankruptcy quickly if incurring catastrophic shocks. Moreover, we try to verify that the
performance difference between good firms and bad firms will become larger if catastrophic risks are more volatile.

VI Current Conclusions and Future Empirical Works

In this paper, we construct a series of cash flow models to simulate the insurance firm’s catastrophic risk intermediating process in costly external capital market, focusing on the firm’s decision-making of its underwritings and capital structure. The two-period model concludes that the insurance firm has optimal capital structure, and the positive relation of catastrophic losses and insurance price can be observed in the situation of tight capital supply and high insurance demand. It also explores how the insurance firm can benefit from catastrophic risk underwritings in a long run. The following two dynamic models with stochastic catastrophic shocks shows us that temporary catastrophic shocks impact on the insurance firm’s cash flows, in persistent and amplifying way, by dynamic interaction between underwriting capability and external capital capacity. And they suggest that catastrophe events do accelerate the industrial organization developing process in insurance market.

The future empirical works will study the hypothesis developed from these three models in both of the industry level and the firm level.

In the industry level, with capital shortage due to a series of catastrophe shocks, the positive relation of losses and next-period premium in the insurance industry can be observed. Higher price spike and larger capacity swing in related insurance lines can be
observed when a more severe catastrophe event occurs. This relation will be strengthened when consumers cannot tolerate such catastrophic risk and have no other effective mechanism to mitigate such losses; the effect can be larger for disasters occurring at the first time, such as 9/11; this association would also be stronger in the tighter reinsurance or equity capital market. There would be more bankruptcy firms when a more severe catastrophe event occurs, especially in the hard reinsurance or equity capital market.

In the firm level, under some circumstances, high-quality firms may benefit from catastrophe events by taking advantage of price spikes and the insured’ loyalty in post-catastrophe underwritings, and controlling the penalties of costly reinsurance rate and external capital cost after catastrophic shocks. High-quality and low-quality firms differ from each other in terms of catastrophe risk underwriting technology (i.e. time and regional diversification, catastrophe prediction modeling, and underwriting history) and capital raising capability (i.e. loss reserve, solvency ratio, investment strategy, capital structure, rating rank, and information transparency).
References:


Kunreuther and Michel-Kerjan with Doherty, Grace, Klein and Pauly, 2009, at the war with weather, the MIT Press.


