

When Can Expected Utility Handle First-order Risk Aversion?

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Abstract

Expected utility functions are limited to second-order (conditional) risk aversion, while non-expected utility functions can exhibit either first-order or second-order (conditional) risk aversion. We extend the concept of orders of conditional risk aversion to orders of conditional dependent risk aversion. We show that first-order conditional dependent risk aversion is consistent with the framework of the expected utility hypothesis. Our theoretical result proposes new insights into some economic and financial applications.

Keywords: Expected utility theory; first-order conditional dependent risk aversion; background risk; equity premium puzzle; social security; public investment; consumption risk.

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1 Introduction

The concepts of second-order and first-order risk aversion were coined by Segal and Spivak (1990). For an actuarially fair random variable $\tilde{\varepsilon}$, second-order risk aversion means that the risk premium the agent is willing to pay to avoid $k\tilde{\varepsilon}$ is proportional to k^2 as $k \rightarrow 0$. Under first-order risk aversion, the risk premium is proportional to k . Loomes and Segal (1994) extend this notion to preferences about uninsured events, such as independent additive background risks. They introduce the concept of orders of conditional risk aversion. Define \tilde{y} as an independent additive risk. The conditional risk premium is the amount of money the decision maker is willing to pay to avoid $\tilde{\varepsilon}$ in the presence of \tilde{y} . The preference relation satisfies first-order conditional risk aversion if the risk premium the agent is willing to pay to avoid $k\tilde{\varepsilon}$ is proportional to k as $k \rightarrow 0$. It satisfies second-order conditional risk aversion if the risk premium is proportional to k^2 .

First-order (conditional) risk aversion implies that small risks matter. It is well known from Arrow (1974) and Borch (1974) that differentiable expected utility (EU) is second order. Only non differentiable EU exhibits first order risk aversion at points of nondifferentiability. Because expected utility theory is limited to second-order (conditional) risk aversion, it cannot take into account many real world results. Several non-EU models that can predict first-order risk aversion behavior (recursive utility, rank-dependent EU and loss aversion) are becoming an alternative in the economic and financial literatures to explain different puzzles that EU cannot handle (see, Epstein and Zin 1990; Quiggin 1982; Tversky and Kahneman 1992).

In this paper, we reinvestigate whether first-order conditional risk aversion appears in the framework of the expected utility hypothesis; the general answer to this question is positive with some weak restrictions: expected utility theory exhibits first-order risk aversion when there is a dependent background risk (but not otherwise). We extend the concepts of orders of conditional risk aversion to orders of conditional dependent risk aversion, for which $\tilde{\varepsilon}$ and the background risk \tilde{y} are dependent and \tilde{y} may enter the agent's utility function arbitrarily. We thus propose a new source of first-order risk aversion over general preferences. Since EU theory is simple, parsimonious and quite successful in explaining a wide set of empirical facts, our explanation of first-order risk aversion is easy to interpret, and the assumptions are plausible (i.e. an income risk which changes over the economic cycles can be interpreted as a dependent background risk for many economic agents).

We propose conditions on the stochastic structure between $\tilde{\varepsilon}$ and \tilde{y} that guarantee first-

order conditional dependent risk aversion for expected utility agents with a certain type of risk preference, i.e., with correlation aversion. Eeckhoudt, Rey and Schlesinger (2007) provide an economic interpretation of correlation aversion: a higher level of the background variable mitigates the detrimental effect of a reduction in wealth. It turns out that the concept of expectation dependence, proposed by Wright (1987), is a key element to such stochastically structure.

The paper proceeds as follows: Section 2 sets up the model. Section 3 discusses the concept of orders of conditional risk aversion and investigates the orders of conditional dependent risk aversion. Section 4 applies the results to different economic and financial examples. Section 6 concludes the paper.

2 The model

We consider an agent whose preference for a random wealth, \tilde{w} , and a random outcome, \tilde{y} , can be represented by a bivariate expected utility function. Let $u(w, y)$ be the utility function. We assume that all partial derivatives required for any definition exist. We make the standard assumption that $u_1 > 0$.

Let us assume that $\tilde{z} = k\tilde{\varepsilon}$ is the risk faced by a risk averse agent in the absence of a background risk. Parameter k can be interpreted as the size of the risk. One way to measure an agent's degree of risk aversion for \tilde{z} is to ask her how much she is ready to pay to get rid of \tilde{z} . The answer to this question will be referred to as the risk premium $\pi(k)$ associated with that risk. For an agent with utility function u , $E\tilde{y}$ the expected value of another risk \tilde{y} , and non-random initial wealth w , the risk premium $\pi(k)$ must satisfy the following condition:

$$u(w + Ek\tilde{\varepsilon} - \pi(k), E\tilde{y}) = Eu(w + k\tilde{\varepsilon}, E\tilde{y}). \quad (1)$$

Segal and Spivak (1990) give the following definitions of first and second-order risk aversion:

Definition 2.1 (Segal and Spivak, 1990) *The agent's attitude towards risk at w is of first order if for every $\tilde{\varepsilon}$ with $E\tilde{\varepsilon} = 0$, $\pi'(0) \neq 0$. The agent's attitude towards risk at w is of second order if for every $\tilde{\varepsilon}$ with $E\tilde{\varepsilon} = 0$, $\pi'(0) = 0$ but $\pi''(0) \neq 0$.*

They provide the following results linking properties of a utility function to its order of risk aversion given the level of wealth w_0 :

(a) If a risk averse von Neumann-Morgenstern utility function u is not differentiable at w_0 but has well-defined and distinct left and right derivatives at w_0 , then the agent exhibits first-order risk aversion at w_0 .

(b) If a risk averse von Neumann-Morgenstern utility function u is twice differentiable at w_0 with $u_{11} < 0$, then the agent exhibits second-order risk aversion at w_0 .

Segal and Spivak (1997) point out that if the von Neumann-Morgenstern utility function is increasing, then it must be differentiable almost everywhere, and one may convincingly argue that non-differentiability is seldom observed in the expected utility model. Alternatively, concave functions may still have a countable set of points of nondifferentiability (the same as increasing functions).

Loomes and Segal (1994) introduced the order of conditional risk aversion by examining the characteristic of $\pi(k)$ in the presence of an independent uninsured risk \tilde{y} . For an agent with utility function u and initial wealth w , the conditional risk premium $\pi_c(k)$ must satisfy the following condition:

$$Eu(w + Ek\tilde{\varepsilon} - \pi_c(k), \tilde{y}) = Eu(w + k\tilde{\varepsilon}, \tilde{y}), \quad (2)$$

where $\tilde{\varepsilon}$ and \tilde{y} are independent.

Definition 2.2 (Loomes and Segal, 1994) *The agent's attitude towards risk at w is first-order conditional risk aversion if for every $\tilde{\varepsilon}$ with $E\tilde{\varepsilon} = 0$, $\pi'_c(0) \neq 0$. The agent's attitude towards risk at w is second-order conditional risk aversion if for every $\tilde{\varepsilon}$ with $E\tilde{\varepsilon} = 0$, $\pi'_c(0) = 0$ but $\pi''_c(0) \neq 0$.*

It is obvious that the definitions of first- and second-order conditional risk aversion are more general than the definitions of first- and second-order risk aversion. We can extend the above definitions to the case $E\tilde{\varepsilon} \neq 0$. Since u is differentiable, fully differentiating (2) with respect to k yields

$$E\{[E\tilde{\varepsilon} - \pi'_c(k)]u_1(w + Ek\tilde{\varepsilon} - \pi_c(k), \tilde{y})\} = E[\tilde{\varepsilon}u_1(w + k\tilde{\varepsilon}, \tilde{y})]. \quad (3)$$

Since $\tilde{\varepsilon}$ and \tilde{y} are independent,

$$\pi'_c(0) = \frac{E\tilde{\varepsilon}Eu_1(w, \tilde{y}) - E[\tilde{\varepsilon}u_1(w, \tilde{y})]}{Eu_1(w, \tilde{y})} = 0. \quad (4)$$

Therefore, not only does $\pi_c(k)$ approach zero as k approaches zero, but also $\pi'_c(0) = 0$. This implies that at the margin, accepting a small risk has no effect on the welfare of risk-averse

agents. This is an important property of expected-utility theory: “in the small”, the expected-utility maximizers are risk-neutral in presence of an independent background risk.

Using a Taylor expansion of π_c around $k = 0$, we obtain that¹

$$\pi_c(k) = \pi_c(0) + \pi'_c(0)k + O(k^2) = O(k^2). \quad (5)$$

This result is the Arrow-Pratt approximation, which states that the conditional risk premium is approximately proportional to the square of the size of the risk.

In conclusion, within the von Neumann-Morgenstern expected-utility model, if the random outcome and the background risk are independent, then second-order conditional risk aversion relies on the assumption that the utility function is differentiable. Hence, with an independent background risk, utility functions in the von Neumann-Morgenstern expected utility class can generically exhibit only second-order conditional risk aversion and cannot explain the rejection of a small, independent, and actuarially favorable gamble.

3 Order of conditional dependent risk aversion

We now introduce the concept of order of conditional dependent risk aversion. For an agent with utility function u and non-random initial wealth w , the conditional dependent risk premium, $\pi_{cd}(k)$, must satisfy the following condition:

$$Eu(w + Ek\tilde{\varepsilon} - \pi_{cd}(k), \tilde{y}) = Eu(w + k\tilde{\varepsilon}, \tilde{y}), \quad (6)$$

where $\tilde{\varepsilon}$ and \tilde{y} are not necessarily independent risks.

Definition 3.1 *The agent’s attitude towards risk at w is first-order conditional dependent risk aversion if for every $\tilde{\varepsilon}$, $\pi_{cd}(k) - \pi_c(k) = O(k)$.*

Definition 3.2 *The agent’s attitude towards risk at w is second-order conditional dependent risk aversion if for every $\tilde{\varepsilon}$, $\pi_{cd}(k) - \pi_c(k) = O(k^2)$.*

$\pi_{cd}(k) - \pi_c(k)$ measures how dependence between risks affects risk premium. Second-order conditional dependent risk aversion implies that, in the presence of a dependent background risk,

¹In the statistical literature, the sequence b_k is at most of order k^λ , denoted as $b_k = O(k^\lambda)$, if for some finite real number $\Delta > 0$, there exists a finite integer K such that for all $k > K$, $|k^\lambda b_k| < \Delta$ (see White 2000, p16).

small risk has no effect on risk premium, while first-order conditional dependent risk aversion implies that, in the presence of a dependent background risk, small risk affects risk premium.

We denote by $F(\varepsilon, y)$ and $f(\varepsilon, y)$ the joint distribution and density functions of $(\tilde{\varepsilon}, \tilde{y})$, respectively. $F_\varepsilon(\varepsilon)$ and $F_y(y)$ are the marginal distributions.

Wright (1987) introduced the concept of expectation dependence in the economic literature.

Definition 3.3 *If*

$$ED(y) = [E\tilde{\varepsilon} - E(\tilde{\varepsilon}|\tilde{y} \leq y)] \geq 0 \text{ for all } y, \quad (7)$$

and there is at least some y_0 for which a strong inequality holds,

then $\tilde{\varepsilon}$ is positive expectation dependent on \tilde{y} . Similarly, $\tilde{\varepsilon}$ is negative expectation dependent on \tilde{y} if (7) holds with the inequality sign reversed.

Wright (1987, p115) interprets negative first-degree expectation dependence as follows: “when we discover \tilde{y} is small, in the precise sense that we are given the truncation $\tilde{y} \leq y$, our expectation of $\tilde{\varepsilon}$ is revised upward.” Definition 3.3 is useful for deriving an explicit value of $\pi_{cd}(k)$.

Lemma 3.4

$$\pi_{cd}(k) = -k \frac{\int_{-\infty}^{\infty} ED(y) u_{12}(w, y) F_y(y) dy}{Eu_1(w, \tilde{y})} + O(k^2). \quad (8)$$

Proof See Appendix.

Lemma 3.4 shows the general condition for first-order risk aversion. The condition involves two important concepts u_{12} , the cross-derivative of the utility function, and $ED(y)$, the expectation dependence between the two risks. The sign of u_{12} indicates how this first concept acts on utility u . Eeckhoudt et al. (2007) provide a context-free interpretation of the sign of u_{12} . They show that $u_{12} < 0$ is necessary and sufficient for “correlation aversion,” meaning that a higher level of the background variable mitigates the detrimental effect of a reduction in wealth. Condition (8) also captures the welfare interaction between the two risks. The sign of expectation dependence indicates whether the movements on background risk tend to reinforce the movements of \tilde{y} on wealth (positive expectation dependence) or to counteract them (negative expectation dependence). Lemma 3.4 allows a quantitative treatment of the direction and size of the effect of expectation dependence on first order risk aversion. To clarify this, consider the following cases: (1) Assume the agent is correlation neutral ($u_{12} = 0$) or the background

risk is independent ($ED(y) = 0$), then the agent's attitude towards risk is only second-order conditional dependent risk aversion; (2) Assume $u_{12} < 0$ and $ED(y) > 0$ ($ED(y) < 0$), then the agent's attitude towards risk is also first-order conditional dependent risk aversion and her marginal risk premium for a small risk is positive (negative) (i.e., $\lim_{k \rightarrow 0^+} \pi'_{cd}(k) > (<)0$).

From Lemma (3.4) and Equation (5), we obtain²

Proposition 3.5 (i) *If $\tilde{\varepsilon}$ is positive expectation dependent on \tilde{y} and $u_{12} < 0$, then the agent's attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) - \pi_c(k) = |O(k)|$;*

(ii) *If $\tilde{\varepsilon}$ is negative expectation dependent on \tilde{y} and $u_{12} > 0$, then the agent's attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) - \pi_c(k) = |O(k)|$;*

(iii) *If $\tilde{\varepsilon}$ is positive expectation dependent on \tilde{y} and $u_{12} > 0$, then the agent's attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) - \pi_c(k) = -|O(k)|$;*

(iv) *If $\tilde{\varepsilon}$ is negative expectation dependent on \tilde{y} and $u_{12} < 0$, then the agent's attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) - \pi_c(k) = -|O(k)|$.*

The economic intuition behind Proposition 3.5 is as follows: In absence of a dependent background risk, differentiable EU is only second order because the derivative is taken around the certainty line ($k = 0$ meaning no risk). In small neighborhoods differentiable functions behave like linear functions (in the present context, expected value, hence risk neutrality). But suppose now that risk $\tilde{\varepsilon}$ is added to risk \tilde{y} and the two are not independent. For example, suppose that $\tilde{y} = (-x, H; x, T)$ and $k\tilde{\varepsilon} = (-k, H; k, T)$; H and T are the same for \tilde{y} and $k\tilde{\varepsilon}$ and represent two states of nature. From now, when we take the derivative of $\pi(k)$ at $k = 0$, we take derivatives of the utility function at two different points: $-x$ and x . Since these derivatives are typically different under risk aversion, the derivative with respect to k will not be zero but a function of the difference between $u(w, -x)$ and $u(w, x)$.

We consider two special cases to illustrate Proposition 3.5.

Case 1. Consider an additive background risk with $u(x, y) = U(x + y)$. Here x may be the random wealth of an agent and y may be a random income risk which cannot be insured. Since

²Dionne and Li (2011) show that the more information that we have about the sign of higher cross derivatives of the utility, the weaker the dependence conditions on distribution we need. These weaker dependence conditions, which demonstrate the applicability of a weak version of expectation dependence induce weaker dependence conditions between $\tilde{\varepsilon}$ and \tilde{y} to guarantee first-order conditional dependent risk aversion.

$u_{12} < 0 \Leftrightarrow U'' < 0$, part (i) and (iv) of Proposition 3.5 imply that if the agent is risk averse and $\tilde{\varepsilon}$ is positive (negative) expectation dependent on the background risk \tilde{y} , then the agent's attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) > (<)\pi_c(k)$.

Case 2. Consider a multiplicative background risk with $u(x, y) = U(xy)$. Here x may be the random wealth of an agent and y may be a random real interest rate risk which cannot be hedged. Since $u_{12} < 0 \Leftrightarrow -xy \frac{U''(xy)}{U'(xy)} > 1$ (relative risk aversion greater than 1), Proposition 3.5 implies that (i) if $-xy \frac{U''(xy)}{U'(xy)} > 1$ and $\tilde{\varepsilon}$ is positive (negative) expectation dependent on the background risk \tilde{y} , then the agent's attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) > (<)\pi_c(k)$; (ii) if $-xy \frac{U''(xy)}{U'(xy)} < 1$ and $\tilde{\varepsilon}$ is positive (negative) expectation dependent on the background risk \tilde{y} , then the agent's attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) < (>)\pi_c(k)$.

A person is defined as a coward gambler if she does not make a sufficiently small bet when you offer her two-to-one odds and let her choose her side (see, Samuelson 1963). The assumption that people are cowards has some clear implications to financial markets (see Segal and Spivak, 1990). The following proposition yields the conditions for people to be not coward (or coward) in the presence of background risk.

Proposition 3.6 (i) *Let $E[\tilde{\varepsilon}] > 0$. If the decision maker's attitude towards risk is second-order conditional dependent risk aversion, then for a sufficiently small $k > 0$, $Eu(w + k\tilde{\varepsilon}, \tilde{y}) > Eu(w, \tilde{y})$.*

(ii) *Let $E[\tilde{\varepsilon}] > 0$. If her attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) - \pi_c(k) = -|O(k)|$, then for a sufficiently small $k > 0$, $Eu(w + k\tilde{\varepsilon}, \tilde{y}) > Eu(w, \tilde{y})$.*

(iii) *Let $E[\tilde{\varepsilon}] > 0$. If her attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) - \pi_c(k) = |O(k)|$, and if $E[\tilde{\varepsilon}]$ is small enough, then for a sufficiently small $k > 0$, $Eu(w, \tilde{y}) > Eu(w + k\tilde{\varepsilon}, \tilde{y})$.*

(iv) *Let $E[\tilde{\varepsilon}] < 0$. If her decision maker's attitude towards risk is second-order conditional dependent risk aversion, then for a sufficiently small $k > 0$, $Eu(w, \tilde{y}) > Eu(w + k\tilde{\varepsilon}, \tilde{y})$.*

(v) *Let $E[\tilde{\varepsilon}] < 0$. If her attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) - \pi_c(k) = |O(k)|$, then for a sufficiently small $k > 0$, $Eu(w, \tilde{y}) > Eu(w + k\tilde{\varepsilon}, \tilde{y})$.*

(vi) *Let $E[\tilde{\varepsilon}] < 0$. If the his attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) - \pi_c(k) = -|O(k)|$, and if $E[\tilde{\varepsilon}]$ is small enough, then for a sufficiently small $k > 0$, $Eu(w + k\tilde{\varepsilon}, \tilde{y}) > Eu(w, \tilde{y})$.*

Proof See Appendix. Q.E.D.

Consider a portfolio problem. The agent has a sure wealth w . The risk-free return over the period is r_f . The return of the risky asset over the period is a random variable \tilde{r} . The problem of the agent is to determine how much to invest in risky asset, where $w - \alpha$ is invested in riskless asset and α is invested in risky asset. The value of the portfolio at the end of the period may be written as

$$(w - \alpha)(1 + r_f) + \alpha(1 + \tilde{r}) = w(1 + r_f) + \alpha(\tilde{r} - r_f) = w_0 + \alpha\tilde{x}, \quad (9)$$

where $w_0 = w(1 + r_f)$ is future wealth obtained with the risk-free strategy and $\tilde{x} = \tilde{r} - r_f$. The agent wants to choose α to maximize expected utility.

$$V(\alpha) = Eu(\tilde{w}, \tilde{y}) = Eu(w_0 + \alpha\tilde{x}, \tilde{y}). \quad (10)$$

Define α^* as the solution to this problem. We have the following result (see Li, 2011).

Proposition 3.7 $\alpha^* \geq 0$, if and only if $V'(0) \geq 0$.

A sufficient condition for $V'(0) \geq 0$ is $Eu(w_0 + \alpha\tilde{x}, \tilde{y}) > Eu(w_0, \tilde{y})$ for small $\alpha > 0$. From part (i) of Proposition 3.6, we know that if the decision maker's attitude towards risk is second-order conditional dependent risk aversion, then he will invest some positive amount in the risky asset if it has a positive mean. This recovers Arrow's (1974) famous results. However, part (iii) of Proposition 3.6 tells us that this conclusion fails when the decision maker is first-order conditional dependent risk averse. In that case we get the much more plausible result that if the expected yield is positive but sufficiently small, a correlation averse decision maker will not invest in the risky asset if it is positive expectation dependent on the background risk.

4 Applications

In this section we illustrate the applicability of our results. In particular, we demonstrate the practical relevance of the distinction between first- and second-order risk aversion.

4.1 Pareto-Improving Social Security Reform

When the introduction of social security represents a Pareto-improving reform? This question is controversial. We examine it by the simple two-period partial equilibrium model introduced

by Krueger and Kubler (2006). The agent lives for two periods and earns a wage w in the first period on which she pays a payroll tax τ . The remainder of her wage is invested into a risky saving technology with stochastic gross return \tilde{R} . In the second period of her life she receives social security payments of $\tau w \tilde{G}$, where \tilde{G} is the stochastic gross return of the social security system. The agent consumption \tilde{c} in the second period of her life is equal to

$$\tilde{c} = (1 - \tau)w\tilde{R} + \tau w\tilde{G}. \quad (11)$$

Lifetime utility, as a function of the size of the social security system, is therefore given by

$$U(\tau) = Eu((1 - \tau)w\tilde{R} + \tau w\tilde{G}) \quad (12)$$

where $u(\cdot)$ is a differentiable utility function and where $E(\cdot)$ is the expected utility with respect to the uncertainty in the second period of life. We want to verify when a marginal introduction of a social security system is welfare improving. In other words, we seek a sufficient condition under which $U(\tau) > U(0)$ for small τ .

Since

$$U(\tau) = Eu((1 - \tau)w\tilde{R} + \tau w\tilde{G}) = Eu(w\tilde{R} + \tau w(\tilde{G} - \tilde{R})), \quad (13)$$

we can define $\tilde{\varepsilon} = w(\tilde{G} - \tilde{R})$, $\tilde{y} = w\tilde{R}$, and have

$$U(\tau) = Eu(\tau\tilde{\varepsilon} + \tilde{y}). \quad (14)$$

Under the assumption that $u(c) = \ln c$ and that \tilde{G} and \tilde{R} are jointly lognormal, Krueger and Kubler (2006) show that (i) the introduction of a marginal social security system is welfare improving if $E[\tilde{G}] > E[\tilde{R}]$; (ii) But even if $E[\tilde{G}] < E[\tilde{R}]$, the introduction of social security may still be justified if the stochastic saving returns are very volatile ($var(\tilde{R})$ big) or the correlation between private saving returns and returns to social security is small. From Proposition 4.1 below we obtain that the introduction of a marginal social security system is welfare improving if (i) $E[\tilde{G}] > E[\tilde{R}]$, and $\tilde{G} - \tilde{R}$ and \tilde{R} are independent; or (ii) $E[\tilde{G}] > E[\tilde{R}]$. and $\tilde{G} - \tilde{R}$ is negative expectation dependent on \tilde{R} . But even if $E[\tilde{G}] < E[\tilde{R}]$, the introduction of social security may still be justified if $\tilde{G} - \tilde{R}$ is negative expectation dependent on \tilde{R} , and $|E[\tilde{G} - \tilde{R}]|$ is very small. Therefore, for very general utility functions and probability distributions, our result provides more general conditions for welfare improving.

Indeed from Propositions 3.5 and 3.6, we obtain the following results:

Proposition 4.1 *Suppose the agent is risk averse.*

(i) *Let $E[\tilde{G}] > E[\tilde{R}]$. If $\tilde{G} - \tilde{R}$ and \tilde{R} are independent, then for a sufficiently small $\tau > 0$, $U(\tau) > U(0)$.*

(ii) *Let $E[\tilde{G}] > E[\tilde{R}]$. If $\tilde{G} - \tilde{R}$ is negative expectation dependent on \tilde{R} , then for a sufficiently small $\tau > 0$, $U(\tau) > U(0)$.*

(iii) *Let $E[\tilde{G}] > E[\tilde{R}]$. If $\tilde{G} - \tilde{R}$ is positive expectation dependent on \tilde{R} , and $E[\tilde{G} - \tilde{R}]$ is small enough, then for a sufficiently small $\tau > 0$, $U(0) > U(\tau)$.*

(iv) *Let $E[\tilde{G}] < E[\tilde{R}]$. If $\tilde{G} - \tilde{R}$ and \tilde{R} are independent, then for a sufficiently small $k > 0$, $U(0) > U(\tau)$.*

(v) *Let $E[\tilde{G}] < E[\tilde{R}]$. If $\tilde{G} - \tilde{R}$ is positive expectation dependent on \tilde{R} , then for a sufficiently small $k > 0$, $U(0) > U(\tau)$.*

(vi) *Let $E[\tilde{G}] < E[\tilde{R}]$. If $\tilde{G} - \tilde{R}$ is negative expectation dependent on \tilde{R} , and if $|E[\tilde{G} - \tilde{R}]|$ is small enough, then for a sufficiently small $\tau > 0$, $U(\tau) > U(0)$.*

4.2 The Arrow-Lind theorem (1970) revisited

Arrow and Lind (1970) propose that, for risky public projects, under certain conditions, the social cost of the risk tends to zero as the population tends to infinite, so we should neglect the risk premium. Consider the case where all individuals have the same preferences $U(\cdot)$, and their disposable incomes are identically distributed random variables represented by \tilde{A} . Suppose that the government undertakes an investment with returns represented by \tilde{B} , which are independent of \tilde{A} . Let $\bar{B} = E\tilde{B}$. Consider a specific taxpayer and denote his fraction of this investment by s with $0 \leq s \leq 1$. Suppose that each taxpayer has the same tax rate and that there are n taxpayers, then $s = \frac{1}{n}$. Arrow and Lind (1970) show that

$$EU\left(\tilde{A} + \frac{\tilde{B}}{n} + r(n)\right) = EU\left(\tilde{A} + \frac{\tilde{B}}{n}\right), \quad (15)$$

where $r(n)$ is the risk premium of the representative individual. They demonstrate that not only does $r(n)$ vanish when $n \rightarrow \infty$, but so does the total of the risk premiums for all individuals: $nr(n)$ approaches zero as n rises. This result implies that the total cost of risk-bearing ($nr(n)$) goes to zero as the population of taxpayers increases and the expected value of net benefit (\bar{B}) closely approximates the correct measure of net benefits in terms of willingness to pay.

Arrow and Lind (1970) conclude that, *by spreading the risk across large number of people, the total cost of risk-bearing is insignificant. Therefore, the government should ignore uncertainty*

in evaluating public investments. Similarly, the choice of the rate of discount should in the case be independent of considerations of risk. This theorem has been a point of reference in the discussion about the efficiency of public vs. private financing and risk bearing. Foldes and Rees (1977) state that: *this theorem, if generally applicable, would have important practical consequences. It would tend to support an extension of public sector investment by justifying the use of a riskless discount rate applied to expected returns. It would also argue in favor of state participation in private investments where this allows risks to be spread over a larger number of persons.*

However, Arrow-Lind theorem depends on the assumption of independence between \tilde{A} and \tilde{B} . Laffont (1989, p54) points that if the return of the public is dependent on the income risk, Arrow and Lind's result can be invalidated. Magill (1984) shows that the total cost of risk-bearing is positive (negative) if income is positively (negatively) quadrant dependent on the return from the public project.

We now apply our result to the cases where \tilde{A} and \tilde{B} can be dependent. Since (15) can be rewritten as

$$EU(\tilde{A} + \frac{\tilde{B}}{n} + r(n)) = EU(\tilde{A} + \frac{\tilde{B}}{n}), \quad (16)$$

from Proposition 3.5, we obtain:

Proposition 4.2 *Suppose the representative individual is risk averse.*

(i) *If \tilde{B} and \tilde{A} are independent, then the representative individual's attitude towards risk is second-order conditional dependent risk aversion and $r(n) = |O(\frac{1}{n^2})|$*

(ii) *If \tilde{B} is positive expectation dependent on \tilde{A} , then the representative individual's attitude towards risk is first-order conditional dependent risk aversion and $r(n) = |O(\frac{1}{n})|$;*

(iii) *If \tilde{B} is negative expectation dependent on \tilde{A} , then the representative individual's attitude towards risk is first-order conditional dependent risk aversion and $r(n) = -|O(\frac{1}{n})|$.*

Therefore, when \tilde{A} and \tilde{B} are expectation dependent, $nr(n)$ cannot vanish as n increases. Proposition 4.2 shows that if the return on the investment and the disposable incomes are positive (negative) expectation dependent and the society is risk-averse, then, as the population of taxpayers increases, the total cost of risk-bearing will remain less (greater) than zero and the expected value of net benefit (\bar{B}) overestimates (underestimates) the correct measure of net benefits in terms of willingness to pay.

Proposition 4.2 implies that (i) by spreading the risk across large number of people, the total cost of risk-bearing is insignificant if the income risk and the return of public investment are independent. Therefore, the government should ignore uncertainty in evaluating public investments. The choice of the rate of discount should in the case be independent of considerations of risk; (ii) if the income risk and the return of public investment are expectation dependent, the total cost of risk-bearing is significant. Therefore, the government should not ignore uncertainty in evaluating public investments. Our result provides more exact conditions for when discount rate of public investment should be larger or smaller than the risk-free rate. Since expectation dependence is weaker than quadrant dependence (see Wright 1987), Proposition 4.2 provides a more general results than Magill (1984)'s.

4.3 The welfare cost of business cycles

We consider the following question: What is the effect on welfare of eliminating all consumption variability could be eliminated. Consider a single consumer, endowed with the stochastic consumption stream $\tilde{c} = \bar{c} + k\tilde{\varepsilon}$. Preferences over such consumption are assumed to be

$$Eu(\tilde{c} + \tilde{y}), \tag{17}$$

where \tilde{y} is income risk.

A risk-averse consumer would prefer a deterministic consumption to a risky one with the same mean under certain conditions. We quantify this utility difference by multiplying the risky consumption by the constant factor $1 + \lambda$. We choose λ so that the household is indifferent between the deterministic consumption and the compensated risky one. That is, λ^* solves

$$Eu((1 + \lambda^*)\tilde{c} + \tilde{y}) = Eu(E\tilde{c} + \tilde{y}).$$

Representative agent models with expected utility suggest welfare costs of business cycles are small (see Lucas 1987; 2003). Epstein-Zin recursive utility variations give bigger numbers (Pemberton 1996; Dolmas 1998; Epaulard and Pommeret 2003). Incomplete market models offer bigger numbers. Our result shows that dependent idiosyncratic risk would appear to be a key factor.

Lucas (1987; 2003) defines this compensation parameter λ^* as the welfare gain (or welfare loss) from eliminating consumption risk. When there is no uninsurable risk ($\tilde{y} \equiv E\tilde{y}$), Lucas (1987; 2003) argues that the welfare costs of business cycles are likely to be very small, and

that the potential gains from counter-cyclical stabilization policy are negligible. Therefore we should look for ways to attain higher growth rates rather than for economic policies to reduce fluctuation in consumption. However, this conclusion is in conflict with actual practice of short-term economic policies.

We now apply our results to this issue. Since for $\forall \lambda$, we have

$$\begin{aligned}
Eu((1 + \lambda)\tilde{c} + \tilde{y}) &= Eu((1 + \lambda)\bar{c} + (1 + \lambda)k\tilde{\varepsilon} + \tilde{y}) & (18) \\
&= Eu((1 + \lambda)\bar{c} + (1 + \lambda)(kE\tilde{\varepsilon} - \pi_{cd}(k)) + \tilde{y}) \\
&= Eu(E\tilde{c} + \lambda\bar{c} + \lambda kE\tilde{\varepsilon} - (1 + \lambda)\pi_{cd}(k) + \tilde{y}),
\end{aligned}$$

hence $\lambda^* = \frac{\pi_{cd}(k)}{\bar{c} + kE\tilde{\varepsilon} - \pi_{cd}(k)}$ and we obtain the following proposition:

Proposition 4.3 *Suppose the consumer is risk averse and k is small (i.e. $\bar{c} > O(k)$).*

(i) *If \tilde{c} and \tilde{y} are independent, then the consumer's attitude towards risk is second-order conditional dependent risk aversion and $\lambda^* = |O(k^2)|$;*

(ii) *If \tilde{c} is positive expectation dependent on \tilde{y} , then the consumer's attitude towards risk is first-order conditional dependent risk aversion and $\lambda^* = |O(k)|$;*

Proposition 4.3 states that, when risk associated to the annual change of consumption is small, (i) if consumption risk and income risk are independent, then the welfare costs of business cycles is very small, and therefore the potential gains from counter-cyclical stabilization policy are negligible; (ii) If consumption risk is positive expectation dependent on income risk, then the welfare gains of business cycles is not very small, and therefore the potential gains from counter-cyclical stabilization policy are significant.

4.4 The effect of introducing a background risk on equilibrium asset price

We consider a static Lucas (1978) “tree economy” which consists of individuals, all of whom may be portrayed by a “representative agent”. The economy is competitive in that individuals maximize expected utility with prices taken as given. Initial wealth consists of one unit of the risky asset plus an allocation of a risk-free asset. We assume the risk-free rate is zero. We denote w as the value of wealth that is initially invested in the risk-free asset and define \tilde{x} as the final value of the risky asset. Agents' preferences are representable by a von Neumann-Morgenstern utility function u . The agent can adjust her portfolio via buying and selling the two assets. Let

P represent the price of the risky asset and β denote the demand for additional units of \tilde{x} . We assume the agent faces the following optimization program:

$$\beta^* \in \arg \max_{\beta} Eu(w + \tilde{x} + \beta(\tilde{x} - P)). \quad (19)$$

Gollier and Schlesinger (2002) show that the equilibrium asset price (i.e. $\beta^* = 0$) is

$$P^* = \frac{E[\tilde{x}u'(w + \tilde{x})]}{Eu'(w + \tilde{x})}. \quad (20)$$

One way to explain the equity premium puzzle in the theoretical model is to recognize that there are other sources of risk on final wealth than the riskiness of assets returns. To capture the effects of these types of risks, we introduce a labor income risk, \tilde{y} , which can not be fully insured. This yields the following optimization program:

$$\beta^{**} \in \arg \max_{\beta} Eu(w + \tilde{x} + \beta(\tilde{x} - P) + \tilde{y}), \quad (21)$$

and modified equilibrium asset price:

$$P^{**} = \frac{E[\tilde{x}u'(w + \tilde{x} + \tilde{y})]}{Eu'(w + \tilde{x} + \tilde{y})}. \quad (22)$$

We want to compare P^{**} with P^* .

Proposition 4.4 *Define $\tilde{x} = \bar{x} + k\tilde{\varepsilon}$ with $E\tilde{\varepsilon} = 0$. Suppose $\tilde{x} > 0$ and $\tilde{y} > 0$ almost surely.*

- (i) *If $\tilde{\varepsilon}$ and \tilde{y} are independent, then $P^{**} - P^* = O(k^2)$;*
- (ii) *If $\tilde{\varepsilon}$ is positive expectation dependent on \tilde{y} , and relative prudence coefficient is larger than 2 ($-x \frac{u'''(x)}{u''(x)} \geq 2$ for $\forall x$), then $P^{**} - P^* = O(k)$.*

Proof See Appendix.

Propositions 4.4 shows how a dependent background risk affects equilibrium asset price and how it can be related to first order risk aversion and relative prudence.

The equity premium puzzle is a manifestation that second-order risk aversion and is not sufficient to explain asset prices. Epstein and Zin (1990) explain it with non-expected utility (presumably first-order risk aversion). Another alternative has explored incomplete market models with idiosyncratic risk. Works by Constantinidis and Duffie (1996) and Krueger and Lustig (2010) suggest that making idiosyncratic risk dependent on aggregate conditions (i.e. dependent background risk) is key to generate the right equity premium implications. Krueger

and Lustig (2010) show that independent background risk is not sufficient. Our result shows that independent idiosyncratic risk can only generate second-order risk aversion while expectation dependent idiosyncratic risk can generate first-order risk aversion, and hence offers a better understanding of these incomplete markets results, and how they relate to the non-standard preference models by dependent background risks.

5 Concluding remarks

The paper shows that differentiable expected utility can be compatible with first order risk behavior if there exists some background dependent risk that cannot be eliminated (eg, uncertain income).

6 Appendix: Proof of Lemma 3.4

From the definition of $\pi_{cd}(k)$, we know that

$$Eu(w + Ek\tilde{\varepsilon} - \pi_{cd}(k), \tilde{y}) = Eu(w + k\tilde{\varepsilon}, \tilde{y}). \quad (23)$$

Differentiating with respect to k yields

$$\pi'_{cd}(k) = \frac{E\tilde{\varepsilon}Eu(w + Ek\tilde{\varepsilon} - \pi_{cd}(k), \tilde{y}) - E[\tilde{\varepsilon}u_1(w + k\tilde{\varepsilon}, \tilde{y})]}{Eu_1(w - \pi_{cd}(k), \tilde{y})}. \quad (24)$$

Since $\pi_{cd}(0) = 0$, we have

$$\pi'_{cd}(0) = \frac{E\tilde{\varepsilon}Eu_1(w, \tilde{y}) - E[\tilde{\varepsilon}u_1(w, \tilde{y})]}{Eu_1(w, \tilde{y})}. \quad (25)$$

Note that

$$E[\tilde{\varepsilon}u_1(w, \tilde{y})] = E\tilde{\varepsilon}Eu_1(w, \tilde{y}) + Cov(\tilde{\varepsilon}, u_1(w, \tilde{y})) \quad (26)$$

and the covariance can always be written as (see Cuadras (2002), Theorem 1)

$$Cov(\tilde{\varepsilon}, u_1(w, \tilde{y})) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F(\varepsilon, y) - F_{\varepsilon}(\varepsilon)F_y(y)]d\varepsilon du_1(w, y). \quad (27)$$

Since we can always write (see e.g. Tesfatsion (1976), Lemma 1)

$$\int_{-\infty}^{\infty} [F_{\varepsilon}(\varepsilon|\tilde{y} \leq y) - F_{\varepsilon}(\varepsilon)]d\varepsilon = E\tilde{\varepsilon} - E(\tilde{\varepsilon}|\tilde{y} \leq y),$$

hence, by straightforward manipulations, we find

$$\begin{aligned}
Cov(\tilde{\varepsilon}, u_1(w, \tilde{y})) &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F(\varepsilon, y) - F_{\varepsilon}(\varepsilon)F_y(y)]u_{12}(w_0, y)d\varepsilon dy & (28) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [F_{\varepsilon}(\varepsilon|\tilde{y} \leq y) - F_{\varepsilon}(\varepsilon)]d\varepsilon F_y(y)u_{12}(w, y)dy \\
&= \int_{-\infty}^{\infty} [E\tilde{\varepsilon} - E(\tilde{\varepsilon}|\tilde{y} \leq y)]F_y(y)u_{12}(w, y)dy & \text{(by (??))} \\
&= \int_{-\infty}^{\infty} ED(y)u_{12}(w, y)F_y(y)dy.
\end{aligned}$$

Finally, we get

$$\pi'_{cd}(0) = -\frac{\int_{-\infty}^{\infty} ED(y)u_{12}(w, y)F_y(y)dy}{Eu_1(w, \tilde{y})}. \quad (29)$$

Using a Taylor expansion of π around $k = 0$, we obtain that

$$\pi_{cd}(k) = \pi_{cd}(0) + \pi'_{cd}(0)k + O(k^2) = -k\frac{\int_{-\infty}^{\infty} ED(y)u_{12}(w, y)F_y(y)dy}{Eu_1(w, \tilde{y})} + O(k^2). \quad (30)$$

Q.E.D.

7 Appendix: Proof of Proposition 3.6

(i) Suppose the decision maker's attitude towards risk is second-order conditional dependent risk aversion, then $\pi_{cd}(k) - \pi_c(k) = O(k^2)$. >From (6), we have

$$\begin{aligned}
&Eu(w + k\tilde{\varepsilon}, \tilde{y}) & (31) \\
&= Eu(w + Ek\tilde{\varepsilon} - \pi_{cd}(k), \tilde{y}) \\
&= Eu(w + kE\tilde{\varepsilon} - O(k^2) - \pi_c(k), \tilde{y}) \\
&= Eu(w + kE\tilde{\varepsilon} - O(k^2), \tilde{y}) & \text{by (5)} \\
&> Eu(w, \tilde{y}) & \text{for a sufficiently small } k > 0.
\end{aligned}$$

(ii) Suppose the decision maker's attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) - \pi_c(k) = -|O(k)|$. >From (6), we have

$$\begin{aligned}
&Eu(w + k\tilde{\varepsilon}, \tilde{y}) & (32) \\
&= Eu(w + Ek\tilde{\varepsilon} - \pi_{cd}(k), \tilde{y}) \\
&= Eu(w + kE\tilde{\varepsilon} + |O(k)| - \pi_c(k), \tilde{y}) \\
&= Eu(w + kE\tilde{\varepsilon} + |O(k)| - O(k^2), \tilde{y}) & \text{by (5)} \\
&> Eu(w, \tilde{y}) & \text{for sufficiently small } k > 0.
\end{aligned}$$

(iii) Suppose the decision maker's attitude towards risk is first-order conditional dependent risk aversion and $\pi_{cd}(k) - \pi_c(k) = |O(k)|$. From (6), we have

$$\begin{aligned}
& Eu(w + k\tilde{\varepsilon}, \tilde{y}) \tag{33} \\
&= Eu(w + Ek\tilde{\varepsilon} - \pi_{cd}(k), \tilde{y}) \\
&= Eu(w + kE\tilde{\varepsilon} - |O(k)| - \pi_c(k), \tilde{y}) \\
&= Eu(w + kE\tilde{\varepsilon} - |O(k)| - O(k^2), \tilde{y}) \quad \text{by (5)} \\
&< Eu(w, \tilde{y}) \quad \text{for sufficiently small } k > 0 \text{ and sufficiently small } E[\tilde{\varepsilon}].
\end{aligned}$$

(iv), (v) and (vi): We can prove them using the same approach as in (i), (ii) and (iii).
Q.E.D.

8 Appendix: Proof of Proposition 4.4

(i) Using a Taylor expansion of π around $k = 0$, we obtain that

$$\begin{aligned}
& u'(w + \bar{x} + k\tilde{\varepsilon}) \tag{34} \\
&= u'(w + \bar{x}) + \tilde{\varepsilon}u''(w + \bar{x})k + O(k^2),
\end{aligned}$$

$$\begin{aligned}
& (\bar{x} + k\tilde{\varepsilon})u'(w + \bar{x} + k\tilde{\varepsilon}) \tag{35} \\
&= \bar{x}u'(w + \bar{x}) + [\tilde{\varepsilon}u'(w + \bar{x}) + (\bar{x} + k\tilde{\varepsilon})\tilde{\varepsilon}u''(w + \bar{x})]k + O(k^2) \\
&= \bar{x}u'(w + \bar{x}) + \tilde{\varepsilon}u'(w + \bar{x})k + \bar{x}\tilde{\varepsilon}u''(w + \bar{x})k + \tilde{\varepsilon}^2u''(w + \bar{x})k^2 + O(k^2) \\
&= \bar{x}u'(w + \bar{x}) + \tilde{\varepsilon}u'(w + \bar{x})k + \bar{x}\tilde{\varepsilon}u''(w + \bar{x})k + O(k^2),
\end{aligned}$$

$$\begin{aligned}
& u'(w + \bar{x} + k\tilde{\varepsilon} + \tilde{y}) \tag{36} \\
&= u'(w + \bar{x} + \tilde{y}) + \tilde{\varepsilon}u''(w + \bar{x} + \tilde{y})k + O(k^2)
\end{aligned}$$

and

$$\begin{aligned}
& (\bar{x} + k\tilde{\varepsilon})u'(w + \bar{x} + k\tilde{\varepsilon} + \tilde{y}) \tag{37} \\
&= \bar{x}u'(w + \bar{x} + \tilde{y}) + [\tilde{\varepsilon}u'(w + \bar{x} + \tilde{y}) + (\bar{x} + k\tilde{\varepsilon})\tilde{\varepsilon}u''(w + \bar{x} + \tilde{y})]k + O(k^2) \\
&= \bar{x}u'(w + \bar{x} + \tilde{y}) + \tilde{\varepsilon}u'(w + \bar{x} + \tilde{y})k + \bar{x}\tilde{\varepsilon}u''(w + \bar{x} + \tilde{y})k + \tilde{\varepsilon}^2u''(w + \bar{x} + \tilde{y})k^2 + O(k^2) \\
&= \bar{x}u'(w + \bar{x} + \tilde{y}) + \tilde{\varepsilon}u'(w + \bar{x} + \tilde{y})k + \bar{x}\tilde{\varepsilon}u''(w + \bar{x} + \tilde{y})k + O(k^2).
\end{aligned}$$

Since $\tilde{\varepsilon}$ and \tilde{y} are independent, we have

$$Eu'(w + \bar{x} + k\tilde{\varepsilon}) = u'(w + \bar{x}) + O(k^2), \quad (38)$$

$$E(\bar{x} + k\tilde{\varepsilon})u'(w + \bar{x} + k\tilde{\varepsilon}) = \bar{x}u'(w + \bar{x}) + O(k^2), \quad (39)$$

$$Eu'(w + \bar{x} + k\tilde{\varepsilon} + \tilde{y}) = Eu'(w + \bar{x} + \tilde{y}) + O(k^2) \quad (40)$$

and

$$E(\bar{x} + k\tilde{\varepsilon})u'(w + \bar{x} + k\tilde{\varepsilon} + \tilde{y}) = \bar{x}Eu'(w + \bar{x} + \tilde{y}) + O(k^2). \quad (41)$$

Hence,

$$\begin{aligned} & P^{**} - P^* \quad (42) \\ &= \frac{E[\tilde{x}u'(w + \tilde{x} + \tilde{y})]}{Eu'(w + \tilde{x} + \tilde{y})} - \frac{E[\tilde{x}u'(w + \tilde{x})]}{Eu'(w + \tilde{x})} \\ &= \frac{\bar{x}Eu'(w + \bar{x} + \tilde{y}) + O(k^2)}{Eu'(w + \bar{x} + \tilde{y}) + O(k^2)} - \frac{\bar{x}u'(w + \bar{x}) + O(k^2)}{u'(w + \bar{x}) + O(k^2)} = O(k^2) \end{aligned}$$

(ii) From (36), we have

$$\begin{aligned} & Eu'(w + \bar{x} + k\tilde{\varepsilon} + \tilde{y}) \quad (43) \\ &= Eu'(w + \bar{x} + \tilde{y}) + E\tilde{\varepsilon}u''(w + \bar{x} + \tilde{y})k + O(k^2) \\ &= Eu'(w + \bar{x} + \tilde{y}) + \text{cov}(\tilde{\varepsilon}, u''(w + \bar{x} + \tilde{y}))k + O(k^2) \\ &= Eu'(w + \bar{x} + \tilde{y}) + k \int_{-\infty}^{\infty} ED(y)u'''(w + \bar{x} + y)F_y(y)dy + O(k^2) \\ &= Eu'(w + \bar{x} + \tilde{y}) + |O(k)|. \end{aligned}$$

From (37), we have

$$\begin{aligned} & E(\bar{x} + k\tilde{\varepsilon})u'(w + \bar{x} + k\tilde{\varepsilon} + \tilde{y}) \quad (44) \\ &= \bar{x}Eu'(w + \bar{x} + \tilde{y}) + E\tilde{\varepsilon}u'(w + \bar{x} + \tilde{y})k + E\tilde{\varepsilon}u''(w + \bar{x} + \tilde{y})\bar{x}k + O(k^2) \\ &= \bar{x}Eu'(w + \bar{x} + \tilde{y}) + \text{cov}(\tilde{\varepsilon}, u'(w + \bar{x} + \tilde{y}))k + \text{cov}(\tilde{\varepsilon}, u''(w + \bar{x} + \tilde{y}))\bar{x}k + O(k^2) \\ &= \bar{x}Eu'(w + \bar{x} + \tilde{y}) + k \int_{-\infty}^{\infty} ED(y)u''(w + \bar{x} + y)F_y(y)dy + \bar{x}k \int_{-\infty}^{\infty} ED(y)u'''(w + \bar{x} + y)F_y(y)dy \\ & \quad + O(k^2) \\ &= \bar{x}Eu'(w + \bar{x} + \tilde{y}) + k \int_{-\infty}^{\infty} ED(y)[u''(w + \bar{x} + y) + \bar{x}u'''(w + \bar{x} + y)]F_y(y)dy + O(k^2). \end{aligned}$$

Since

$$\begin{aligned}
& -x \frac{u'''(x)}{u''(x)} \geq 2 \text{ for } \forall x > 0 \quad (\text{see Chiu et al., 2012, Lemma 2}) \quad (45) \\
\Leftrightarrow & -x \frac{u'''(x+y)}{u''(x+y)} \geq 2 \text{ for } \forall x > 0 \text{ and } y \geq 0 \\
\Rightarrow & u''(w + \bar{x} + y) + \bar{x}u'''(w + \bar{x} + y) > 0,
\end{aligned}$$

we obtain

$$E(\bar{x} + k\tilde{\varepsilon})u'(w + \bar{x} + k\tilde{\varepsilon} + \tilde{y}) = \bar{x}Eu'(w + \bar{x} + \tilde{y}) + |O(k)|. \quad (46)$$

Therefore,

$$\begin{aligned}
& P^{**} - P^* \quad (47) \\
= & \frac{E[\tilde{x}u'(w + \tilde{x} + \tilde{y})]}{Eu'(w + \tilde{x} + \tilde{y})} - \frac{E[\tilde{x}u'(w + \tilde{x})]}{Eu'(w + \tilde{x})} \\
= & \frac{\bar{x}Eu'(w + \bar{x} + \tilde{y}) + |O(k)|}{Eu'(w + \bar{x} + \tilde{y}) + |O(k)|} - \frac{\bar{x}u'(w + \bar{x}) + O(k^2)}{u'(w + \bar{x}) + O(k^2)} = O(k).
\end{aligned}$$

Q.E.D.

9 References

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