Higher-Order Risk Attitudes toward Correlation

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Covariance

- $\tilde{x}$ and $\tilde{y}$ are valued in the intervals $[a, b]$ and $[c, d]$.

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$$Cov(\tilde{x}, \tilde{y}) = E\tilde{x}\tilde{y} - E\tilde{x}E\tilde{y}. \quad (1)$$

- $E\tilde{x}\tilde{y} \geq E\tilde{x}E\tilde{y}$ if and only if $\tilde{x}$ and $\tilde{y}$ co-vary positively.
**Equity premium**

- $u$: the bivariate utility of the representative agent
- $\bar{x}$: the GDP per capita
- $\bar{y}$: the background risk
- Equity premium, $\varphi$:

$$\varphi = \frac{E\bar{x}Eu^{(1,0)}(\bar{x}, \bar{y})}{E\bar{x}u^{(1,0)}(\bar{x}, \bar{y})} - 1. \quad (2)$$

- $u^{(k_1, k_2)}$: the $(k_1, k_2)$th cross derivative of $u$, i.e.,

$$u^{(k_1, k_2)} = \frac{\partial^{k_1+k_2}}{\partial x^{k_1} \partial y^{k_2}} u(x, y)$$
Signing equity premium

- Full information: e.g., $u(x, y) = \log(x + y)$ and $(\tilde{x}, \tilde{y})$ is joint-normal distributed $\Rightarrow sign(\varphi)$

- Partial information: e.g., risk aversion and $(\tilde{x}, \tilde{y})$ is affiliated $\Rightarrow sign(\varphi)$ ?

- Sign the covariance between functions
Covariance between monotonic functions

Definition (Esary et al. 1967, p1466) $(\tilde{x}, \tilde{y})$ is said to be associated if for all functions $\alpha, \beta$ which are increasing in each component,

$$\text{Cov}(\alpha(\tilde{x}, \tilde{y}), \beta(\tilde{x}, \tilde{y})) \geq 0.$$ (3)

Risk averse in $x$ ($u^{(2,0)} < 0$), correlation averse ($u^{(1,1)} \leq 0$) and $(\tilde{x}, \tilde{y})$ is associated $\Rightarrow \varphi \geq 0$
Higher-order risk attitudes

- Higher-order risk attitudes (e.g., prudence and temperance)\(\Leftrightarrow\) signing the higher-order derivative of the utility function (expected utility framework)

- Weaker dependence structures of \((\tilde{x}, \tilde{y})\)
Definition Define $F^1(y|x) = F(y|x)$ and $F^{n+1}(y|x) = \int_{c}^{y} F^n(t|x)dt$. We say that $\tilde{y}$ is $N^{th}$-order stochastic dominance dependent on $x$ ($N^{th}$ SDD($\tilde{y}|x$)) if

(i) $F^N(y|x') \leq F^N(y|x)$ for all $y$ and $x' \geq x$;

(ii) $F^n(d|x') \leq F^n(d|x)$ for all $y$, $x' \geq x$ and $n = 1, \ldots, N - 1$.

Eeckhoudt and Kimball (1992): Third-order stochastic dominance dependence
Covariance between functions beyond monotonicity

- **Definition** (Denuit et al. 1999) The class $\mathcal{U}_{(s_1,s_2)-icv}$ of the regular $(s_1, s_2)$-increasing concave functions defined as the class of all the functions $u$, for $s_1$ and $s_2$ are positive integers, such that $(-1)^{k_1+k_2+1}u^{(k_1,k_2)} \geq 0$ for all $k_1 = 0, 1, ..., s_1$, $k_2 = 0, 1, ..., s_2$ with $k_1 + k_2 \geq 1$.

- **Proposition**
  The following statements are equivalent.
  (i) 
  $$\text{Cov}(\alpha(\tilde{x}, \tilde{y}), \beta(\tilde{x}, \tilde{y})) \geq 0$$
  (4)
  for all $\alpha$ and $\beta$ such that $\alpha^{(1,0)} \geq 0$, $\beta^{(1,0)} \geq 0$, $\alpha \in \mathcal{U}_{(0,I)-icv}$ and $\beta \in \mathcal{U}_{(0,J)-icv}$;
  (ii) $N^{th} \text{SDD}(\tilde{y}|x)$ where $N = \min(I, J)$.  

Various concepts of bivariate dependence

\[(\tilde{x}, \tilde{y})\] is associated \hspace{1cm} (5)

\[\Rightarrow N^{th} SDD(\tilde{y} | x)\]

\[\Rightarrow \text{positive } N^{th} ED(\tilde{y} | x)\]

positive SED on \(\tilde{y}\) \(\Rightarrow (\tilde{x}, \tilde{y})\) is positive correlated \hspace{1cm} (6)
Risk aversion in the presence of another risk

► **Proposition** (Finkelshtain et al. 1999, part (a) and (b) of Theorem 2)

The following statements are equivalent.

(i) \( \text{Eu}(\tilde{x}, \tilde{y}) \leq \text{Eu}(E\tilde{x}, \tilde{y}) \) for \((\tilde{x}, \tilde{y})\) such that \(E(\tilde{x}|\tilde{y})\) is increasing in \(y\);

(ii) \( u^{(2,0)} \leq 0 \) and \( u^{(1,1)} \leq 0 \).
Risk aversion in the presence of another risk

**Proposition**

Suppose $N^{th}SDD(\tilde{y}|x)$, $u^{(2,0)} \leq 0$ and $-u^{(1,0)} \in \mathcal{U}_{(0,N)-icv}$, then

$$Eu(\tilde{x}, \tilde{y}) \leq Eu(E\tilde{x}, \tilde{y}). \quad (7)$$

If one of the following conditions is satisfied, then an agent is risk averse for $\tilde{x}$ in the presence of $\tilde{y}$:

(i) she is risk averse in $x$ ($u^{(2,0)} \leq 0$), correlation averse ($u^{(1,1)} \leq 0$) and $FSDD(\tilde{y}|x)$;

(ii) she is risk averse in $x$ ($u^{(2,0)} \leq 0$), correlation averse $u^{(1,1)} \leq 0$, cross-prudent in $x$ ($u^{(1,2)} \geq 0$) and $SSDD(\tilde{y}|x)$;

(iii) she is risk averse in $x$ ($u^{(2,0)} \leq 0$), correlation averse $u^{(1,1)} \leq 0$, cross-prudent and cross-temperate in $x$ ($u^{(1,2)} \geq 0$ and $u^{(1,3)} \leq 0$), and $TSDD(\tilde{y}|x)$. 
Application

\[
U(s) = u_0(x_0 - s, h_0) + \frac{1}{1 + \rho} Eu_1(s(1 + r) + \tilde{x}, \tilde{h})
\] (8)

\[
u_0^{(1,0)}(x_0 - s^*, h_0) = \frac{1 + r}{1 + \rho} Eu_1^{(1,0)}(s^*(1 + r) + \tilde{x}, \tilde{h})
\] (9)

\(s_{x,E}\): the solution with \((\tilde{x}, E\tilde{h})\) substituted for \((\tilde{x}, \tilde{h})\)
\(s_{E,h}\): the solution with \((E\tilde{x}, \tilde{h})\) substituted for \((\tilde{x}, \tilde{h})\).

Proposition

(i) \(u_0^{(2,0)} \leq 0, u_1^{(2,0)} \leq 0, u_1^{(1,2)} \geq 0, u_1^{(1,1)} \in \mathcal{U}_{(N,0)-icv}\) and 
\(N^{th} SDD(\tilde{x}|h) \Rightarrow s^* \geq s_{x,E}\);
(ii) \(u_0^{(2,0)} \leq 0, u_1^{(3,0)} \geq 0, u_1^{(2,0)} \in \mathcal{U}_{(0,N)-icv}\) and 
\(N^{th} SDD(\tilde{h}|x) \Rightarrow s^* \geq s_{E,h}\).
V(a) = u_0(x_0 - a, h_0) + \frac{1}{1 + \rho} Eu_1(\tilde{x}, \tilde{h} + a)  \quad (10)

The optimal amount of investment \( a^* \) is determined by

\[
u_0^{(1,0)}(x_0 - a^*, h_0) = \frac{m}{1 + \rho} Eu_1^{(0,1)}(\tilde{x}, \tilde{h} + ma^*) \quad (11)
\]

- \( a_{x,E} \): the solution with \((\tilde{x}, E\tilde{h})\) substituted for \((\tilde{x}, \tilde{h})\)
- \( a_{E,h} \): the solution with \((E\tilde{x}, \tilde{h})\) substituted for \((\tilde{x}, \tilde{h})\).

**Proposition**

(i) \( u_0^{(2,0)} \leq 0, \ u_1^{(0,3)} \geq 0, \ u_1^{(0,2)} \in \mathcal{U}_{(N,0) - icv} \) and \( N^{th} SDD(\tilde{x}|h) \)  
\( \Rightarrow \ a^* \geq a_{x,E}; \)

(ii) \( u_0^{(2,0)} \leq 0, \ u_1^{(0,2)} \leq 0, \ u_1^{(2,1)} \geq 0, \ u_1^{(1,1)} \in \mathcal{U}_{(0,N) - icv} \) and 
\( N^{th} SDD(\tilde{h}|x) \Rightarrow \ a^* \geq a_{E,h}. \)
A class of bivariate stochastic orderings

- $(\tilde{x}_1, \tilde{y}_1)$ and $(\tilde{x}_2, \tilde{y}_2)$ are two 2-dimensional random vectors with density functions $f$ and $g$

\[
Eu(\tilde{x}_1, \tilde{y}_1) = \int_a^b \int_c^d u(x, y) f(x, y) \, dx \, dy
\]

\[
= \int_a^b \int_c^d u(x, y) \frac{f(x, y)}{g(x, y)} g(x, y) \, dx \, dy
\]

\[
= E[u(\tilde{x}_2, \tilde{y}_2)] \frac{f(\tilde{x}_2, \tilde{y}_2)}{g(\tilde{x}_2, \tilde{y}_2)}
\]

\[
= E[u(\tilde{x}_2, \tilde{y}_2)] + Cov[u(\tilde{x}_2, \tilde{y}_2), \frac{f(\tilde{x}_2, \tilde{y}_2)}{g(\tilde{x}_2, \tilde{y}_2)}],
\]

If $(\tilde{x}_2, \tilde{y}_2)$ is associated and $\frac{f}{g}$ is increasing in $x$ and $y$, then $Eu(\tilde{x}_1, \tilde{y}_1) \geq E[u(\tilde{x}_2, \tilde{y}_2)]$ (see e.g., Shaked and Shanthikumar 2007, Theorem 6.B.8).
A class of bivariate stochastic orderings

**Proposition**

The following statements are equivalent.

(i) \( Eu(\tilde{x}_1, \tilde{y}_1) \geq E[u(\tilde{x}_2, \tilde{y}_2)] \) for all \( u, f \) and \( g \) such that \( u^{(1,0)} \geq 0, \left( \frac{f}{g} \right)^{(1,0)} \geq 0, u \in \mathcal{U}_{(0,I)-icv}, \left( \frac{f}{g} \right) \in \mathcal{U}_{(0,J)-icv} \);

(ii) \( N^{th} SDD(\tilde{y}\mid x) \) where \( N = \min(I, J) \).

When \( I = J = 2 \), \( Eu(\tilde{x}_1, \tilde{y}_1) \geq E[u(\tilde{x}_2, \tilde{y}_2)] \) if \( u \) is monotonic \( (u^{(1,0)} \geq 0 \text{ and } u^{(0,1)} \geq 0) \) and risk averse in \( y \) \( (u^{(0,2)} \leq 0) \), \( \frac{f}{g} \) is increasing in \( x \) and \( y \), concave in \( y \), and \( SDD(\tilde{y}\mid x) \).
Application

\[ U(s) = u_0(x_0 - s, h_0) + \frac{1}{1 + \rho} Eu_1(s(1 + r) + \tilde{x}, \tilde{h}) \quad (13) \]

\[ u_0^{(1,0)}(x_0 - s^*, h_0) = \frac{1 + r}{1 + \rho} Eu_1^{(1,0)}(s^*(1 + r) + \tilde{x}, \tilde{h}) \quad (14) \]

\( s' \): the solution with \((\tilde{x}', \tilde{h}')\) substituted for \((\tilde{x}, \tilde{h})\)

**Proposition** \( u_0^{(2,0)} \leq 0, u_1^{(2,0)} \leq 0, (\frac{f}{g})^{(1,0)} \geq 0, \)

\(-u_1^{(1,0)} \in \mathcal{U}_{(0,I) - icv}, (\frac{f}{g}) \in \mathcal{U}_{(0,J) - icv} \) and \( N^{th} \text{SDD}(\tilde{h}'|x') \)

where \( N = \min(I, J) \Rightarrow s^* \leq s' \);
Application

\[ V(a) = u_0(x_0 - a, h_0) + \frac{1}{1 + \rho} Eu_1(\tilde{x}, \tilde{h} + a) \quad (15) \]

The optimal amount of investment \( a^* \) is determined by

\[ u_0^{(1,0)}(x_0 - a^*, h_0) = \frac{m}{1 + \rho} Eu_1^{(0,1)}(\tilde{x}, \tilde{h} + ma^*) \quad (16) \]

\( a \): the solution of (16) with \((\tilde{x}', \tilde{h}')\) substituted for \((\tilde{x}, \tilde{h})\)

\textbf{Proposition} \( u_0^{(2,0)} \leq 0, u_1^{(0,2)} \leq 0, u_1^{(1,1)} \leq 0, (\frac{f}{g})^{(1,0)} \geq 0, -u_1^{(0,1)} \in \mathcal{U}_{(0,1) - icv}, (\frac{f}{g}) \in \mathcal{U}_{(0,J) - icv} \) and \( N^{th} SDD(\tilde{h}'|x') \Rightarrow a^* \leq a' \).
Justify the first-order approach to bi-signal principal-agent problems

\[ U(a) = \int_a^b \int_c^d u(s(x, y)) f(x, y | a) dx dy - a \quad (17) \]

- \( U(a) \) is concave \( \Rightarrow \) First-order-approach (FOA)

- Monotone likelihood ratio condition (MLRC) and the concavity of the distribution function condition (CDFC) (Rogerson, 1985) \( \Rightarrow \) \( U(a) \) is concave

- Most of the distribution functions do not have the CDFC property
Justify the first-order approach to bi-signal principal-agent problems

- Jewitt: Define $H(x, y) = u(s(x, y))$.

$$\frac{d^2}{da^2} U(a) = -\text{Cov}(H(\tilde{x}, \tilde{y})), -\frac{f_{aa}(\tilde{x}, \tilde{y} | a)}{f(\tilde{x}, \tilde{y} | a)}, \quad (18)$$

- $(\tilde{x}, \tilde{y})$ is affiliated, $H(x, y)$ and $-\frac{f_{aa}(x, y | a)}{f(x, y | a)}$ are increasing functions $\Rightarrow \frac{d^2}{da^2} U(a) \leq 0$
Proposition: The following statements are equivalent.

(i) \[ \frac{d^2}{da^2} U(a) \leq 0 \] (19)

for all \( H(x, y) \) and \( f(x, y | a) \) such that \( H^{(1,0)} \geq 0 \) \( (\frac{f_{aa}}{f})^{(1,0)} \leq 0 \), \( H \in \mathcal{U}(0, I)_{icv} \) and \( -\frac{f_{aa}}{f} \in \mathcal{U}(0, J)_{icv} \);

(ii) \( N^{th} SDD(\tilde{y} | x) \) where \( N = \min(I, J) \).