Optimal Risk Sharing in the Presence of Moral Hazard under Market Risk and Jump Risk

Takashi MISUMI, Hisashi NAKAMURA
and Koichiro TAKAOKA

Hitotsubashi University

ARIA
August 6, 2013
Motivation

- **IT** is well known that moral hazard is important in economics and finance.
  - Lots of studies in the literature on economic theory (e.g., Holmström (1979), Mas-Colell et al. (1995), and many others).

- **Surprisingly**, however, there has been a big gap in the research of moral hazard between the theory and the practice in financial engineering.
  - E.g., fixed-income investment, the term structure of interest rates, corporate risk management.

- **Question**: How much compensation should an investor pay to a hedge-fund manager?
  - The manager may be lazy. However, very hard for the investor to verify the manager’s high-tech activities.
Purpose and results

**Purpose:**
- Bridges the gap by providing a tractable framework to study optimal risk sharing between the investor and the manager with general utility forms in the presence of moral hazard under diffusive risk and jump risk.
  - Diffusive risk: Regular market risk.
  - Jump risk: Rare-event risk.

**Results:**
- Shows that, for a two-date discrete-time moral hazard model, there exists a continuous-time model that obtains the same optimal result.
- Moreover, characterizes explicitly the optimal risk sharing, in particular, the structural effect of the jump risk on the optimal allocations.
- Larger allocation of the firm to provide an incentive to take the optimal effort.
Relationships to previous literatures

The gap between theory and practice:

1. Physical environments are often too naive for practical applications to finance.
   - Continuous-time: Exponential utility (Holmström and Milgrom (1987), Schättler and Sung (1993)); diffusive shock (Cvitanić and Zhang (2007), Nakamura and Takaoka (2013)).

2. The agent controls the drift rate while continuing to forget how he has controlled it until then (weak formulation).
Most close to Cvitanić and Zhang (2007) and Nakamura and Takaoka (2013).

Similarities to them:
1. 2 risk-averse players: 1 investor and 1 firm.
2. Continuous time $[0, T], \ T > 0$.
3. The firm produces wealth with an effort cost in the presence of moral hazard and shares the outcome with the investor at $T$. 
Departures of this paper

- **DEPARTURES:**
  - Two types of risk: not only Brownian motions as market risk but also Poisson processes as jump risk.
  - Correspondingly, need further modifications:
    - The firm controls directly the probability measure, rather than the drift rate.
      - In the spirit of standard discrete-time moral hazard models (e.g., Holmström (1979), Mas-Colell et al. (1995)).
    - The effort cost is characterized by relative entropy.
      - A measure of statistical discrimination between the reference (i.e., original) measure $\mathbb{P}$ and the controlled probability measure $\mathbb{Q}$.
Outline of this presentation

1. Model setup
2. Optimal risk sharing in the presence of moral hazard
3. Characterization
4. Concluding remarks
Set-up

- 2 risk-averse players: 1 investor and 1 firm.
- Continuous time $[0, T]$, $T > 0$.
- Filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$.
  - $\{B_1(t), \cdots, B_n(t)\}_{0 \leq t \leq T}$: $n$ independent 1-dim standard $\mathbb{F}$-Brownian motions (BM).
  - $\{N_1(t), \cdots, N_m(t)\}_{0 \leq t \leq T}$: $m$ independent Poisson processes w/ its intensity $\lambda_i > 0$ ($i = 1, \cdots, m$). Indep of the $n$ BMs.
    - Compensated Poisson process $M_i(t) := N_i(t) - \lambda_i t$.
  - $\mathbb{F}$ is generated by the $n$ BMs and the $m$ Poisson processes.
- The firm produces wealth with an effort cost in the presence of moral hazard and shares the outcome with the investor at $T$. 
• The firm produces the wealth $X$ characterized by SDEs:

$$
\begin{align*}
dX(t) &= X(t-) \, dR(t), \quad X(0) = x > 0 \\
dR(t) &= c \, dt + \sum_{j=1}^{n} \sigma_j \, dB_j(t) + \sum_{i=1}^{m} z_i \, dM_i(t), \quad R(0) = a \in \mathbb{R},
\end{align*}
$$

where $c, \sigma_j, z_i \forall i, j$ are constants, $\sigma_j > 0 \forall j$, $z_i > -1 \forall i$, and $z_{i_1} \neq z_{i_2}$ if $i_1 \neq i_2$.

- $\{B_j; j = 1, \cdots, n\}$ stands for market risk
- $\{N_i; i = 1, \cdots, m\}$ stands for rare-event (i.e., jump) risk.
- For each $i = 1, \cdots, m$, $z_i$ denotes the size of the jump.
Set-up (cont.)

- **The** firm can control the probability measure.
  - In the spirit of the standard moral hazard literature in economics (Holmström (1979), Mas-Colell et al. (1995))
  - Changes the probability measure from the original (reference) $P$ into $Q$ such that $Q \ll P$.
  - Effort cost: separable cost characterized by relative entropy:
    
    \[
    \mathcal{H}(Q \Vert P) := E_P \left[ \frac{dQ}{dP} \left( \log \frac{dQ}{dP} \right) 1_{\{ \frac{dQ}{dP} > 0 \}} \right] = E_Q \left[ \left( \log \frac{dQ}{dP} \right) 1_{\{ \frac{dQ}{dP} > 0 \}} \right] = E_Q \left[ \log \frac{dQ}{dP} \right].
    \]

  - Assume that $\mathcal{H}(Q \Vert P) < \infty$. 

Example of relative entropy

- Consider the case of finite scenarios: $\Omega = \{\omega_1, \cdots, \omega_l\}$.
- Under $\mathbb{P}$, a r.v. $Y$: its realizations $\{y_1, \cdots, y_l\}$ with the probabilities $\{p_1, \cdots, p_l\}$ where $p_s > 0 \ \forall \ s$ and $\sum_{s=1}^l p_s = 1$.
- The new probabilities are $\{q_1, \cdots, q_l\}$ for $\{y_1, \cdots, y_l\}$ where $q_s \geq 0 \ \forall \ s$ and $\sum_{s=1}^l q_s = 1$.
- Then $\mathcal{H}(Q \mid \mid \mathbb{P}) = \sum_{s=1}^l p_s \frac{q_s}{p_s} \log \frac{q_s}{p_s} = \sum_{s=1}^l q_s \log \frac{q_s}{p_s}$.
  - When $\{q_1, \cdots, q_l\}$ are more distant from $\{p_1, \cdots, p_l\}$, $\mathcal{H}(Q \mid \mid \mathbb{P})$ becomes larger.
  - For example, $\{p_1, p_2, p_3\} = \{\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\}$ and $\{q_1, q_2, q_3\} = \{\frac{2}{3}, \frac{1}{3}, 0\}$.
  - Then $dQ/d\mathbb{P} = \{2, 1, 0\}$ and $d\mathbb{P}/dQ = \{\frac{1}{2}, 1, \text{n.a.}\}$. Thus $\mathbb{E}^Q[d\mathbb{P}/dQ] < 1$. 

Example of relative entropy
Set-up (cont.)

- A measure $\mathcal{Q}$ is absolutely continuous w.r.t. $\mathcal{P}$, written as $\mathcal{Q} \ll \mathcal{P}$, i.e., $\mathcal{P}(A) = 0$ implies $\mathcal{Q}(A) = 0$ for $A \in \mathcal{F}$.
- Define $Z(t) := \frac{d\mathcal{Q}}{d\mathcal{P}} \bigg|_{\mathcal{F}(t)}$.
- By the Martingale Representation Theorem, $\exists \mathcal{F}$-predicable processes $\theta_j$ and $\alpha_i \geq -1$ for all $i, j = 1$ such that

$$dZ(t) = Z(t-) \left\{ \sum_{j=1}^{n} \theta_j(t) dB_j(t) + \sum_{i=1}^{m} \alpha_i(t) dM_i(t) \right\}.$$ 

- $\mathcal{Q}$-Brownian motion $\tilde{B}_j(t) := B_j(t) - \int_0^t \theta_j(s) \, ds$
- $\mathcal{Q}$-(local) martingale $\tilde{M}_i(t) := N_i(t) - \int_0^t \tilde{\lambda}_i(s) \, ds$ where $\tilde{\lambda}_i(s) := \lambda_i \{ \alpha_i(s) + 1 \}$.

- We do NOT assume that the firm controls $\theta_j$ and $\tilde{\lambda}_i \forall i, j$.
  - By MRT, they are $\mathcal{F}$-predicable, i.e., would be controlled based on the information set that continues to lose the information of a history of the controls over time.
  - Rather, $\exists$ a pair of $(\theta, \tilde{\lambda})$ that corresponds to the controlled $\mathcal{Q}$.
Set-up (cont.)

- **Risk** aversion of the firm and the investor.
  - $U_i : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is player $i$’s utility function of his or her own wealth $\gamma_i \in \mathbb{R}$ at time $T$ ($i = 1, 2$),
    - $i = 1$ and $i = 2$ denote the firm and the investor, resp.
    - $U_i$ ($i = 1, 2$) is $C^2$ on $\text{dom} \ U_i := \{x \in \mathbb{R} \mid U_i(x) > -\infty\}$: $U_i' > 0$, $U_i'' \leq 0$.
  - The firm’s reservation utility at time 0: a constant $r \in \mathbb{R}$. 
Set-up (cont.)

- The firm enters into a contract with the investor and shares the time-$T$ outcome $X(T)$ with the investor.
- The investor offers a menu of contract payoffs $C_T$ to the firm, and the firm then decides whether or not to accept it.
- A contract (the firm’s wealth allocation) $C_T = C_T(X(\cdot))$
  - Define the set $A_2$ of the contracts $C_T(X(\cdot))$ such that
    - (i) $C_T \in \text{dom } U_1$ and $X(T) - C_T \in \text{dom } U_2$ a.s.,
    - (ii) $C_T(X(\cdot))$ is continuous and is Gâteaux differentiable,
    - (iii) $1 < \exists \, p < \infty$, $\mathbb{E}^p \left[ e^{pU_1(C_T)} \right] < \infty$ and
      $\mathbb{E}^p \left[ |U_2(X(T) - C_T)|^q \right] < \infty$ where $q = \frac{p}{p-1}$.
Firm’s optimization

• For $C_T \in \mathcal{A}_2$, define the firm’s expected utility under the controlled probability measure $Q$, denoted by $V_1$, as:

$$V_1 := \sup_{Q \ll P, \mathcal{H}(Q\|P) < \infty} \left\{ \mathbb{E}_Q^Q [U_1(C_T(X(\cdot)))] - \mathcal{H}(Q\|P) \right\}.$$ 

• We obtain the following proposition:

Proposition

For $C_T \in \mathcal{A}_2$, 

$$V_1 = \log \mathbb{E}_P^P \left[ e^{U_1(C_T(X(\cdot)))} \right].$$

The maximizer, denoted by $Q^*$, is then characterized by

$$\frac{dQ^*}{dP} = \frac{e^{U_1(C_T(X(\cdot)))}}{\mathbb{E}_P^P \left[ e^{U_1(C_T(X(\cdot)))} \right]}.$$
Proof

Taking exponential of $\mathbb{E}^Q[U_1(C_T(X(\cdot))))] - \mathcal{H}(Q \mid \mid P)$,

\[
\begin{align*}
\exp \left( U_1(C_T(X(\cdot))) \right) - \mathcal{H}(Q \mid \mid P) &= \exp \left( U_1(C_T(X(\cdot))) - \log \frac{dQ}{dP} \right) \\
&\leq \mathbb{E}^Q \left[ e^{U_1(C_T(X(\cdot)))} - \log \frac{dQ}{dP} \right] \quad \text{(by Jensen’s inequality)} \\
&= \mathbb{E}^Q \left[ e^{U_1(C_T(X(\cdot)))} \frac{dP}{dQ} \right] = \mathbb{E}^P \left[ e^{U_1(C_T(X(\cdot)))} \mathbf{1}_{\left\{ \frac{dQ}{dP} > 0 \right\}} \right] \\
&\leq \mathbb{E}^P \left[ e^{U_1(C_T(X(\cdot)))} \right]
\end{align*}
\]

with equality if and only if $U_1(C_T(X(\cdot))) - \log \frac{dQ}{dP}$ is a constant, i.e., \[
\frac{dQ}{dP} = \frac{e^{U_1(C_T(X(\cdot)))}}{\mathbb{E}^P \left[ e^{U_1(C_T(X(\cdot)))} \right]}. \quad \text{Thus } Q^* \text{ is obtained.}
\]

\[\square\]
Example

- In the case of the finite-scenario case above, setting $Y = U_1(C_T)$.

$$V_1 = \max_{\{q_1, \cdots, q_l\}} \sum_{s=1}^{l} y_s q_s - q_s \log \frac{q_s}{p_s}$$

subject to $\sum_{s=1}^{l} q_s = 1$ and $q_s \geq 0$ ($s = 1, \cdots, l$).

- Assume that $q_s \geq 0$ is satisfied for each $s = 1, \cdots, l$.

- Let the Lagrangian multiplier associated with $\sum_{s=1}^{l} q_s = 1$ be denoted by $\kappa$.

$$L(\{q_1, \cdots, q_l\}; \kappa) = \sum_{s=1}^{l} q_s y_s - q_s \log \frac{q_s}{p_s} + \kappa \left( \sum_{s=1}^{l} q_s - 1 \right).$$

- Differentiating with respect to $q_s$, $y_s - \log \frac{q_s}{p_s} - 1 + \kappa = 0$. Hence, $\frac{q_s}{p_s} = \frac{e^{y_s}}{\mathbb{E}[e^Y]}$ is confirmed. The optimal probability distribution $(q_1^*, \cdots, q_l^*)$ is obtained.
Implementability condition

- As usual in hidden action problems, impose $V_1 = r$ on the set $A_2$
- Define the set $A'_2$ of the contracts $C_T \in A_2$ such that $C_T$ satisfies $V_1 = r$.
- Implementability condition: $\frac{dQ^*}{dP} = e^{-r}e^{U_1(C_T(X(\cdot)))}$
- Due to the characteristics of the Radon-Nikodym derivative,

Corollary

For a two-date (i.e., $\{0, T\}$) discrete-time moral hazard model, there exists a continuous-time model that obtains the same optimal result.
Investor’s optimization

- Formulates the investor’s optimization problem with respect to $C_T \in A'_2$ as follows:

$$\sup_{C_T \in A'_2} \mathbb{E}^{Q^*} \left[ U_2(X(T) - C_T(X(\cdot))) \right].$$

- Although the investor cannot observe the true probability measure $Q$ directly, she can verify the optimal $Q^*$ via the implementability condition.

$$\sup_{C_T \in A'_2} \mathbb{E}^{Q^*} \left[ U_2(X(T) - C_T(X(\cdot))) \right] = \sup_{C_T \in A'_2} \mathbb{E}^{P} \left[ \frac{dQ^*}{dP} U_2(X(T) - C_T(X(\cdot))) \right]$$

$$= \sup_{C_T \in A'_2} \mathbb{E}^{P} \left[ e^{-r} e^{U_1(C_T(X(\cdot)))} U_2(X(T) - C_T(X(\cdot))) \right].$$
Investor’s optimization (cont.)

- With the Lagrangian multiplier associated with $V_1 = r$ as $\mu$, the constrained optimization problem is:

$$\sup_{C_T \in A_2'} \left\{ e^{-r} \mathbb{E}_P \left[ e^{U_1(C_T(X(\cdot)))} \left\{ U_2 \left( X(T) - C_T(X(\cdot)) \right) + \mu \right\} \right] \right\}.$$

- A necessary and sufficient condition for optimality of $C_T$ is:

**Proposition**

$$\frac{U'_2(X(T) - C_T)}{U'_1(C_T)} - U_2(X(T) - C_T) = \mu \quad a.s..$$

- Obviously,

**Corollary**

$$C_T(X(\cdot)) = C_T(X(T)).$$
Characterization

• From the optimality condition, obtains the uniqueness of the optimal contract if it exists.

• Also,

\[ 0 < \frac{dC_T}{dX(T)} = 1 - \frac{U'_2 U''_1}{U''_2 U'_1 + U'_2 U''_1 - U'_2 (U'_1)^2} \leq 1. \]

• For the reference, in the case of no moral hazard, from the standard Borch rule \( \frac{U'_2}{U'_1} = \mu \),

\[ 0 \leq \frac{dC_T}{dX(T)} = 1 - \frac{U'_2 U''_1}{U''_2 U'_1 + U'_2 U''_1} \leq 1. \]
Example

Consider the case of $U_1(z) = \log z$ and $U_2(z) = z$: more precisely, $U_1(z) := \log z$ for $z > 0$ and $U_1(z) := -\infty$ for $z \leq 0$.

- When the optimality condition holds, $C_T = \frac{X(T) + \mu}{2}$.
  → Denote this by $C^\mu_T$.
- From the implementability condition,

\[
e^r = \mathbb{E}^P[e^{U_1(C^\mu_T)}] = \frac{\mathbb{E}^P[X(T)] + \mu}{2}
\]

- Therefore,

\[
\mu = 2e^r - \mathbb{E}^P[X(T)].
\]

Let it be denoted by $\mu^*$.
Example (cont.)

- We assume that $\mu^* \geq 0$, or equivalently, $r \geq \log \frac{E^P[X(T)]}{2}$.
- The optimal contract $C_T^{\mu^*} = \frac{X(T)+\mu^*}{2} = e^r + \frac{X(T)-E^P[X(T)]}{2}$.
- The investor’s optimal expected utility is then obtained as:

$$e^{-r} E^P \left[ e^{U_1(C_T^{\mu^*})} U_2(X(T) - C_T^{\mu^*}) \right]$$

$$= e^{-r} E^P \left[ \frac{X(T) + \mu^*}{2} \cdot \frac{X(T) - \mu^*}{2} \right] = \frac{1}{4e^r} E^P [(X(T))^2 - (\mu^*)^2]$$

$$= \frac{E^P [(X(T) - E^P[X(T)])^2]}{4e^r} + E^P[X(T)] - e^r$$

$$= \frac{\text{Var}^P[X(T)]}{4e^r} + E^P[X(T)] - e^r$$

where $\text{Var}^P[X(T)]$ denotes variance of $X(T)$ under $P$.

- $E^P[X(T)] = x e^{cT}$.
- $\text{Var}^P[X(T)] = x^2 \exp\{2cT\} \left( \exp\{(\sum_j \sigma_j^2 + \sum_i \lambda_i z_i)^T\} - 1 \right)$. 

\[23/27\]
Concluding remarks

• Provides a tractable framework to study optimal risk sharing between an investor and a firm with general utility forms in the presence of moral hazard under both market risk and jump risk.

• For future work,
  ▶ Applies this framework to the practice in finance.
  ▶ A companion work of this paper, namely Misumi et al. (2013), does.
Thank you!