Higher-Order Risk Attitudes toward Correlation

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Abstract

Higher-order risk attitudes other than risk aversion (e.g., prudence and temperance) play vital roles both in theoretical and empirical work. While the literature has mainly focused on how they entail a preference for combining "good" outcomes with "bad" outcomes, We consider here an alternative approach which relates higher-order risk attitudes to the sign of correlation. The theoretical result in this paper proposes new insights into economic and financial applications such as risk aversion in the presence of another risk, bivariate stochastic dominance and justifying the first-order approach to moral hazard principal-agent problems.

Key words: higher-order risk attitudes; stochastic dominance dependence; correlation; covariance

JEL classification: D81.

1 Introduction

Covariance is perhaps the most common notion of dependence between two random variables. Consider two random variables $\tilde{x}$ and $\tilde{y}$ valued in the intervals $[a, b]$ and $[c, d]$ respectively. It is well known that one of the most used formula in the economics of uncertainty is the covariance rule:

$$Cov(\tilde{x}, \tilde{y}) = E\tilde{x}\tilde{y} - E\tilde{x}E\tilde{y}, \tag{1}$$

which implies that $E\tilde{x}\tilde{y} \geq E\tilde{x}E\tilde{y}$ if and only if $\tilde{x}$ and $\tilde{y}$ co-vary positively.

A direct application of above covariance rule is to sign the equity premium: Denote u as the bivariate utility of the representative agent, $\tilde{x}$ as the GDP per capita and $\tilde{y}$ as the background risk. From Gollier (2001, page 65-68), we know that the equity premium, φ, can be written as[1]:

$$\varphi = \frac{E\tilde{x}Eu^{(1,0)}(\tilde{x}, \tilde{y})}{E\tilde{x}u^{(1,0)}(\tilde{x}, \tilde{y})} - 1. \tag{2}$$

If there is full information on the agent's preference (e.g., $u(x, y) = \log(x + y)$) and the distribution of $(\tilde{x}, \tilde{y})$ (e.g., $(\tilde{x}, \tilde{y})$ is joint-normal distributed), then we can simply sign φ. Generally, however, we have only partial information on preferences (e.g., risk aversion) and distributions of random variables (e.g., affiliation). The rules of covariance between functions employ partial information on the agent's preferences and the random variables and, therefore, they sign the correlation. We recall a well-developed concept for covariance between monotonic functions.

Definition (Esary et al. 1967, p1466) $(\tilde{x}, \tilde{y})$ is said to be associated if for all functions α, β which are increasing in each component,

$$Cov(\alpha(\tilde{x}, \tilde{y}), \beta(\tilde{x}, \tilde{y})) \geq 0. \tag{3}$$

Suppose that all that we know is that the agent risk averse in x ($u^{(2,0)} < 0$) and correlation averse ($u^{(1,1)} \leq 0$)[2], then the above definition and (2) imply that , the equity premium, φ, is positive when $(\tilde{x}, \tilde{y})$ is associated. Thus, we have partial information on the sign of the equity premium.

Recently, prudence and temperance, as well as even higher-order risk attitudes, have become important both in theoretical and empirical work. Eeckhoudt and Schlesinger (2006) examine

[1] $u^{(k_1,k_2)}$ denotes the (k_1, k_2)th cross derivative of, that is, $u^{(k_1,k_2)} = \frac{\partial^{k_1+k_2}}{\partial x^{k_1} \partial y^{k_2}} u(x, y)$.

[2] For the interpretation of correlation averse, please see Richard (1975), Epstein and Tanny (1980), Eeckhoudt et al. (2007).

these higher-order risk attitudes toward particular classes of lottery pairs. They show how higher-order risk attitudes can be fully characterized by a preference relation over these lotteries. They call such preference as "risk apportionment" and show that, if preferences are defined in an expected utility framework with differentiable utility, the direction of preference for a particular class of lottery pairs is equivalent to signing the higher-order derivative of the utility function. Since then, the concept of "risk apportionment" has been extended by Eeckhoudt et al. (2007), Tsetlin and Winkler (2009), Jokung (2011) and Denuit et al. (2013) to higher orders of multivariate risk attitudes. Eeckhoudt (2012) and Eeckhoudt and Schlesinger (2012) provide excellent surveys on this line of research.

The agents with higher-order risk attitudes might be somewhat similar when they sign the correlation of their utility functions with some types of random variables. However, there is relatively little theory on how to sign covariance between functions beyond monotonicity. This paper concentrates on covariance rule that can be applied to higher-order risk attitudes.

Although association is one of the basic conditions that describe positive dependence, when we sign correlation, finding other conditions of dependence which are weaker than association is still useful. Generally speaking, as we shall see in this paper, the more information we know for the agent's preference, the weaker dependence of random variables we need to sign covariance.

We start by building a theory of covariance between functions with higher-order derivatives. Then there are three applications to illustrate it, including risk aversion in the presence of another risk, bivariate stochastic orderings and justifying the first-order approach to moral hazard principal-agent problems.

The paper is organized as follows. Sections 2 proposes a covariance rule between functions with higher-order derivatives. Section 3 contains three applications of the covariance rule in an expected-utility framework. Section 4 concludes the paper.

2 A Sign for Covariance of functions with higher-order derivatives

Eeckhoudt and Kimball (1992) propose a positive dependence concept to study the demand for insurance in the presence of a background risk: The distribution of background risk conditional upon a given level of insurable loss deteriorates in the sense of third-order stochastic dominance as the amount of insurable loss increases. Denote by $F(y|x)$ the conditional distribution of

$\tilde{y}$ given $\tilde{x} = x$. The following dependent structure extends Eeckhoudt and Kimball (1992)'s concept to N^{th}-order stochastic dominance.

Definition Define $F^1(y|x) = F(y|x)$ and $F^{n+1}(y|x) = \int_c^y F^n(t|x)dt$. We say that $\tilde{y}$ is N^{th}-order stochastic dominance dependent on x ($N^{th}SDD(\tilde{y}|x)$) if

(i) $F^N(y|x') \leq F^N(y|x)$ for all y and $x' \geq x$;

(ii) $F^n(d|x') \leq F^n(d|x)$ for all y, $x' \geq x$ and $n = 1, ...N-1$.

$FSDD(\tilde{y}|x)$ (N=1) states that x increases $\tilde{y}$ via first-order stochastic dominance (FSD). $SSDD(\tilde{y}|x)$ (N=2) means that x increases $\tilde{y}$ via second-order stochastic dominance (SSD) which includes a "mean-preserving increase in risk" as defined by Rothschild and Stiglitz (1970). $TSDD(\tilde{y}|x)$ (N=3) implies that x increases $\tilde{y}$ via third-order stochastic dominance (TSD). Combing any SSD shift with any "increase in downside risk" as defined by Menezes et al. (1980) yields a TSD. Eeckhoudt and Schlesinger (2008) explain how the different metrics in the extant empirical literature can be put into a stochastic dominance framework.

Now we are going to sign $Cov(\alpha(\tilde{x}, \tilde{y}), \beta(\tilde{x}, \tilde{y}))$ by $N^{th}SDD(\tilde{y}|x)$ and the signs of higher-order partial derivatives of α and β. Denuit et al. (1999) introduce the class $\mathcal{U}_{(s_1,s_2)-icv}$ of the regular (s_1, s_2)-increasing concave functions defined as the class of all the functions u, for s_1 and s_2 are positive integers, such that $(-1)^{k_1+k_2+1}u^{(k_1,k_2)} \geq 0$ for all $k_1 = 0, 1, ..., s_1$, $k_2 = 0, 1, ..., s_2$ with $k_1 + k_2 \geq 1$. Many higher-order risk attitudes are in $\mathcal{U}_{(s_1,s_2)-icv}$ class. For more details, we refer the interested readers to Eeckhoudt et al. (2007). The following proposition extends Esary et al.'s result (1967, Theorem 4.3) from monotonic functions to functions with higher-order derivatives.

Proposition 2.1 *The following statements are equivalent.*

(i)

$$Cov(\alpha(\tilde{x}, \tilde{y}), \beta(\tilde{x}, \tilde{y})) \geq 0 \tag{4}$$

for all α and β such that $\alpha^{(1,0)} \geq 0$, $\beta^{(1,0)} \geq 0$, $\alpha \in \mathcal{U}_{(0,I)-icv}$ and $\beta \in \mathcal{U}_{(0,J)-icv}$;

(ii) $N^{th}SDD(\tilde{y}|x)$ where $N = \min(I, J)$.

Proof See appendix. Q.E.D.

When $\tilde{x}$ is the wealth, $\tilde{y}$ is another risk that cannot be hedged and $\beta(x, y) = u(x, y)$ is the agent's utility function, $Cov(\tilde{x}, u(\tilde{x}, \tilde{y})) \geq 0$ implies wealth and utility go in the same direction.

For example, Proposition 2.1 shows that: (i) when $(\tilde{x}, \tilde{y})$ is $FSDD(\tilde{y}|x)$, for a monotonic agent ($u^{(1,0)} \geq 0$ and $u^{(0,1)} \geq 0$), wealth and utility move in the same direction; (ii) when $(\tilde{x}, \tilde{y})$ is $SSDD(\tilde{y}|x)$, for an agent who is monotonic in x and y, and risk averse in y ($u^{(1,0)} \geq 0$, $u^{(0,1)} \geq 0$ and $u^{(0,2)} \leq 0$), wealth and utility go into the same direction; (iii) when $(\tilde{x}, \tilde{y})$ is $TSDD(\tilde{y}|x)$, for an agent who is monotonic in x and y, risk averse and prudent in y ($u^{(1,0)} \geq 0$, $u^{(0,1)} \geq 0$, $u^{(0,2)} \leq 0$ and $u^{(0,3)} \geq 0$), wealth and utility move together.

When $\tilde{x}$ is the GDP per capita, $\tilde{y}$ is the background risk, and $\beta(x, y) = u^{(1,0)}(x, y)$ is the agent's marginal utility function. $Cov(\tilde{x}, u^{(1,0)}(\tilde{x}, \tilde{y})) \leq 0$ implies GDP per capita and marginal utility move in the opposite directions. Proposition 2.1 and (2) show that, if one of the following conditions is satisfied, then the equity premium is positive: (i) $FSDD(\tilde{y}|x)$, the agent is risk averse in x and correlation averse ($u^{(2,0)} < 0$ and $u^{(1,1)} \leq 0$); (ii) $SSDD(\tilde{y}|x)$, the agent is risk aversion in x, correlation averse and cross-prudent in x ($u^{(2,0)} < 0$, $u^{(1,1)} \leq 0$ and $u^{(1,2)} \geq 0$); (iii) $TSDD(\tilde{y}|x)$, the agent is risk aversion in x, correlation averse, cross-prudent and cross-temperate in x ($u^{(2,0)} < 0$, $u^{(1,1)} \leq 0$, $u^{(1,2)} \geq 0$ and $u^{(1,3)} \leq 0$). For more discussions of correlation averse, cross-prudent and cross-temperate, please see Eeckhoudt et al. (2007).

To close this section, we note that Proposition 2.1 also adds knowledge of comparing stochastic dependence structures. From Proposition 2.1 and the definition of association, we know that "$(\tilde{x}, \tilde{y})$ is associated" implies $N^{th}SDD(\tilde{y}|x)$. Proposition 2.1 can also help us to show that $N^{th}SDD(\tilde{y}|x)$ implies " $\tilde{x}$ is positive $N^{th}ED$ on $\tilde{y}$" (The discussion of the relationship between $N^{th}SDD(\tilde{y}|x)$ and $N^{th}ED$ is given in Appendix). Now we can state formally the relationships between the various concepts of bivariate dependence:

$$\begin{aligned} & (\tilde{x}, \tilde{y}) \textit{ is associated} \\ \Rightarrow \quad & N^{th}SDD(\tilde{y}|x) \\ \Rightarrow \quad & \tilde{x} \textit{ is positive } N^{th}ED \textit{ on } \tilde{y} \\ \Rightarrow \quad & (\tilde{x}, \tilde{y}) \textit{ is positive correlated.} \end{aligned} \tag{5}$$

3 Some economic consequences

This part of the paper demonstrates the usefulness of Proposition 2.1 for economic and financial applications such as risk aversion in the presence of another risk, bivariate stochastic dominance and justifying the first-order approach to moral hazard principal-agent problems.

3.1 Risk aversion in the presence of another risk

An agent is risk averse if she prefers the expected value of a random amount of money than the random amount of money. Finkelshtain et al. (1999) study the question of risk aversion in the presence of background risks in other argument of the utility function. They propose the following result[3];

Theorem 3.1 *(Finkelshtain et al. 1999, part (a) and (b) of Theorem 2) The following statements are equivalent.*

(i) $Eu(\tilde{x},\tilde{y}) \leq Eu(E\tilde{x},\tilde{y})$ *for* $(\tilde{x},\tilde{y})$ *such that* $E(\tilde{x}|\tilde{y})$ *is increasing in* y*;*

(ii) $u^{(2,0)} \leq 0$ *and* $u^{(1,1)} \leq 0$.

Proposition 2.1 can extend risk aversion in the presence of background risks to higher-order risk attitudes. We propose the following result.

Proposition 3.2 *Suppose* $N^{th}SDD(\tilde{y}|x)$, $u^{(2,0)} \leq 0$ *and* $-u^{(1,0)} \in \mathcal{U}_{(0,N)-icv}$*, then*

$$Eu(\tilde{x},\tilde{y}) \leq Eu(E\tilde{x},\tilde{y}). \tag{6}$$

Proof See appendix. Q.E.D.

The above proposition shows that, if one of the following conditions is satisfied, then an agent is risk averse for $\tilde{x}$ in the presence of $\tilde{y}$: (i) she is risk averse in x ($u^{(2,0)} \leq 0$), correlation averse ($u^{(1,1)} \leq 0$) and $FSDD(\tilde{y}|x)$; (ii) she is risk averse in x ($u^{(2,0)} \leq 0$), correlation averse $u^{(1,1)} \leq 0$, cross-prudent in x ($u^{(1,2)} \geq 0$) and $SSDD(\tilde{y}|x)$; (iii) she is risk averse in x ($u^{(2,0)} \leq 0$), correlation averse $u^{(1,1)} \leq 0$, cross-prudent and cross-temperate in x ($u^{(1,2)} \geq 0$ and $u^{(1,3)} \leq 0$), and $TSDD(\tilde{y}|x)$. Propositions 3.1 and 3.2 study risk aversion in the presence of background risks for different stochastic dependence structures.

In the following two examples we will apply Proposition 3.2 to optimal of savings and health investments problems studied by Denuit et al. (2011). The applications consider an agent facing a financial risk in the presence of a non-hedged background risk such as health or environmental risk. A decision has to be made about the amount of an investment (in the financial dimension) resulting in a future benefit either in the same dimension (savings) or in the other dimension (environmental quality or health improvement). The stochastic dependence structures proposed here are different to Denuit et al.(2011)'s.

[3] In statistics literature, the similar result is called as Bakhtin's Lemma (Bulinski and Shashkin 2007, Lemma 1.11; Rao 2012, Theorem 1.2.31).

Example The agent's objective is to select the optimal amount of savings (s) to be transferred from period 0 to period 1. The choice s is made in order to maximize total utility U defined as

$$U(s) = u_0(x_0 - s, h_0) + \frac{1}{1+\rho} E u_1(s(1+r) + \tilde{x}, \tilde{h}) \tag{7}$$

where ρ is the subjective discount rate and r is the rate of return. The optimal amount of savings s^* is determined by

$$u_0^{(1,0)}(x_0 - s^*, h_0) = \frac{1+r}{1+\rho} E u_1^{(1,0)}(s^*(1+r) + \tilde{x}, \tilde{h}) \tag{8}$$

Define $s_{x,E}$ as the solution of (8) with $(\tilde{x}, E\tilde{h})$ substituted for $(\tilde{x}, \tilde{h})$, and $s_{E,h}$ as the solution of (8) with $(E\tilde{x}, \tilde{h})$ substituted for $(\tilde{x}, \tilde{h})$. Proposition 3.2 implies the following result.

Proposition 3.3 *(i)* $u_0^{(2,0)} \leq 0,\ u_1^{(2,0)} \leq 0,\ u_1^{(1,2)} \geq 0,\ u_1^{(1,1)} \in \mathcal{U}_{(N,0)-icv}$ *and* $N^{th} SDD(\tilde{x}|h) \Rightarrow s^* \geq s_{x,E}$;

(ii) $u_0^{(2,0)} \leq 0,\ u_1^{(3,0)} \geq 0,\ u_1^{(2,0)} \in \mathcal{U}_{(0,N)-icv}$ *and* $N^{th} SDD(\tilde{h}|x) \Rightarrow s^* \geq s_{E,h}$.

Proof See appendix. Q.E.D.

The above proposition proposes the stochastic dependence structures between the two risks and higher-risk attitudes which imply the increase of optimal amount of savings.

Example The agent chooses how much of resources x_0 is to be devoted to an investment a that will improve his future health by an amount ma, where m represents the productivity of the current monetary sacrifice expressed in units of the other attribute. The choice a is made in order to maximize total utility V defined as

$$V(a) = u_0(x_0 - a, h_0) + \frac{1}{1+\rho} E u_1(\tilde{x}, \tilde{h} + a) \tag{9}$$

The optimal amount of investment a^* is determined by

$$u_0^{(1,0)}(x_0 - a^*, h_0) = \frac{m}{1+\rho} E u_1^{(0,1)}(\tilde{x}, \tilde{h} + ma^*) \tag{10}$$

Define $a_{x,E}$ as the solution of (10) with $(\tilde{x}, E\tilde{h})$ substituted for $(\tilde{x}, \tilde{h})$, and $a_{E,h}$ as the solution of (10) with $(E\tilde{x}, \tilde{h})$ substituted for $(\tilde{x}, \tilde{h})$.

Proposition 3.4 *(i)* $u_0^{(2,0)} \leq 0,\ u_1^{(0,3)} \geq 0,\ u_1^{(0,2)} \in \mathcal{U}_{(N,0)-icv}$ *and* $N^{th} SDD(\tilde{x}|h) \Rightarrow a^* \geq a_{x,E}$;

(ii) $u_0^{(2,0)} \leq 0,\ u_1^{(0,2)} \leq 0,\ u_1^{(2,1)} \geq 0,\ u_1^{(1,1)} \in \mathcal{U}_{(0,N)-icv}$ *and* $N^{th} SDD(\tilde{h}|x) \Rightarrow a^* \geq a_{E,h}$.

Proof See appendix. Q.E.D.

Proposition 3.4 shows under which conditions optimal investment for health (environmental) improvement is reached.

3.2 A class of bivariate stochastic orderings

Stochastic dominance is a very useful tool in various areas of economics and finance. It has been studied extensively in the univariate case (e.g. Hadar Russell (1969) and Hanoch and Levy (1969)). Denuit et al. (2013) extend the univariate case to multivariate N^{th}-degree concave (convex) stochastic dominance and N^{th}-degree risk. In this section we use a stochastic order that can be related to characteristics such as higher-order risk attitudes. We study bivariate stochastic dominance for this stochastic order.

Let $(\tilde{x}_1, \tilde{y}_1)$ and $(\tilde{x}_2, \tilde{y}_2)$ be two 2-dimensional random vectors with density functions f and g. Since

$$\begin{aligned} Eu(\tilde{x}_1, \tilde{y}_1) &= \int_a^b \int_c^d u(x,y) f(x,y) dxdy \\ &= \int_a^b \int_c^d u(x,y) \frac{f(x,y)}{g(x,y)} g(x,y) dxdy \\ &= E[u(\tilde{x}_2, \tilde{y}_2) \frac{f(\tilde{x}_2, \tilde{y}_2)}{g(\tilde{x}_2, \tilde{y}_2)}] \\ &= E[u(\tilde{x}_2, \tilde{y}_2)] + Cov[u(\tilde{x}_2, \tilde{y}_2), \frac{f(\tilde{x}_2, \tilde{y}_2)}{g(\tilde{x}_2, \tilde{y}_2)}], \end{aligned} \tag{11}$$

we have the following conclusion: If $(\tilde{x}_2, \tilde{y}_2)$ is associated and $\frac{f}{g}$ is increasing in x and y, then $Eu(\tilde{x}_1, \tilde{y}_1) \geq E[u(\tilde{x}_2, \tilde{y}_2)]$ (see e.g., Shaked and Shanthikumar 2007, Theorem 6.B.8).

We can apply Proposition 2.1 to (11) to get:

Proposition 3.5 *The following statements are equivalent.*

(i) $Eu(\tilde{x}_1, \tilde{y}_1) \geq E[u(\tilde{x}_2, \tilde{y}_2)]$ *for all* u, f *and* g *such that* $u^{(1,0)} \geq 0$, $(\frac{f}{g})^{(1,0)} \geq 0$, $u \in \mathcal{U}_{(0,I)-icv}$, $(\frac{f}{g}) \in \mathcal{U}_{(0,J)-icv}$;

(ii) $N^{th}SDD(\tilde{y}_2|x_2)$ *where* $N = \min(I, J)$.

The advantage of above the proposition is that it extends the bivariate stochastic ordering from associated random variables to $N^{th}SDD$ random variables, while the cost is that it requires more restrictions on the higher-order partial derivatives of u and $\frac{f}{g}$. For example, when $I = J = 2$, Proposition 3.5 implies that, $Eu(\tilde{x}_1, \tilde{y}_1) \geq E[u(\tilde{x}_2, \tilde{y}_2)]$ if u is monotonic ($u^{(1,0)} \geq 0$ and $u^{(0,1)} \geq 0$) and risk averse in y ($u^{(0,2)} \leq 0$), $\frac{f}{g}$ is increasing in x and y, concave in y, and $SSDD(\tilde{y}|x)$.

In the following two examples, we re-examine the optimal of savings and health investments problems again by Proposition 3.5. We suppose f and g are density functions of $(\tilde{x}, \tilde{h})$ and $(\tilde{x}', \tilde{h}')$ respectively

Example We consider (7) again. Define s' as the solution of (8) with $(\tilde{x}', \tilde{h}')$ substituted for $(\tilde{x}, \tilde{h})$. Proposition 3.5 implies the following result.

Proposition 3.6 *$u_0^{(2,0)} \leq 0$, $u_1^{(2,0)} \leq 0$, $(\frac{f}{g})^{(1,0)} \geq 0$, $-u_1^{(1,0)} \in \mathcal{U}_{(0,I)-icv}$, $(\frac{f}{g}) \in \mathcal{U}_{(0,J)-icv}$ and $N^{th} SDD(\tilde{h}'|x')$ where $N = \min(I, J) \Rightarrow s^* \leq s'$.*

Proof See appendix. Q.E.D.

Example We consider (9) again. Define a' as the solution of (10) with $(\tilde{x}', \tilde{h}')$ substituted for $(\tilde{x}, \tilde{h})$. We obtain the following result from Proposition 3.5.

Proposition 3.7 *$u_0^{(2,0)} \leq 0$, $u_1^{(0,2)} \leq 0$, $u_1^{(1,1)} \leq 0$, $(\frac{f}{g})^{(1,0)} \geq 0$, $-u_1^{(0,1)} \in \mathcal{U}_{(0,I)-icv}$, $(\frac{f}{g}) \in \mathcal{U}_{(0,J)-icv}$ and $N^{th} SDD(\tilde{h}'|x') \Rightarrow a^* \leq a'$.*

Proof See appendix. Q.E.D.

The above two examples demonstrate the usefulness of Proposition 3.5 for deriving comparative static effects of changes of risk.

3.3 Justify the first-order approach to bi-signal principal-agent problems

Suppose an agent chooses an effort level $a \geq 0$, and she has a von NeumannVMorgenstern utility function $u(s) - a$, where s is the agents monetary payoff and $u(.)$ is a strictly increasing function. Let $s = s(x, y)$ be the function, chosen by the principal, specifying her payment to the agent as a function of the signal $(\tilde{x}, \tilde{y})$ with probability density function $f(x, y|a)$. The agents expected payoff is

$$U(a) = \int_a^b \int_c^d u(s(x,y)) f(x,y|a) dx dy - a, \tag{12}$$

It is well known that, one way to guarantee that the First-order-approach (FOA) is valid is to show that the $U(a)$ is concave, given the wage contract. The sufficient conditions for this requirement are the monotone likelihood ratio condition (MLRC) and the concavity of the distribution function condition (CDFC) (Rogerson, 1985). However, most of the distribution functions do not have the CDFC property.

Jewitt (see Conlon 2009 p274-275) has suggested another sufficient conditions for the concavity of $U(a)$. Define $H(x, y) = u(s(x, y))$. We can obtain (Conlon 2009, 274-275):

$$\frac{d^2}{da^2} U(a) = -Cov(H(\tilde{x}, \tilde{y})), -\frac{f_{aa}(\tilde{x}, \tilde{y}|a)}{f(\tilde{x}, \tilde{y}|a)}), \tag{13}$$

which implies that, "$(\tilde{x}, \tilde{y})$ is affiliated, $H(x, y)$ and $-\frac{f_{aa}(x,y|a)}{f(x,y|a)}$ are increasing functions" are sufficient conditions for $\frac{d^2}{da^2}U(a) \leq 0$.

Applying Proposition 2.1 to (13), we can propose a sufficient and necessary condition for the concavity of $U(a)$.

Proposition 3.8 *The following statements are equivalent.*

(i)

$$\frac{d^2}{da^2}U(a) \leq 0 \tag{14}$$

for all $H(x, y)$ *and* $f(x, y|a)$ *such that* $H^{(1,0)} \geq 0$ $(\frac{f_{aa}}{f})^{(1,0)} \leq 0$, $H \in \mathcal{U}_{(0,I)-icv}$ *and* $-\frac{f_{aa}}{f} \in \mathcal{U}_{(0,J)-icv}$*;*

(ii) $N^{th}SDD(\tilde{y}|x)$ *where* $N = \min(I, J)$*.*

Since

$$\begin{aligned} & (\tilde{x}, \tilde{y}) \ is \ affiliated \\ \Rightarrow \ & (\tilde{x}, \tilde{y}) \ is \ associated \\ \Rightarrow \ & N^{th}SDD(\tilde{y}|x), \end{aligned} \tag{15}$$

Proposition 3.8 extends our knowledge of the validity of the FOA in the principal-agent problems from affiliated bi-signal to $N^{th}SDD$ bi-signal.

4 Conclusion

The rule of covariance has been widely studied in the monotonic functions case. The monotonic functions are consistent with some basic preference conditions such as monotonicity. However, many higher-order risk attitudes, that are easy to explain to decision makers, are beyond the monotonic functions. This paper fills this gap by linking higher-order risk attitudes to the sign of correlation. We present some economic and financial applications that are useful in applying the result.

5 Appendix

5.1 Proof of Proposition 2.1

(ii) $\Rightarrow$ (i): From Esary et al. (1967, p1471), we know that

$$\begin{aligned} & Cov[\alpha(\tilde{x},\tilde{y}),\beta(\tilde{x},\tilde{y})] \\ = \; & E\{Cov[\alpha(\tilde{x},\tilde{y}),\beta(\tilde{x},\tilde{y})|\tilde{x}]\} + Cov\{E[\alpha(\tilde{x},\tilde{y})|\tilde{x}], E[\beta(\tilde{x},\tilde{y})|\tilde{x}]\}. \end{aligned} \tag{16}$$

$\alpha^{(0,1)} \geq 0$ and $\beta^{(0,1)} \geq 0$ imply $Cov[\alpha(x,\tilde{y}),\beta(x,\tilde{y})] \geq 0$, and hence $E\{Cov[\alpha(\tilde{x},\tilde{y}),\beta(\tilde{x},\tilde{y})|\tilde{x}]\} \geq 0$. It is well known that if $\frac{dE[\alpha(\tilde{x},\tilde{y})|\tilde{x}]}{dx} \geq 0$, and $\frac{dE[\beta(\tilde{x},\tilde{y})|\tilde{x}]}{dx} \geq 0$, then

$$Cov\{E[\alpha(\tilde{x},\tilde{y})|\tilde{x}], E[\beta(\tilde{x},\tilde{y})|\tilde{x}]\} \geq 0, \tag{17}$$

From $E[\alpha(\tilde{x},\tilde{y})|\tilde{x}] = \int_c^d \alpha(x,y)dF(y|x)$ and note that $F(d|x) = 1$ and $F(c|x) = 0$ for all $x \Rightarrow F_x(d|x) = F_x(c|x) = 0$, we have

$$\begin{aligned} \frac{dE[\alpha(\tilde{x},\tilde{y})|\tilde{x}]}{dx} &= \int_c^d \alpha^{(1,0)}(x,y)dF(y|x) + \int_c^d \alpha(x,y)dF_x(y|x) \\ &= \int_c^d \alpha^{(1,0)}(x,y)dF(y|x) - \int_c^d \alpha^{(0,1)}(x,y)F_x(y|x)dy. \end{aligned} \tag{18}$$

Then, applying integration by parts to $\int_c^d \alpha^{(0,1)}(x,y)F_x(y|x)dy$, we obtain, for $I \geq 1$ (we define $\sum_{i=1}^{0}(.) = 0$),

$$\begin{aligned} & \int_c^d \alpha^{(0,1)}(x,y)F_x(y|x)dy \\ = \; & \sum_{i=1}^{I-1}(-1)^{i+1}\alpha^{(0,i)}(x,d)F_x^{i+1}(d|x) + (-1)^{I+1}\int_c^d \alpha^{(0,I)}(x,y)F_x^I(y|x)dy. \end{aligned} \tag{19}$$

So, $\alpha^{(1,0)} \geq 0$, $\alpha \in \mathcal{U}_{(0,I)-icv}$ and $I^{th}SDD(\tilde{y}|x)$ imply

$$\int_c^d \alpha^{(0,1)}(x,y)F_x(y|x)dy \leq 0,$$

and hence $\frac{dE[\alpha(\tilde{x},\tilde{y})|\tilde{x}]}{dx} \geq 0$.

By the same approach we can show that, $\beta^{(1,0)} \geq 0$, $\beta \in \mathcal{U}_{(0,J)-icv}$ and $J^{th}SDD(\tilde{y}|x)$ imply $\frac{dE[\beta(\tilde{x},\tilde{y})|\tilde{x}]}{dx} \geq 0$.

Finally, we can conclude that, $\alpha^{(1,0)} \geq 0$, $\beta^{(1,0)} \geq 0$, $\alpha \in \mathcal{U}_{(0,I)-icv}$, $\beta \in \mathcal{U}_{(0,J)-icv}$ and $N^{th}SDD(\tilde{y}|x)$ where $N = \min(I,J)$, imply (17).

(i) $\Rightarrow$ (ii): We prove this by contradictions.

Suppose that $F^N(y|x_2) > F^N(y|x_1)$ for $x_2 \geq x_1$. Due to the continuity of $F^N(y|x)$, we have $F_x^N(y|x) > 0$ for some $x \in (x_1, x_2)$. Choose the following function:

$$\bar{\beta}(x,y) = -\int_y^d dt_1 \int_{t_1}^d dt_2 ... \int_{t_{N-1}}^d 1_{[x_1,x_2]}(t_N) dt_N, \tag{20}$$

then

$$\bar{\beta}^{(1,0)}(x,y) = 0, \tag{21}$$

$$\bar{\beta}^{(0,k-1)}(x,y) = (-1)^k \int_y^d dt_k \int_{t_k}^d dt_{k+1} ... \int_{t_{N-1}}^d 1_{[x_1,x_2]}(t_N) dt_N, \; for \; k = 2, ..., N-1, \tag{22}$$

$$\bar{\beta}^{(0,N-1)}(x,y) = (-1)^N \int_y^d 1_{[x_1,x_2]}(t_N) dt_N, \tag{23}$$

and

$$\bar{\beta}^{(0,N)}(x,y) = (-1)^{N+1} 1_{[x_1,x_2]}(y). \tag{24}$$

Thus, $(-1)^{N+1}\bar{\beta}^{(0,N)} \geq 0$, and $\bar{\beta}^{(0,k-1)}(x,d) = 0$ for $k = 2, ..., N$. From (18) and (19) we get

$$\frac{dE[\bar{\beta}(\tilde{x},\tilde{y})|\tilde{x}]}{dx} = -(-1)^{N+1} \int_c^d \bar{\beta}^{(0,N)}(x,y) F_x^N(y|x) dy < 0, \tag{25}$$

and hence, form (16), we obtain

$$\begin{aligned} & Cov[\tilde{x}, \bar{\beta}(\tilde{x},\tilde{y})] \qquad (26) \\ = \; & E\{Cov[\tilde{x}, \bar{\beta}(\tilde{x},\tilde{y})|\tilde{x}]\} + Cov\{E[\tilde{x}|\tilde{x}], E[\bar{\beta}(\tilde{x},\tilde{y})|\tilde{x}]\} \\ = \; & Cov\{\tilde{x}, E[\bar{\beta}(\tilde{x},\tilde{y})|\tilde{x}]\} < 0 \end{aligned}$$

which is a contradiction.

Now suppose $F^j(d|x_2) > F^j(d|x_1)$ for $x_2 > x_1$ and $1 \leq j \leq N-1$. Due to the continuity of $F^j(d|x)$, we have $F_x^j(d|x) > 0$ for some $x \in (x_1, x_2)$. Choose the following function:

$$\bar{\beta}(x,y) = \frac{(-1)^j}{(j-1)!}(y-d)^{j-1} \tag{27}$$

then

$$\bar{\beta}^{(1,0)}(x,y) = 0, \tag{28}$$

$$\bar{\beta}^{(0,k-1)}(x,y) = \begin{cases} \frac{(-1)^j}{(j-k)!}(y-d)^{j-k} & k = 2,3,..,j \\ 0 & k > j \end{cases} \tag{29}$$

and

$$(-1)^k \bar{\beta}^{(0,k-1)}(x,y) = \begin{cases} \frac{1}{(j-k)!}(d-y)^{j-k} & k = 2,3,..,j \\ 0 & k > j, \end{cases} \tag{30}$$

and hence, $(-1)^k\bar{\beta}^{(0,k-1)}(x,y) \geq 0$ and

$$(-1)^k\bar{\beta}^{(0,k-1)}(x,d) = \begin{cases} 0 & k = 2,3,..,j-1 \\ 1 & k = j \\ 0 & k > j. \end{cases} \tag{31}$$

Therefore, (18) and (19) we get

$$\frac{dE[\bar{\beta}(\tilde{x},\tilde{y})|\tilde{x}]}{dx} = -F_x^j(d|x) < 0, \tag{32}$$

and hence, form (16), we obtain

$$\begin{aligned} & Cov[\tilde{x},\bar{\beta}(\tilde{x},\tilde{y})] \\ = \; & E\{Cov[\tilde{x},\bar{\beta}(\tilde{x},\tilde{y})|\tilde{x}]\} + Cov\{E[\tilde{x}|\tilde{x}], E[\bar{\beta}(\tilde{x},\tilde{y})|\tilde{x}]\} \\ = \; & Cov\{\tilde{x}, E[\bar{\beta}(\tilde{x},\tilde{y})|\tilde{x}]\} < 0 \end{aligned} \tag{33}$$

which is a contradiction.

5.2 Various concepts of bivariate dependence

The expectation dependence concept is introduced by Wright (1987). Li (2011) proposes the higher-order extensions. The definition is recalled next. Define

$$ED_1(\tilde{x}|y) = [E\tilde{x} - E(\tilde{x}|\tilde{y} \leq y)] \geq 0 \; for \; all \; y, \tag{34}$$

then $\tilde{x}$ is positive first-degree expectation dependent on $\tilde{y}$. Define $ED_2(\tilde{x}|y) = \int_c^y ED_1(\tilde{x}|t)F_Y(t)dt$, repeated integrals defined by

$$ED_N(\tilde{x}|y) = \int_c^y ED_{N-1}(\tilde{x}|t)dt, \; for \; N \geq 3. \tag{35}$$

Definition If $ED_K(\tilde{x}|d) \geq 0$, for $k = 2, ..., N-1$ and

$$ED_N(\tilde{x}|y) \geq 0 \; for \; all \; y \in [c,d], \tag{36}$$

then $\tilde{x}$ is positive N^{th}-order expectation dependent (N^{th}ED) on $\tilde{y}$.

Expectation dependence and its higher-order extensions have been shown to play a key role in many economic and financial problems, such as asset allocation (Wright, 1987; Hadar and Seo, 1988), demand for risky asset under background risk (Li, 2011), first-order risk aversion

(Dionne and Li 2011, 2013) and asset pricing (Dionne et al., 2013). It is interesting to find the relationship between $N^{th}SDD(\tilde{y}|x)$ and $ED_N(\tilde{x}|y)$. Denuit et al. (2013, Equation (19)) propose the following result.

$$ED_N(\tilde{x}|Y) = -\frac{1}{(N-1)!}Cov[\tilde{x}, (Y-\tilde{y})_+^{N-1}]. \tag{37}$$

Define $g(x,y) = -(Y-y)_+^{N-1}$, then $g^{(1,0)} = 0$ and $(-1)^i g^{(0,i)} \le 0$ for $i = 1, 2, ...,$ and from Proposition 2.1 we obtain:

Proposition 5.1 $N^{th}SDD(\tilde{y}|x) \Rightarrow \tilde{x}$ *is positive* $N^{th}ED$ *on* $\tilde{y}$

Proposition 5.1 shows that, when $(\tilde{x}, \tilde{y})$ is $N^{th}SDD(\tilde{y}|x)$, the results derived in above applications of N^{th}ED (Wright, 1987; Hadar and Seo, 1988; Li, 2011; Dionne and Li 2011; Dionne and Li 2013; Dionne et al., 2013) still hold.

5.3 Proof of Proposition 3.2

We are going to use an approach motivated by Bulinski and Shashkin (2007, Lemma 1.11) and Rao (2012, Theorem 1.2.31). Let

$$\beta(x,y) = \begin{cases} \frac{u(x,y)-u(E\tilde{x},y)}{x-E\tilde{x}} & x \neq E\tilde{x} \\ u^{(1,0)}(E\tilde{x},y) & x = E\tilde{x} \end{cases} \tag{38}$$

Then

$$\beta^{(1,0)}(x,y) = \frac{u^{(1,0)}(x,y)(x-E\tilde{x}) - [u(x,y)-u(E\tilde{x},y)]}{(x-E\tilde{x})^2} \le 0 \quad (since\ u^{(2,0)}(x,y) \le 0) \tag{39}$$

and, for $i = 1, ..., N$,

$$(-1)^i\beta^{(0,i)}(x,y) = (-1)^i \frac{u^{(0,i)}(x,y) - u^{(0,i)}(E\tilde{x},y)}{x-E\tilde{x}} \ge 0 \quad (since\ (-1)^i u^{(1,i)}(x,y) \ge 0). \tag{40}$$

Hence, from Proposition 2.1, we obtain $Cov(\tilde{x}, \beta(\tilde{x},\tilde{y})) \le 0$ for $N^{th}SDD(\tilde{y}|x)$. Since

$$\begin{aligned} & Cov(\tilde{x}, \beta(\tilde{x},\tilde{y})) \\ = \; & Cov(\tilde{x}-E\tilde{x}, \beta(\tilde{x},\tilde{y})) \\ = \; & E[\beta(\tilde{x},\tilde{y})(\tilde{x}-E\tilde{x})] - E[\beta(\tilde{x},\tilde{y})]E[(\tilde{x}-E\tilde{x})] \\ = \; & E[u(\tilde{x},\tilde{y}) - u(E\tilde{x},\tilde{y})], \end{aligned} \tag{41}$$

we obtain $E[u(\tilde{x},\tilde{y}) - u(E\tilde{x},\tilde{y})] \le 0$.

5.4 Proof of Proposition 3.3

(i) Because U is concave in s, $s^* \geq s_{x,E}$ if and only if $U'(s^*)$, with $(\tilde{x}, E\tilde{h})$ substituted for $(\tilde{x}, \tilde{h})$, is negative. In other words, there will be a precautionary demand for savings if and only if

$$\begin{aligned} U'(s^*) &= -u_0^{(1,0)}(x_0 - s^*, h_0) + \frac{1+r}{1+\rho} E u_1^{(1,0)}(s^*(1+r) + \tilde{x}, E\tilde{h}) \\ &= \frac{1+r}{1+\rho}[E u_1^{(1,0)}(s^*(1+r) + \tilde{x}, E\tilde{h}) - E u_1^{(1,0)}(s^*(1+r) + \tilde{x}, \tilde{h})] \\ &\leq 0, \end{aligned} \tag{42}$$

where the second equality is obtained by using Proposition 3.2.

(ii) Because U is concave in s, $s^* \geq s_{E,h}$ if and only if $U'(s^*)$, with $(E\tilde{x}, \tilde{h})$ substituted for $(\tilde{x}, \tilde{h})$, is negative. In other words, there will be a precautionary demand for savings if and only if

$$\begin{aligned} U'(s^*) &= -u_0^{(1,0)}(x_0 - s^*, h_0) + \frac{1+r}{1+\rho} E u_1^{(1,0)}(s^*(1+r) + E\tilde{x}, \tilde{h}) \\ &= \frac{1+r}{1+\rho}[E u_1^{(1,0)}(s^*(1+r) + E\tilde{x}, \tilde{h}) - E u_1^{(1,0)}(s^*(1+r) + \tilde{x}, \tilde{h})] \\ &\leq 0, \end{aligned} \tag{43}$$

where the second equality is obtained by using Proposition 3.2.

5.5 Proof of Proposition 3.4

(i) Because V is concave in a, $a^* \geq a_{x,E}$ if and only if $V'(a^*)$, with $(\tilde{x}, E\tilde{h})$ substituted for $(\tilde{x}, \tilde{h})$, is negative. In other words, there will be a precautionary investment if and only if

$$\begin{aligned} V'(a^*) &= -u_0^{(1,0)}(x_0 - a^*, h_0) + \frac{m}{1+\rho} E u_1^{(0,1)}(\tilde{x}, E\tilde{h} + ma^*) \\ &= \frac{m}{1+\rho}[E u_1^{(0,1)}(\tilde{x}, E\tilde{h} + ma^*) - E u_1^{(0,1)}(\tilde{x}, \tilde{h} + ma^*)] \\ &\leq 0, \end{aligned} \tag{44}$$

where the second equality is obtained by using Proposition 3.2.

(ii) Because V is concave in a, $a^* \geq a_{E,h}$ if and only if $V'(a^*)$, with $(E\tilde{x}, \tilde{h})$ substituted for $(\tilde{x}, \tilde{h})$, is negative. In other words, there will be a precautionary investment if and only if

$$\begin{aligned} V'(a^*) &= -u_0^{(1,0)}(x_0 - a^*, h_0) + \frac{m}{1+\rho} E u_1^{(0,1)}(E\tilde{x}, \tilde{h} + ma^*) \\ &= \frac{m}{1+\rho}[E u_1^{(0,1)}(E\tilde{x}, \tilde{h} + ma^*) - E u_1^{(0,1)}(\tilde{x}, \tilde{h} + ma^*)] \\ &\leq 0, \end{aligned} \tag{45}$$

where the second equality is obtained by using Proposition 3.2.

5.6 Proof of Proposition 3.6

Because U is concave in s, $s^* \leq s'$ if and only if $U'(s^*)$, with $(\tilde{x}', \tilde{h}')$ substituted for $(\tilde{x}, \tilde{h})$, is positive. In other words, there will be a precautionary demand for savings if and only if

$$\begin{aligned} U'(s^*) &= -u_0^{(1,0)}(x_0 - s^*, h_0) + \frac{1+r}{1+\rho} E u_1^{(1,0)}(s^*(1+r) + \tilde{x}', \tilde{h}') \\ &= \frac{1+r}{1+\rho}[E u_1^{(1,0)}(s^*(1+r) + \tilde{x}', \tilde{h}') - E u_1^{(1,0)}(s^*(1+r) + \tilde{x}, \tilde{h})] \\ &\geq 0, \end{aligned} \tag{46}$$

where the second equality is obtained by using Proposition 3.5.

5.7 Proof of Proposition 3.7

Because V is concave in a, $a^* \leq a'$ if and only if $V'(a^*)$, with $(\tilde{x}', \tilde{h}')$ substituted for $(\tilde{x}, \tilde{h})$, is positive. In other words, there will be a precautionary investment if and only if

$$\begin{aligned} V'(a^*) &= -u_0^{(1,0)}(x_0 - a^*, h_0) + \frac{m}{1+\rho} E u_1^{(0,1)}(\tilde{x}, \tilde{h}' + m a^*) \\ &= \frac{m}{1+\rho}[E u_1^{(0,1)}(\tilde{x}', \tilde{h}' + m a^*) - E u_1^{(0,1)}(\tilde{x}, \tilde{h} + m a^*)] \\ &\geq 0, \end{aligned} \tag{47}$$

where the second equality is obtained by using Proposition 3.5.

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