

# Optimal consumption and the timing of the resolution of uncertainty

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### **Abstract**

We explore the effect of the timing of the resolution of income uncertainty on consumption. An agent faces uncertainty about his income at date  $t + 2$ . What is the effect of being informed that the uncertainty will be resolved at date  $t + 1$  on the consumption at date  $t$ ? We show that the effect is positive if and only if marginal utility is convex (prudence), when the risk free rate is equal to the rate of pure preference for the present. The intuition is that an early resolution of uncertainty allows for time-diversifying the risk. When the risk free rate is larger than the rate of pure preference for the present, a sufficient condition is that the agent be prudent with a concave absolute risk tolerance, as is the case for the HARA family.

# 1 Introduction

A consumer faces income uncertainty at date  $t + 2$ . In this paper, we examine the effect of the expectation of an early resolution of this uncertainty at date  $t + 1$  on the optimal consumption at date  $t$ . We show that the timing of income risks matters when determining a consumption plan. Moreover, we show that we can in general sign this effect, and we give an intuition for it.

The introduction of uncertainty into the classical consumption life cycle model has generated a large amount of research over the last three decades. We know since Leland (1968) that an increase in future income risks à la Rothschild-Stiglitz (1970) reduces initial consumption if the marginal utility of consumption is convex. Kimball (1990) coined the term "prudence" for this assumption. It is widely recognized as a sensible assumption on consumers' preferences. Because these models have two dates, they cannot address questions related to the timing of the resolution of uncertainty. Skinner (1988), Deaton (1992) and Carroll (1994, 1997) provide clear expositions of the multi-period theory of life-cycle consumption under uncertainty. Yet they do not investigate explicitly the question of the temporal resolution of uncertainty. The only exception is a recent paper by Blundell and Stocker (1999). Using a three-date framework, they provide approximate solutions for optimal consumption choices under constant relative risk aversion. Their simulations show that there is more precautionary savings when the uncertainty is in the middle period than when the uncertainty is in the distant period. Observe that moving the risk from one period to another has two different effects. There is first an income effect, since moving incomes over time affects their net present value. The second effect comes from the change in the timing of the resolution of the uncertainty. In this paper, we focus only on this second effect.

As initially stressed by Drèze and Modigliani (1972) and Spence and Zeckhauser (1972), it is intuitive that consumers will react differently to the bearing of an immediate lottery than to a delayed one. There is a simple intuition for why an early resolution of uncertainty should *reduce* precautionary savings. This intuition relies on the notion of "time-diversification", or "self-insurance". Risks that are realized earlier can be disseminated over more periods. Namely, a risk  $\tilde{x}$  affecting wealth that is realized with time horizon of  $n$  periods can be allocated to  $n$  consumption risks  $\tilde{x}/n$ . This smoothing of the shock plays exactly the same role as a diversification device. The reduction of the risk induces prudent agents to reduce their precautionary

savings. However, this reasoning holds only when the risk free rate in the economy equals the rate of pure preference for the present, which is the only case where it is indeed optimal to allocate equally a risk on wealth to risk on consumption over the remaining lifetime.

Our work is also related to the theory of natural resources. An exhaustible natural resource whose stock is unknown is extracted and consumed by successive generations. This is the so-called "cake-eating" problem. There is a substantial literature on the cake-eating problem under uncertainty. Kemp (1976), Loury (1978) and Gilbert (1979) first explored this problem and gave the conditions under which it is possible to derive the optimal path of consumption. In these models, the consumer updates in a continuous way his beliefs about the size of cake conditional on the observation that the size of the cake can be no less than what has been effectively consumed. That is, the only observable signal is whether the resource is extinguished or not. This is a restrictive view of the process that drives arrival of information. Yet the subsequent literature on exhaustible resource has devoted considerable attention to the modelling of arrival of information through extraction, exploration and discoveries processes (see e.g. Deskmukh and Pliska (1980), Dasgupta and Stiglitz (1981), Hoel (1984)). In this paper, we solve a problem that has been left aside by this literature, i.e. the situation where the social planner gets exogenous information about the stock of the resource.

The course of the paper is the following. In the next section, we introduce the model. Section 3 is devoted to the benchmark case where the risk free rate of the economy equals the rate of pure preference for the present. We show that positive prudence is necessary and sufficient for an early resolution of uncertainty to raise initial consumption. We also show that this result does not hold when the risk free rate is not equal to the rate of pure preference for the present. In section 4, we examine the case of a small risk on wealth, whereas the general necessary and sufficient condition is extracted in section 5. Because this condition is quite technical and unintuitive, we provide simpler sufficient condition in section 6. We also explain why our results differ from those of Blundell and Stocker (1999). Section 7 is devoted to an extension where the resolution of uncertainty is only partial. Section 8 presents briefly other extensions to our model.

## 2 The model

We consider the standard consumption-saving problem with three dates. The felicity function of the consumer is denoted  $u(\cdot)$ , and is assumed to be increasing and concave. The discount factor of utility is denoted  $\beta$ . The agent is endowed with a flow of income  $\tilde{y}_t$ ,  $t = 1, 2, 3$ . Only  $\tilde{y}_3$  is uncertain. Let  $R > 0$  denote one plus the risk free rate, and  $\tilde{w} = R^2y_1 + Ry_2 + \tilde{y}_3$  is the future value of the flow of incomes. In the absence of any early resolution of uncertainty, the problem of the consumer is written as

$$c_1^* \in \arg \max_{c_1} u(c_1) + \beta \left\{ \max_{c_2} u(c_2) + \beta E u(\tilde{w} - R^2c_1 - Rc_2) \right\}. \quad (1)$$

Assuming that  $u$  is differentiable, the first-order conditions to this program yield

$$u'(c_1^*) = \beta R u'(c_2^*) = (\beta R)^2 E u'(\tilde{w} - R^2c_1^* - Rc_2^*). \quad (2)$$

The special case of the cake-eating problem and its applications to non-renewable resources are obtained when  $R = 1$ : the stock of the resource is not productive. In this case the problem is to determine the socially efficient rate of extraction when the stock of the resource is uncertain. Since utility is increasing, we have directly incorporated into the program the fact that the cake will be completely consumed within the three periods. We also implicitly assumed that it is never optimal to run the risk of consuming entirely the cake before date 3. The assumption  $u(0) = -\infty$  is sufficient to guarantee that this is the case. The reader is referred to Kemp (1976), Loury (1978) and Gilbert (1979) for a derivation of the solutions when there is a possibility of 'premature exhaustion'.

The objective of the paper is to compare the optimal initial consumption with a late resolution of uncertainty,  $c_1^*$ , to the optimal initial consumption  $c_1^{**}$  when  $\tilde{w}$  is revealed between date  $t = 1$  and  $t = 2$ . In this case of early resolution of uncertainty, the consumption problem becomes

$$c_1^{**} \in \arg \max_{c_1} u(c_1) + \beta E \left\{ \max_{c_2} u(c_2) + \beta u(\tilde{w} - R^2c_1 - Rc_2) \right\}. \quad (3)$$

We can solve this problem by backward induction. For each net future value  $z = w - R^2 c_1$ , we obtain  $c_2(z)$  by solving

$$u'(c_2^{**}(z)) = \beta R u'(z - R c_2^{**}(z)) \quad (4)$$

for each  $z$ . The optimal early consumption is then obtained by solving the following Euler equation:

$$u'(c_1^{**}) = \beta R E u'(c_2^{**}(\tilde{w} - R^2 c_1^{**})). \quad (5)$$

The intuition suggests that  $c_1^{**}$  is larger than  $c_1^*$ : an earlier resolution of uncertainty makes the future less problematic, something that should induce the agent to reduce his precautionary savings. Because problem (3) is concave in  $c_1$ , this is the case if and only if

$$u'(c_1^*) \geq \beta R E u'(c_2^{**}(\tilde{w} - R^2 c_1^*)), \quad (6)$$

or, equivalently, if and only if

$$E u'(c_2^{**}(\tilde{w} - R^2 c_1^*)) \leq u'(c_2^*). \quad (7)$$

In short, the expectation of an early resolution of uncertainty increases initial consumption if and only if it reduces the marginal value of future wealth, which is given by the expected marginal utility of future optimal consumption.

### 3 The benchmark case: $\beta R = 1$

This problem is easiest to solve in the special case with  $\beta R = 1$ , i.e., when the rate of pure preference for the present is equal to the risk free rate of the economy. In the cake-eating problem ( $R = 1$ ), this is the case when the social planner allocates the same utility weight to the successive generations ( $\beta = 1$ ).

When  $\beta R = 1$ , we see from condition (4) that with an early resolution of uncertainty, consumption smoothing is optimal for the last two dates,  $c_2^{**}(z) = z - R c_2^{**}(z) = z/(R + 1)$ . Let

$$\tilde{z}^* = \tilde{w} - R^2 c_1^* \quad (8)$$

denote the future wealth net of the future value of initial consumption  $c_1^*$ . In the case of a late resolution of uncertainty, the date 2 optimal consumption is given by

$$u'(c_2^*) = Eu'(\tilde{z}^* - Rc_2^*). \quad (9)$$

Thus, the comparative static condition (7) can be rewritten as

$$Eu' \left( \frac{\tilde{z}^*}{R+1} \right) \leq u'(c_2^*). \quad (10)$$

Now, observe that

$$\begin{aligned} u'(c_2^*) &= \frac{R}{R+1}u'(c_2^*) + \frac{1}{R+1}Eu'(\tilde{z}^* - Rc_2^*) \\ &= E \left[ \frac{R}{R+1}u'(c_2^*) + \frac{1}{R+1}u'(\tilde{z}^* - Rc_2^*) \right] \end{aligned} \quad (11)$$

The first equality is directly derived from the first order condition (9) for  $c_2^*$ . The expectation operator at the end is on a weighted sum of marginal utility, which can itself be interpreted as an expected marginal utility conditional to  $\tilde{z}^* = z^*$ . This reinterpretation is a crucial point, as we will see below. Observe that, if  $u'$  is convex, this conditional expectation satisfies the following Jensen's inequality:

$$\frac{R}{R+1}u'(c_2^*) + \frac{1}{R+1}u'(z^* - Rc_2^*) \geq u' \left( \frac{z^*}{1+R} \right). \quad (12)$$

Taking the expectation with respect to  $\tilde{z}^*$  directly implies necessary and sufficient condition (10). This proves the sufficiency part of the following Proposition.

**Proposition 1** *Suppose that the rate of pure preference for the present is equal to the risk free rate:  $\beta R = 1$ . Then, an earlier resolution of uncertainty raises initial consumption if and only if the consumer is prudent ( $u'$  convex).*

*Sketch of the proof of necessity:* Suppose that  $u'$  is locally concave around  $y$ . Take  $R = 1$  and  $\tilde{z}^* = 2y + k\tilde{\varepsilon}$ , with  $E\tilde{\varepsilon} = 0$ . For  $k$  small enough,  $c_2^*$ ,  $\tilde{z}^*/(R+1)$  and  $\tilde{z}^* - Rc_2^*$  are in the neighborhood of  $y$ . Using Jensen's inequality directly yields the comparative statics condition opposite to (10). ■

This Proposition provides a new definition of the concept of prudence alternative to Kimball (1990). An agent is prudent if and only if adding a zero mean risk to his future income reduces his initial consumption. It is well known from Leland (1968) that a necessary and sufficient condition for a risk-averse agent to make positive precautionary savings is the convexity of marginal utility. We showed that the same condition is necessary and sufficient for an earlier resolution of uncertainty to yield an increase in initial consumption, when  $\beta R = 1$ . In the following, we establish a simple intuition for this equivalence. The underlying idea is that information reduces risk. It thus reduces the need for precautionary savings, under prudence.

There is an intuitive argument for why perfect information reduces risk and thus should increase consumption under prudence. If uncertainty is realized at the intermediary date  $t = 2$ , rather than at the last date, the consumer can "time diversify" the shock over the last two periods. This means splitting every dollar of loss or gain in wealth into a fifty cents loss or gain in consumption at each date. Our interpretation of the concept of time diversification is as follows.<sup>1</sup> The marginal value of wealth after date  $t = 1$  is the discounted value of the expected marginal utility of optimal future consumption  $(\tilde{c}_2, \tilde{c}_3)$ . It is given by

$$Eu'(\tilde{c}_2) + \beta Eu'(\tilde{c}_3) = (1 + \beta)E \left[ \frac{1}{1 + \beta} u'(\tilde{c}_2) + \frac{\beta}{1 + \beta} u'(\tilde{c}_3) \right]. \quad (13)$$

The bracketed term of the right-hand-side of this equality can be interpreted as the expected marginal utility of a random variable which is distributed as  $(c_2, (1 + \beta)^{-1}; c_3, \beta(1 + \beta)^{-1})$ . For a given state  $\tilde{z}^* = z^*$ , this random variable is degenerated at  $c_2^{**} = c_3^{**} = z^*/(1 + R)$  with an early resolution of uncertainty. It takes values  $c_2^*$  and  $z^* - Rc_2^*$  when the information is not revealed before  $t = 3$ . The point is that under  $\beta R = 1$  the second "lottery" is a mean-preserving spread of the first. It implies that under prudence, the expected – or discounted – marginal utility is increased by this absence of time diversification, *state by state*. Taking the expectation with respect to the states of nature yields the result.

Of course, an upper bound to the optimal initial consumption with a resolution of the uncertainty in the middle period is the one that would be optimal without any uncertainty. This is because the value function on

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<sup>1</sup>For an exposition of the fallacious interpretations of this concept, see Samuelson (1963). More details are provided in Gollier and Zeckhauser (1997) and Gollier (2000).

wealth inherits prudence from the utility function on consumption.<sup>2</sup>

At this stage, it is important to remember that we only considered the particular case where  $\beta R = 1$ . This is a particular case since when  $\beta R \neq 1$  perfect consumption smoothing through time is no more optimal and the time diversification effect cannot be used without an adaptation.

### 3.1 A counterexample

Let us consider the following numerical example. First take the utility function

$$u(c) = \begin{cases} -4 + 3c - \frac{c^2}{2} & \text{if } c < 2 \\ \ln(c - 1) & \text{if } c \geq 2. \end{cases} \quad (14)$$

It yields the following marginal utility function:

$$u'(c) = \begin{cases} 3 - c & \text{if } c < 2 \\ (c - 1)^{-1} & \text{if } c \geq 2. \end{cases} \quad (15)$$

This marginal utility function is decreasing and convex, i.e., the consumer is prudent. Suppose that the future wealth is distributed as  $\tilde{w} = 8.329 + \tilde{\varepsilon}$ , with  $\tilde{\varepsilon} = (-0.5, 1/2; 0.5, 1/2)$ . Finally, we suppose that  $R = 1$  and  $\beta = 4/9$ . Thus, we cannot use Proposition 1 to determine the effect of an early resolution of uncertainty on the initial consumption. We solve this problem numerically. In the economy with a late resolution of uncertainty, the optimal stochastic consumption path is as follows:

$$c_1^* = 4.351 \quad - \quad c_2^* = 2.489 \quad \begin{matrix} \hat{A} \\ \hat{A} \end{matrix} \quad \begin{matrix} c_{31}^* = 1.989 \\ c_{32}^* = 0.989 \end{matrix} \quad (16)$$

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<sup>2</sup>With the resolution of uncertainty at the middle period, the problem is to maximize  $u(c_1) + \beta E v(\tilde{w} - Rc_1)$ , where  $v(z) = \max_c u(c) + \beta u(z - Rc)$ . It is easy to check that  $v^0$  is convex if  $u^0$  is convex. See Carroll and Kimball (1996).

In the case of an early resolution of uncertainty, the optimal stochastic consumption path is written as

$$\begin{array}{r}
c_1^{**} = 4.329 \\
\hat{A} \\
c_{21}^{**} = 2.771 \quad - \quad c_{31}^{**} = 1.729 \\
\hat{A} \\
c_{22}^{**} = 2.271 \quad - \quad c_{32}^{**} = 1.229
\end{array} \tag{17}$$

We see that  $c_1^{**}$  is smaller than  $c_1^*$ , in spite of positive prudence.

## 4 The case of small risks

The previous example has demonstrated that prudence is not sufficient to sign the effect of an early resolution of uncertainty for any probability distribution when  $\beta R \neq 1$ . In this section, we derive the necessary and sufficient condition that permit to sign this effect when  $\beta R$  is arbitrary, but  $\tilde{w}$  is a small risk.

It will be convenient to define  $\tilde{\varepsilon}$ ,  $k$  and  $z^*$  so that

$$\tilde{z}^*(k) = z^* + k\tilde{\varepsilon} \text{ with } E\tilde{\varepsilon} = 0. \tag{18}$$

The marginal value of wealth before date  $t = 2$  with an early resolution of uncertainty is denoted

$$j^{**}(k) \equiv Eu'(c_2^{**}(\tilde{z}^*(k))), \tag{19}$$

where function  $c_2^{**}(z)$  is defined by (4). With a late resolution of uncertainty, the marginal value of wealth before  $t = 2$  equals

$$j^*(k) = u'(c_2^*(k)) = \beta REu'(\tilde{z}^*(k) - Rc_2^*(k)), \tag{20}$$

where  $c_2^*(k)$  is the optimal consumption at  $t = 2$  when the net wealth at  $t = 3$  is distributed as  $\tilde{z}^*(k)$ .

The early resolution of uncertainty raises initial consumption when the future risk is small if and only if  $j^{**}(k)$  is smaller than  $j^*(k)$  in the neighborhood of  $k = 0$ . It is easy to check that

$$j^{**}(0) = j^*(0) = u'(c_{20}), \tag{21}$$

with  $c_{20} = c_2^{**}(z^*) = c_2^*(0)$ . Turning to the first derivatives, we obtain

$$\left. \frac{\partial j^{**}}{\partial k} \right|_{k=0} = \left. \frac{\partial j^*}{\partial k} \right|_{k=0} = 0. \quad (22)$$

Thus, we are forced to examine the second-order effect of risk. After tedious manipulations, we obtain the following Proposition. It relies on the indexes of absolute risk aversion and absolute prudence which are defined as

$$A(c) = \frac{-u''(c)}{u'(c)} \text{ and } P(c) = \frac{u'''(c)}{-u''(c)}. \quad (23)$$

We hereafter assume that these functions exists, i.e., that  $u$  is thrice differentiable.

**Proposition 2** *Suppose that the risk on future wealth is small. Then, an early resolution of uncertainty raises initial consumption if and only if*

$$P(c_2) - RP(c_3) \leq 2 \frac{A(c_2)}{A(c_3)} P(c_3), \quad (24)$$

for any pair  $(c_2, c_3)$  that satisfies the first-order condition  $u'(c_2) = \beta R u'(c_3)$ .

*Proof:* Take  $c_2 = c_{20}$  and so  $c_3 = z^* - Rc_2$ . Fully differentiating condition (19) twice around  $k = 0$  yields

$$\left. \frac{\partial^2 j^{**}}{\partial k^2} \right|_{k=0} = \sigma^2 \beta R \frac{u''(c_2)u'''(c_3) [A(c_2)]^2 + \beta R u''(c_3)u'''(c_2) [A(c_3)]^2}{[A(c_2) + RA(c_3)]^2 [u''(c_2) + \beta R^2 u''(c_3)]}, \quad (25)$$

where  $\sigma^2$  is the variance of  $\tilde{\varepsilon}$ . The equivalent manipulation on  $j^*$  yields

$$\left. \frac{\partial^2 j^*}{\partial k^2} \right|_{k=0} = \sigma^2 \frac{u''(c_2)u'''(c_3)A(c_3)}{[A(c_2) + RA(c_3)]u''(c_3)}. \quad (26)$$

Using first-order condition  $\beta R = u'(c_2)/u'(c_3)$ , it is easy to check that the right-hand-side of (25) is smaller than the right-hand-side of (26) if and only if condition (24) is satisfied. ■

When  $\beta R = 1$ , condition (24) becomes  $(1 - R)P(c_2) \leq 2P(c_2)$ , since  $\beta R = 1$  implies that  $c_2 = c_3$ . Because  $R > -1$ , this condition is equivalent to nonnegative prudence. But this is true only when  $\beta R = 1$ . For example,

utility function (14) does not satisfy this condition when  $\beta R \neq 1$  in spite of nonnegative prudence. Indeed, for  $\beta R < 1$ , as in the counterexample, the rate of pure preference for the present is larger than the interest rate, which implies that the relevant domain of  $(c_2, c_3)$  is such that  $c_2 \geq c_3$ . Now, observe that, for utility function (14), prudence is zero for small  $c_3$ . Thus, the right-hand side of inequality (24) is zero in this region, whereas the left-hand-side is positive, thereby violating the condition.

We can extract from this an intuition for why prudence is not sufficient when  $\beta R$  is different from unity. The early resolution of uncertainty allows for transferring part of the risk to the second period. This is a time-diversification device, which tends to reduce the precautionary saving under positive prudence. But when  $\beta R \neq 1$ , it transfers risk to different consumption levels. If it happens that the degree of prudence is much larger at date 2 than at date 3 ( $P(c_2) \gg P(c_3)$ ), this risk transfer may generate an *increase* in the marginal value of wealth, which is measured by the expected marginal utility of future consumption. This is exactly the way by which we built the counterexample. Observe that the early resolution of uncertainty transfers half of the risk from a region where prudence is zero ( $c_3 < 2$ ) to a region where prudence is positive ( $c_2 > 2$ ). Condition (24) thus puts an upper bound to the degree of prudence in the region where part of the risk is transferred thanks to the early resolution of uncertainty. Note also that it is necessary that the utility function exhibits increasing absolute prudence to generate such a counterexample.

## 5 The necessary and sufficient condition

The general problem that we have to solve takes the following form. We want to guarantee that, for any distribution of the net wealth  $\tilde{z}$ , and for any  $c_2$ , the following condition is satisfied:

$$\beta R E u'(\tilde{z} - R c_2) - u'(c_2) = 0 \Rightarrow E u'(c_2^{**}(\tilde{z})) - u'(c_2) \leq 0, \quad (27)$$

where function  $c_2^{**}(\cdot)$  is defined in (4). The equality to the left of this condition states that  $c_2 = c_2^*$  is the optimal consumption at date 2 without information. The inequality to the right states that the marginal value of wealth is smaller with information than without information.

Consider a specific  $c_2$ . Using the hyperplane separation theorem, as in Pratt and Zeckhauser (1986) and Gollier and Kimball (1996), this condition

holds for any  $\tilde{z}$  if and only if there exists a scalar  $\lambda = \lambda(c_2)$  such that

$$G(z, c_2, \lambda) = u'(c_2^{**}(z)) - u'(c_2) - \lambda[\beta Ru'(z - Rc_2) - u'(c_2)] \leq 0 \quad (28)$$

for all  $z$ . Define  $\hat{z}$  such that  $\beta Ru'(\hat{z} - Rc_2) = u'(c_2)$ . We can interpret  $\hat{z}$  as the precautionary equivalent wealth to  $\tilde{z}$ . Observe that, by definition,  $c_2^{**}(\hat{z}) = c_2$ . This implies that  $G(\hat{z}, c_2, \lambda) = 0$ . Therefore, in order to guarantee that  $G$  is nonpositive in the neighborhood of  $z = \hat{z}$ , we need that

$$0 = \frac{\partial G}{\partial z}(\hat{z}, c_2, \lambda) = u''(c_2)c_2^{**0}(\hat{z}) - \lambda\beta Ru''(\hat{z} - Rc_2). \quad (29)$$

This allows us to extract the only possible candidate for  $\lambda$  that could satisfy condition (28). Using condition  $\beta Ru'(\hat{z} - Rc_2) = u'(c_2)$ , we get

$$\lambda = \frac{A(c_2)}{A(\hat{z} - Rc_2)}c_2^{**0}(\hat{z}) = c_3^{**0}(\hat{z}), \quad (30)$$

where  $c_3^{**}(z) = z - Rc_2^{**}(z)$ . Indeed, fully differentiating condition (4) yields

$$c_2^{**'}(z) = \frac{A(z - Rc_2^{**}(z))}{A(c_2^{**}(z)) + RA(c_3^{**}(z))} \text{ and } c_3^{**'}(z) = \frac{A(c_2^{**}(z))}{A(c_2^{**}(z)) + RA(c_3^{**}(z))}. \quad (31)$$

Notice that conditions (31) determine the optimal allocation of risk  $\tilde{z}$  over dates  $t = 2$  and 3. As suggested by the intuition, it is optimal to allocate a larger share of the risk at time where risk aversion is lower. Combining these observations allows us to write the following Proposition.

**Proposition 3** *Consider a given pair  $(\beta, R)$  and a twice differentiable, increasing and concave utility function  $u$ . An early resolution of uncertainty raises initial consumption for any distribution of net wealth  $\tilde{z}$  if and only if for all  $z, \hat{z}$ , we have*

$$u'(c_2^{**}(z)) \leq (1 - c_3^{**0}(\hat{z}))u'(c_2^{**}(\hat{z})) + c_3^{**0}(\hat{z})\beta Ru'(z - Rc_2^{**}(\hat{z})), \quad (32)$$

where  $c_2^{**}(\cdot)$  is defined in (4) and  $c_3^{**0}(\cdot)$  is defined by condition (31).

Notice first that the sufficiency of (32) is immediately obtained by taking its expectation with respect to  $z = \tilde{z}$ , and with a constant  $\hat{z}$  such that  $c_2^{**}(\hat{z}) = c_2^*$ . It yields

$$\begin{aligned} Eu'(c_2^{**}(\tilde{z})) &\leq (1 - c_3^{**0}(\hat{z}))u'(c_2^*) + c_3^{**0}(\hat{z})\beta REu'(\tilde{z} - Rc_2^*) \\ &= u'(c_2^*), \end{aligned} \quad (33)$$

where we used the first-order condition (2) for  $c_2^*$ . This shows that the early resolution of uncertainty reduces the marginal value of wealth if condition (32) is satisfied. The difficult part of the proof above was to show that this condition is also necessary.

Keep in mind that we can interpret  $c_2^{**}(\hat{z})$  as  $c_2^*$ , the optimal consumption at  $t = 2$  without information. The necessary and sufficient condition states that, in each state of nature  $z$ , the marginal utility of optimal consumption with information must be less than a weighted average of discounted marginal utilities at date  $t = 2$  and 3 without information. The discount rate is  $\beta R$ , and the implicit probabilities are  $1 - c_3^{**}(\hat{z})$  and  $c_3^{**}(\hat{z})$ . Again, it is easy to get back the case  $\beta R = 1$ , since it implies  $c_2^{**}(z) = c_3^{**}(z) = z/(1 + R)$  and  $c_3^{**0} = 1/(1 + R)$ . In such a case, condition (32) is formally equivalent to condition (12), which holds if and only if  $u'$  is convex.

Our necessary and sufficient condition is still complex, as it requires that a bivariate function  $F(., .)$  be nonpositive, with

$$F(z, \hat{z}) = u'(c_2^{**}(z)) - (1 - c_3^{**0}(\hat{z}))u'(c_2^{**}(\hat{z})) - c_3^{**0}(\hat{z})\beta R u'(z - Rc_2^{**}(\hat{z})). \quad (34)$$

By construction, we have that  $F(\hat{z}, \hat{z}) = F_1(\hat{z}, \hat{z}) = 0$ . Thus, a necessary condition is that  $F_1(\hat{z}, \hat{z})$  be nonpositive. After some manipulations, it can be checked that this necessary condition is nothing else than condition (24), the necessary and sufficient condition for small risk on net wealth.

Because of the complexity of the necessary and sufficient condition (32), the next section is devoted to deriving simpler sufficient condition from it.

## 6 Simple sufficient conditions

The necessary and sufficient condition requires that  $F(z, \hat{z})$  be uniformly nonpositive. Our sufficient condition implies that  $F_1(z, \hat{z})$  have the same sign as  $\hat{z} - z$ . Because we know that  $F(\hat{z}, \hat{z}) = 0$ , this is stronger than necessary. Replacing  $u'(c_2^{**}(z))$  by  $\beta R u'(c_3^{**}(z))$  in (34), we have

$$F_1(z, \hat{z}) = \beta R \left[ u''(z - Rc_2^{**}(z))c_3^{**}(z) - u''(z - Rc_2^{**}(\hat{z}))c_3^{**0}(\hat{z}) \right]. \quad (35)$$

Consider first the case of HARA utility functions, with  $u'(c) = (\eta + c/\gamma)^{-\gamma}$ . This family of functions gathers all familiar utility functions as exponential, quadratic, power and logarithmic ones. As it is well-known since Wilson

(1968), HARA utility functions imply that all efficient risk-sharing rules are linear. That implies that  $c_3^{**0}(z) = c_3^{**0}(\hat{z})$ . It yields

$$F_1(z, \hat{z}) = \beta R \left[ u''(z - Rc_2^{**}(z))c_3^{**0}(\hat{z}) - u''(z - Rc_2^{**}(\hat{z}))c_3^{**0}(\hat{z}) \right]. \quad (36)$$

If  $z$  is larger than  $\hat{z}$ ,  $c_2^{**}(z)$  is larger than  $c_2^{**}(\hat{z})$  and

$$u''(z - Rc_2^{**}(z)) - u''(z - Rc_2^{**}(\hat{z})) \leq 0 \quad (37)$$

for the subset of HARA functions that exhibit prudence, as is the case for exponential, power and logarithmic functions. Therefore,  $F_1(z, \hat{z})$  is negative when  $z > \hat{z}$ . Symmetrically,  $F_1(z, \hat{z})$  is positive when  $z < \hat{z}$ . Because  $F(\hat{z}, \hat{z}) = 0$ , this is sufficient for  $F$  to be uniformly nonpositive. By Proposition 3, it implies that an early resolution of uncertainty raises initial consumption.

**Proposition 4** *Suppose that  $u$  is HARA with positive prudence. It implies that an early resolution of uncertainty raises initial consumption.*

Notice that we don't need to have any information about the value of  $\beta$  and  $R$  to conclude in the HARA case.

We now relax the assumption that  $u$  is HARA, but we restrict  $\beta R$  to be smaller than 1. In other words, we assume that the rate of pure preference for the present is larger than the risk free rate. We first prove the following Lemma. It relies on the concept of absolute risk tolerance  $T$ , which is the inverse of absolute risk aversion:  $T(c) = 1/A(c) = -u'(c)/u''(c)$ .

**Lemma 1** *Suppose that  $\beta R \leq 1$  (resp.  $\beta R \geq 1$ ). Then  $c_3^{**'}(\cdot)$  is increasing in  $z$  if absolute risk tolerance  $T(\cdot)$  is concave (resp. convex).*

*Proof:* We can rewrite condition (31) as

$$c_3^{**'}(z) = \frac{T(c_3^{**}(z))}{T(c_3^{**}(z)) + RT(c_2^{**}(z))}. \quad (38)$$

Fully differentiating with respect to  $z$  yields

$$c_3^{**00}(z) = \frac{T'(c_3^{**})c_3^{**0} [T(c_3^{**}) + RT(c_2^{**})] - T(c_3^{**}) [T'(c_3^{**})c_3^{**0} + RT'(c_2^{**})c_2^{**0}]}{[T(c_3^{**}) + RT(c_2^{**})]^2}. \quad (39)$$

Using condition (31) again yields

$$c_3^{**0}(z) = \frac{RT(c_2^{**})T(c_3^{**})[T'(c_3^{**}) - T'(c_2^{**})]}{[T(c_3^{**}) + RT(c_2^{**})]^3}. \quad (40)$$

Suppose that  $\beta R < 1$ , which implies that  $c_3^{**} < c_2^{**}$ . It implies in turn that  $T'(c_3^{**}) - T'(c_2^{**})$  is positive if  $T$  is concave. ■

This Lemma directly generates the following Proposition.

**Proposition 5** *Suppose that the consumer is prudent and  $\beta R \leq 1$  (resp.  $\beta R \geq 1$ ). Then, an early resolution of uncertainty raises initial consumption if absolute risk tolerance  $T(\cdot)$  is concave (resp. convex).*

*Proof:* Consider any  $z$  larger  $\widehat{z}$ . If  $\beta R < 1$  and  $T$  concave, the Lemma implies that  $c_3^{**0}(z)$  is larger than  $c_3^{**0}(\widehat{z})$ . This yields

$$F_1(z, \widehat{z}) \leq \beta R c_3^{**0}(z) [u''(z - Rc_2^{**}(z)) - u''(z - Rc_2^{**}(\widehat{z}))]. \quad (41)$$

Positive prudence implies that the bracketed term in the right-hand-side is negative. Thus  $F_1(z, \widehat{z})$  is negative when  $z$  is larger than  $\widehat{z}$ . The reader can check that, under the same conditions,  $F_1(z, \widehat{z})$  is positive when  $z$  is smaller than  $\widehat{z}$ . ■

Thus, when  $\beta R$  is less than unity, the additional restriction that absolute risk tolerance is concave restores the sufficiency of positive prudence to guarantee the comparative static property. The concavity of absolute risk tolerance is related to decreasing absolute prudence in the following way:

$$T''(c) = P'(c)T(c) + P(c)T'(c). \quad (42)$$

Thus, absolute risk tolerance is concave only if

$$P'(c) \leq -\frac{P(c)T'(c)}{T(c)}. \quad (43)$$

It is widely believed that absolute risk aversion is decreasing, i.e. that  $T'$  is positive. Thus, concave absolute risk tolerance means that absolute prudence is strongly decreasing.<sup>3</sup> Our earlier discussion of why prudence is not sufficient when  $\beta R \neq 1$  provides the intuition for why strongly decreasing

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<sup>3</sup>Decreasing prudence is extensively discussed in Kimball (1990) and Kimball (1993).

absolute prudence is sufficient: when  $\beta R < 1$ , consumption is globally decreasing with time. Decreasing absolute prudence implies that the risk transfer allowed by information reallocates part of the risk toward a low prudence period (date 2).

There is no agreement about whether absolute risk tolerance should be concave, linear or convex. The only relevant paper testing whether absolute risk tolerance is concave or convex is by Guiso and Paeilla (2000), using an Italian panel data where households are requested to estimate the certainty equivalent to a lottery. They obtained that absolute risk tolerance is concave. Let us also list a few properties that we can link to these conditions. The concavity of absolute risk tolerance implies that

1. the demand for risky assets is concave with wealth, at least when the variance of returns is small with respect to the expected excess return;
2. there is a negative relationship between the optimal investment in a risky asset and the length of the time horizon of the investor (see Gollier and Zeckhauser (1997));
3. the equity premium in a Lucas' economy is increased by wealth inequality (see Gollier (1999));
4. under certainty, the marginal propensity to consume out of wealth is increasing.

Recently, Blundell and Stocker (1999) considered the same three-period framework to provide an approximate solution for optimal consumption choice for preferences that display constant relative risk aversion. They used this framework to derive the relation between information and consumption. They gave numerical examples where risk resolving earlier in one's life leads to a *lower* initial consumption. Their numerical results are thus *a priori* contradictory with proposition 4. We hereafter explain why.

Suppose, as in Blundell-Stocker, that total income is made of three parts  $y_1, \tilde{y}_2$  and  $\tilde{y}_3$ , where income  $\tilde{y}_t$  is *received* in period  $t$ . The total future wealth is thus

$$\tilde{w} = R^2 y_1 + R \tilde{y}_2 + \tilde{y}_3.$$

To examine the effect of the resolution of uncertainty, Blundell and Stocker compare scenario 1 ( $\tilde{y}_2 = y + \tilde{\varepsilon}, \tilde{y}_3 = y$ ) to scenario 2 ( $\tilde{y}_2 = y, \tilde{y}_3 = y + \tilde{\varepsilon}$ ),

with  $E\tilde{\varepsilon} = 0$ . Obviously, the uncertainty is revealed earlier in scenario 1. But there is a second effect in the Blundell-Stocker model. The point is that the ex ante distribution of total future wealth is not the same in the two scenarios. By transferring the risk from period 3 to period 2, there is also an increase in risk on wealth by a factor  $R > 1$ . In order to isolate the effect of an early information, it is crucial to let the prior distribution of aggregate wealth unchanged. The effect exhibited by the numerical simulations of Blundell-Stocker is in fact a precautionary effect: transferring uncertain incomes from date 3 to date 2 increases the aggregate future risk, which induces prudent agents to raise their precautionary saving. Our work has shown that Blundell and Stocker would have obtained the opposite result if they would have selected a sufficiently smaller risk free rate.

## 7 Imperfect information revelation

Up to now, we have examined the effect of a complete early resolution of uncertainty. Most of the time, early information is incomplete: some uncertainty remains after the signal is observed. We hereafter show that considering partial resolution of uncertainty adds a new dimension to the complexity of the problem. To simplify, suppose that  $\beta = R = 1$ . We show that prudence is not sufficient to guarantee that any *incomplete* early resolution of uncertainty raises initial consumption.

Between  $t = 1$  and  $t = 2$ , the consumer observes one of the two possible signals  $m = 1$  or  $2$ . Suppose that the net future wealth  $\tilde{z}^*$  conditional to signal  $m = 1$  takes some value  $z_0$  with probability 1. If signal  $m = 2$  is observed, the distribution of  $\tilde{z}^*$  is as some random variable  $\tilde{z}$ . The probability of observing signal  $m = 2$  is  $p$ . We examine the effect of the expectation of observing signal  $\tilde{m}$  on the initial consumption at  $t = 1$ . For a given value of  $p$ , define  $j(p)$  and  $c_2(p)$  as follows:

$$j(p) = u'(c_2(p)) = pEu'(\tilde{z} - c_2(p)) + (1 - p)u'(z_0 - c_2(p)). \quad (44)$$

The second equality is the first-order condition for the consumption-saving problem at date  $t = 2$  when the agent believe that  $\tilde{z}^*$  is distributed as  $(\tilde{z}, p; z_0, 1 - p)$ . Function  $j(\cdot)$  is the marginal value of wealth as a function of this belief  $p$ .

Without information, the marginal value of wealth is  $j^*(p) = j(p)$ . In the case of a partial revelation of uncertainty derived from observing signal  $\tilde{m}$ , the

marginal value of wealth is either  $j(1)$  or  $j(0)$ , respectively with probability  $p$  and  $1 - p$ . It implies that the marginal value of wealth at  $t = 1$  with this information structure is  $j^{**}(p) = pj(1) + (1 - p)j(0)$ . This information structure raises initial consumption if

$$pj(1) + (1 - p)j(0) \leq j(p), \quad (45)$$

i.e., if  $j$  is concave. We focus on the behavior of  $j$  around  $p = 0$ , that is, around certainty. It is easy to check that  $c_2(0) = z_0/2 = c$ . After tedious manipulations, we obtain that

$$j''(0) = u'''(c) [c'_2(0)]^2 + u''(c)c''_2(0), \quad (46)$$

$$c'_2(0) = \frac{Eu'(\tilde{z} - c) - u'(c)}{2u''(c)}, \quad (47)$$

$$c''_2(0) = \frac{[Eu'(\tilde{z} - c) - u'(c)][u''(c) - Eu''(\tilde{z} - c)]}{2[u''(c)]^2}. \quad (48)$$

In consequence,  $j$  is locally concave at  $p = 0$  if

$$\frac{Eu'(\tilde{z} - c) - u'(c)}{2u''(c)} \left[ u'''(c) \frac{Eu'(\tilde{z} - c) - u'(c)}{2u''(c)} + u''(c) - Eu''(\tilde{z} - c) \right] \quad (49)$$

is nonpositive. Assume prudence. Suppose that  $E\tilde{z} = z_0 = 2c$ , which implies that  $E\tilde{z} - c = c$  and  $Eu'(\tilde{z} - c) - u'(c)$  is positive. We would be done if the bracketed term in (49) is positive. This is true if and only if  $Ef(\tilde{z}, c) \geq 0 = f(E\tilde{z}, c)$  with

$$f(z, c) = P(c) [u'(c) - u'(z - c)] + 2[u''(c) - u''(z - c)], \quad (50)$$

for all  $\tilde{z}$  such that  $E\tilde{z} = 2c$ . A necessary condition is that function  $f$  must be convex with respect to its first argument locally at  $z = 2c$ , a condition which requires that

$$\frac{\partial^2 f(2c, c)}{\partial z^2} = -P(c)u'''(c) - 2u''''(c) \quad (51)$$

be nonnegative for all  $z$ . This yields the following result.

**Proposition 6** *Suppose positive prudence and  $\beta = R = 1$ . Any partial early resolution of uncertainty raises initial consumption only if*

$$P(c) \leq 2 \frac{-u''''(c)}{u'''(c)} \quad (52)$$

for all  $c$  in the domain of  $u$ .

The most obvious remark about this result is that prudence is not sufficient when uncertainty is only partially resolved. Second, observe that necessary condition (52) is stronger than  $u'''' < 0$ .<sup>4</sup> However, this condition is weaker than decreasing absolute prudence, which means that  $P(c) \leq -u''''(c)/u'''(c)$ .

## 8 Remarks on extensions

In this last section, we briefly mention some extensions and limits to the work presented above. Assertions are done without proofs (available under request).

### 8.1 Constraints on the level of consumption

We have not taken account of constraints on the level of consumption. What if we assume that there exists a constraint  $c_t \geq k$  on the level of consumption at each date  $t$ ? The answer is that prudence is then no more sufficient for an early uncertainty resolution to decrease consumption even for the benchmark case  $\beta R = 1$  or when considering HARA utility functions.

The intuition behind this result relies on the "irreversibility effect". Considering the constraint  $c_t \geq k$  introduces an element of irreversibility into the model. More consumption in the current period is more irreversible in the sense that it makes the constraint binding more likely in the future periods. As a consequence, there is an (quasi-) option value to reduce current consumption (as in Gollier, Jullien and Treich, (2000)). This effect goes in the opposite direction of the "time diversification" effect.

### 8.2 Time-additivity over $N$ periods

We have considered a three-periods time-additive model. What if we had assumed that there are  $N$  periods? The answer is that most of the insights of the paper would have remained. For instance, proposition 1 and 4 still hold: advancing uncertainty resolution from date 3 to date 2 when there are  $3 < N < \infty$  dates increases consumption at date 1, under prudence. This is very intuitive since the "time-diversification" effect expands over several

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<sup>4</sup>This condition arises as a necessary condition in several papers, as for example Pratt and Zeckhauser (1987), Kimball (1993) and Caballe and Pomansky (1997).

periods. The larger is time-horizon the smaller is the risk on consumption supported in each period (Gollier, (2000)).

Note though that central to the derivation of the results of the paper is the time separability of the lifetime utility function. The discussions provided in the paper have clearly showed that our result are driven by using Jensen's inequality with respect to this time-additivity, rather than with respect to the state-additivity inherent to the expected utility model. There is thus no hope to generate similar results in non-time-additive models, as in recursive utility models for example.

### 8.3 Capital and portfolio risk

We have assumed that the agent faces a risk of revenue at date 3. Suppose that revenue is certain but instead that  $R$  is unknown at date 3. What is the effect of an early resolution of capital risk on initial consumption?

Epstein (1980) examined this problem for constant relative risk aversion utility functions. He proved that consumption decreases with early resolution if and only if relative risk aversion is lower than one. Treich (1997, Ch. 5) found a similar result in the standard portfolio model. Suppose that in the last period the agent may allocate his current wealth between a riskless asset and a risky asset. Treich proved that if preferences belong to the HARA family or if markets are complete, early uncertainty resolution increases current consumption if and only if prudence is lower than twice absolute risk aversion,  $P(\cdot) \leq 2A(\cdot)$ .

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