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On the Demand for Budget Constrained Insurance

by

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Abstract

Much of the traditional economic theory of insurance is based on the assumption that the risk against which insurance is to be purchased is entirely exogenous. This is usually modelled by simply allowing the individual to include insurance as a mechanism of covering risk, without any real analysis of how this insurance is paid for. However, in almost all real-life consumer insurance, the size of the risk is itself a choice variable (the type of car to purchase, the type of employment to take, the amount to invest in an insurable asset, etc.), and decisions are made taking budget constraints explicitly into account. While an enormous number of interesting theorems can be derived in the standard model, these results are typically not robust to the extension of making risk an endogenous choice variable and the explicit inclusion of a budget constraint. Here, we use a simple two state model of the demand for insurance in which we explicitly introduce a budget constraint and in which the insurable risk itself is a choice variable. In the model, we find that the standard result of full coverage being demanded if and only if the premium is such that the insurer earns an expected profit of 0 no longer holds as such, and it turns out that in a simple two state setting some of the ambiguity of the standard model's comparative statics is avoided.

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1 Introduction

Traditionally, the economic theory of insurance is based on the study of preferences over wealth lotteries, using the indirect utility function to describe preferences. In the standard model, two particularly important assumptions are often made. Firstly, that the individual is always able to pay for his insurance contract out of that part of wealth that is not subject to loss. In particular, the implicit assumption is that first the individual uses the non-random part of her wealth to finance the purchase of insurance coverage, and then dedicates whatever terminal wealth ensues for consumption purposes. Secondly, the standard model assumes that the insurable risk itself is fully endowed and non-alterable. This is, of course, a question that is related to the first. If we do allow the individual to make a simultaneous choice of insurance and other goods under a budget constraint, then one of the goods that she could contemplate purchasing is precisely that which pertains to the underlying risk. In this case, decisions on both the amount of risk that impinges upon wealth, and the degree to which that risk is insured, would be made simultaneously (and simultaneously with the purchase of other goods) making the amount of risk an endogenous variable.

In this paper we analyse the demand for insurance in a very simple two-state model that does allow the individual the simultaneous choice of both risk and insurance, with the objective of studying the robustness of certain results concerning the demand for insurance that appear in the standard model.¹ Concretely, we firstly consider the well known standard result that the optimal demand for insurance will only be partial if the premium per unit of coverage is set such that the insurer earns a strictly positive expected profit from the contract. On the other hand, if the insurer earns an expected profit of 0, then the standard model returns the result that the demand for insurance will be necessarily complete.²

Secondly, the standard model returns ambiguous comparative statics of the demand for cov-

¹ Here, we only allow the simultaneous choice of risk and insurance, and not other consumption goods. Thus, the assumption is still that the individual first organizes his financial risk bearing situation, and then dedicates whatever terminal wealth ensues to the consumption of final goods.

² This occurs under standard expected utility maximization when the individual faces only one (insurable) risk. The result is altered when the individual faces multiple sources of risk and the market is incomplete (see, for instance, Doherty and Schlesinger, 1983). Also, the result does not hold when the insurer charges a lump sum loading, instead of a proportional loading.

erage. Specifically, if the total wealth of the individual is increased and the premium is such that the demand for coverage is partial, then if the individual's preferences are characterized by decreasing absolute risk aversion she will respond by purchasing less coverage.³ Thus insurance is an inferior good in the standard model under DARA. For more general utility functions, the result is ambiguous. However, in the standard portfolio selection model, where no insurance *per se* is contemplated, it is easy to show that, under the same assumption of decreasing absolute risk aversion, risk itself is a normal good – that is, a strictly positive part of any increase in wealth will be dedicated to increasing the risky element of the portfolio. Thus, exactly what happens when we allow the individual the option of setting endogenously both the amount of risk and the amount of insurance is unclear. We would expect that an increase in wealth will result in an increase in risk, but that this increased risk position will be insured relatively less, and so it is not clear whether the total purchase of insurance will increase or decrease.

The analysis of comparative statics in models of the demand for insurance is, of course, not new. The seminal paper on the comparative statics of insurance demand is Mossin (1968). In his paper, Mossin studies a variety of models of insurance demand, among which we can find a model of optimal coverage of a bivariate random monetary loss from an initial stock of wealth (i.e. a standard two-state model under indirect utility). In this model, as we have already pointed out, it is straightforward to deduce that the insurance consumer will purchase full coverage if and only if the premium is actuarially fair (and partial coverage if and only if the premium is greater than the actuarially fair premium), and that insurance is an inferior good (and hence may even be Giffen) if the individual's indirect utility function displays decreasing absolute risk aversion (DARA).

Ehrlich and Becker (1972) also studied a two-state model, in which the focus of attention is on the substitutability or complementarity of market insurance and other alternatives that the individual has available (self-insurance and self-protection). However, in their paper Ehrlich and Becker do consider some comparative statics of market insurance alone. The model of this paper is fundamentally the same as the Ehrlich and Becker model, but with one important difference.

³ The increase in wealth implies that the individual is less risk averse, and so will be less willing to insure any given risk.

Our model starts off with a decision on wealth allocation, and we have prices that are paid at the outset (say, at time 0). In contrast, Ehrlich and Becker start with an endowment in the future, and for that reason the only price (for insurance) is paid in the future (at time 1) in the no-loss state. Because Ehrlich-Becker start with future endowments, they get ambiguous comparative statics of wealth increases. It depends on how changes in future wealth are distributed across the two states.⁴

The study of the comparative statics of insurance demand, and in particular the possible inferiority of insurance coverage, was reconsidered by Chesney and Loubergé (1986), Eeckhoudt and Venezian (1990), Meyer and Ormiston (1995), Eeckhoudt, Meyer and Ormiston (1997) and Meyer and Meyer (2004). In diverse settings, these models all reconsider the idea that, when the effect of an increase in wealth is studied, the insurable loss should be held constant.⁵ Chesney and Loubergé (1986) focussed on full insurance coverage and considered how the willingness to buy full insurance coverage is affected when the amount of wealth and its composition are simultaneously affected by some exogenous factor.⁶ Eeckhoudt and Venezian (1990) consider a two-state model in which a risky asset and an insurance contract that is limited to a linear indemnity (that is, pure co-insurance) on that risk are purchased simultaneously. Because this model returns corner solutions for many interesting parameter values, it is not particularly suited to the study of comparative statics. Meyer and Ormiston (1995) study what is a very similar model to Eeckhoudt and Venezian, but in a continuous state environment that avoids the corner solution problem noted by Eeckhoudt and Venezian. Meyer and Ormiston's model assumes that the individual is endowed with risk free wealth and with a unit of a risky asset, and that insurance can only be paid for by selling a part of this risky asset. Hence, the model provides the opposite pole to the Mossin model, in which the individual is also endowed with risk free wealth and with a risk, but

⁴ Naturally, Ehrlich-Becker can be "translated" into our model by a simple redefinition of terms. In the same way, our model can also be similarly translated into the state-preference approach of Hirshleifer (1965, 1966).

⁵ That insurance is not necessarily an inferior good under DARA was pointed out in other models. For example, Dionne and Eeckhoudt (1984) noted this possibility in a two-state model investigating the substitutability between insurance and saving. Much more recently, Cummins and Mahul (2004) also noted this possibility in their study of the optimal demand for insurance with an upper limit on coverage.

⁶ Cleeton and Zellner (1993) extend the analysis of Chesney and Loubergé to partial insurance using the Ehrlich and Becker framework. They conclude that "only in a *very* restricted set of cases can insurance be categorized as a normal good, i.e. the demand for insurance curve shifts outwards at all prices as income increases." (p. 155, their italics).

there insurance is only paid for out of risk free wealth. The change in source of payment for insurance coverage turns out not to affect the comparative statics of the demand for insurance in a particularly important manner. Concretely it is still true that the optimal level of coverage is a decreasing function of risk free wealth conditional upon the utility function displaying DARA, and so once again we get the result that insurance coverage can be inferior.

Next, the paper by Eeckhoudt, Meyer and Ormiston (1997) studies the case of an individual who simultaneously allocates some non-random wealth over the purchase of a safe asset, a risky asset and an insurance product for this risk. Hence, in the model the simultaneous choice of risk and insurance is considered. The risk is determined by a continuous probability density function over finite limits, and the individual's choice is made to maximise the expected utility of final wealth. Both purchases of the risky asset and the insurance premium are paid for from risk free wealth, but clearly at the optimal portfolio there is no requirement for all wealth to be spent on the risky good and insurance, that is, some wealth may be retained for risk free enjoyment. The complexity of the situation assumed in Eeckhoudt, Meyer and Ormiston does not allow for the final solution to be found, but it does allow for the comparative statics to be analysed. In particular, the familiar result that DARA preferences condition the effect of an increase in wealth on the amount of insurance purchased is again found.

Finally, Meyer and Meyer (2004) concentrate on the effects on the demand for insurance of the composite commodity theorem, and they note that when the second good (that which is not the insurance product) is a composite portfolio of risky and riskless assets, and if the amount of this composite good is allowed to change but the relative proportions within this portfolio that are held as risky and non-risky assets is held fixed, then the comparative statics of the demand for insurance, at least as far as normality goes, become far more reasonable. Concretely, insurance becomes normal whenever relative risk aversion is nondecreasing, which is a characteristic that is satisfied by a wide range of often assumed utility types, including many that are decreasingly absolute risk averse. Meyer and Meyer also go on to consider other comparative statics results, some of which will be discussed below.

The paper proceeds in the following way; in the next section a very simple model of insurance

with a budget constraint is explored. In section 3 the actuarially fair premium and the issue of partial coverage are discussed. Then, in section 4 we consider the comparative statics that emerge from the model, and in section 5 these results are contrasted with those that are found in the other models in the literature. Above all, the characteristics of both insurance coverage and the risky good (normality, ordinarity, etc.) are considered using traditional demand theory. Section 6 concludes and offers some suggestions for future research.

2 The model

Consider an indirect utility setting (i.e. the only good present is money). An individual is endowed with a unique opportunity to invest in a productive risky asset, or risky project. Let x be the amount invested in this asset and let v be the price per unit of this asset. Risk impinges upon asset x in the sense that there are two states of nature; in the no-loss state the amount purchased of the asset becomes available as terminal wealth (terminal price per unit is 1), and in the loss state a proportion a of the amount purchased of the asset disappears. For example, the risky asset could be a forest of known terminal value subject to the risk of partial damage due to fire during the period; the risky project could be a foreign investment subject to political risk, or an international trade project subject to export credit risk. The loss state occurs with known probability p , where $0 < p < 1$. In addition, insurance coverage against the loss state can be purchased at an amount c at unit price of π . In the loss state, the individual receives an indemnity payment of c .⁷

The individual is also endowed with a non-random amount of initial wealth, w . This includes her borrowing capacity, which is assumed finite. For simplicity, we also assume an interest rate of zero, the same riskless interest rate being applicable for borrowing and lending. The individual is free to invest wealth w , including the borrowed amount, according to her preferences. She can choose to invest the total amount in the risky asset, with or without insurance. She can also keep part of this wealth, to an amount z , invested in the riskless asset, available at a unit price of 1, and invest the rest in the risky asset, with or without insurance.⁸ Given this, terminal wealth in

⁷ Since the model is restricted to only two states of nature, there is no formal difference between coverage being based on a deductible, or on co-insurance. Either of these options can be accommodated by the model.

⁸ To keep the model simple, we assume away any constraint on investment and insurance decisions that could

state 1 is $x + z$, and terminal wealth in state 2 is $(1 - a)x + c + z$.

We restrict the choice of coverage to be not greater than the proportion of the risky asset that may be lost, i.e. $c \leq ax$. In addition, we impose $x \geq 0$ and $c \geq 0$; there exists no market for short-selling of the risky asset and the insured cannot sell insurance. The problem of the individual is to maximize the expected utility that she gets from the choice of x , c and z subject to her budget constraint. The general problem is

$$\begin{aligned} \max_{x,c,z} & (1-p)u(x+z) + pu((1-a)x+c+z) \\ \text{s.t.} & w \geq vx + \pi c + z \text{ and } c \leq ax \end{aligned}$$

which leads to the following general result, first formally noted by Eeckhoudt, Meyer and Ormiston (1997):⁹

Lemma 1 *In this setting, either $c^* = 0$ or $z^* = 0$, or both. Concretely, if $v + a\pi > 1$, $c^* = 0$, and if $v + a\pi < 1$, $z^* = 0$.*

Proof. See Appendix A. ■

Naturally, there is nothing surprising about this lemma. It simply states that purchasing a unit of wealth to be received in each of the states of nature is equivalent to purchasing a unit of risk free wealth (i.e. these two strategies are perfect substitutes), and so only the least expensive alternative will survive.¹⁰ Either insurance will not be purchased, or the risk free asset will be ignored.

Note that we consider a simple optimisation problem for an individual in a single period. In this period, insurance, the safe asset and the risky asset are available to the individual at prices π , 1, and v respectively. We do not justify their existence, nor the possible existence of other risky assets in the economy. Our objective is to focus on the individual demand for insurance for a single insurable and productive risky asset, or project. Given her investment opportunity, the individual faces two states of nature. As the risky asset and insurance on this asset span the relevant set of states of nature for our individual, the third asset – the riskless asset – is redundant

arise from borrowing. Introducing these kinds of constraints would not change the results of the paper, as long as the individual's own initial wealth is strictly positive.

⁹ See also Eeckhoudt and Venezian (1990) and Meyer and Ormiston (1995).

¹⁰ Indeed, the result generalises to any number of states of nature, so long as the insurance indemnity is a linear function of the loss.

in this situation. If this asset exists nevertheless, and if $v + a\pi \neq 1$, an arbitrage opportunity arises, but this opportunity is restricted by constraints – the individual cannot short-sell the risky asset, or sell insurance. She also has a limited borrowing capacity.

As our interest is in the demand for insurance, we will assume from now on that $v + a\pi \leq 1$. Thus the individual chooses to invest the leveraged amount w in the risky asset and insurance. We can limit our attention to the choice between x and c , as the optimal solution will always set $z^* = 0$. Clearly, the model as presented is an application of the contingent claims scenario where the individual's terminal wealth is $(1 - a)x + c$ in the loss state and x in the no-loss state.¹¹

3 Optimisation and the Actuarially Fair Premium

The problem of the individual is to maximize the utility that she gets from the choice of both x and c , subject to her budget constraint, and to the constraint that coverage is not greater than the insurable loss, $c \leq ax$. Concretely, the problem is

$$\max_{x,c} U(x, c) \equiv (1 - p)u(x) + pu((1 - a)x + c)$$

$$\text{subject to } vx + \pi c \leq w \text{ and } c \leq ax$$

We will indicate the solution to this problem by the vector (x^*, c^*) .

It turns out that the comparative statics of this model are easier to work through if we introduce a simple change of variable. Let the no-accident state (occurs with probability $1 - p$) be state 1, and the accident state (occurs with probability p) be state 2. Let us denote state 2 consumption by y , so that

$$y \equiv (1 - a)x + c$$

In this case, we have $c = y - (1 - a)x$, and so the budget constraint becomes $w \geq vx + \pi(y - (1 - a)x) = (v - \pi(1 - a))x + \pi y$. We interpret $v - \pi(1 - a)$ to be the “net” price of state 1 consumption, and we denote it by $v - \pi(1 - a) \equiv q$. The problem is now to maximise $(1 - p)u(x) + pu(y)$ under

¹¹ To avoid confusions, we stress that our model is an optimisation model, **not** a general equilibrium model. We do **not** assume a two-state economy. Given her opportunity to invest in a productive project, or in a risky asset with positive expected return, the decision-maker faces two states of nature and two contingent claims with given prices. As noted in the introduction, our model is comparable to Ehrlich and Becker's (1972) model, except that we consider a decision made at an initial date 0, leading to two possible states of nature at date 1, instead of starting with state-contingent future endowments.

the conditions that $w \geq qx + \pi y$ and $y \leq x$. We will indicate the solution to this problem by the vector (x^*, y^*) .

Since the objective function is concave, and the restrictions are linear, we know that there exists a unique optimum for the problem. The Lagrangian is:

$$L(x, y, \delta) = (1 - p)u(x) + pu(y) + \delta_1(w - qx - \pi y) + \delta_2(x - y)$$

where δ_1 is the multiplier corresponding to the budget constraint, and δ_2 is the multiplier corresponding to the coverage restriction (naturally, we restrict $\delta_i \geq 0$ for $i = 1, 2$). The first order conditions for the optimum are

$$\frac{\partial L}{\partial x} = (1 - p)u'(x^*) - q\delta_1 + \delta_2 = 0 \quad (1)$$

$$\frac{\partial L}{\partial y} = pu'(y^*) - \pi\delta_1 - \delta_2 = 0 \quad (2)$$

and the complementary slackness conditions are

$$\delta_1(w - qx^* - \pi y^*) = 0 \quad \text{and} \quad \delta_2(x^* - y^*) = 0 \quad (3)$$

Firstly, note that from (1) it must be true that $\delta_1 > 0$, so the budget constraint must always saturate:

$$w = qx^* + \pi y^* \quad (4)$$

Secondly, assuming an interior solution ($\delta_2 = 0$), from (1) and (2) we get the tangency condition in state contingent space:¹²

$$\frac{u'(x^*)}{u'(y^*)} = \frac{pq}{(1 - p)\pi} \quad (5)$$

Clearly, an interior solution can only occur if $q = v - \pi(1 - a) > 0$, which will be assumed from now on.¹³

3.1 Full and partial coverage

We begin by studying the relationship between full/partial coverage, and the insurance premium.

¹² Of course, this is nothing more than the usual condition of equality between the marginal rate of substitution and the ratio of state-contingent prices.

¹³ Recall that, we are already assuming $v + a\pi \leq 1$, and now we also have $v + a\pi > \pi$, so in short, our assumption is $\pi < v + a\pi \leq 1$.

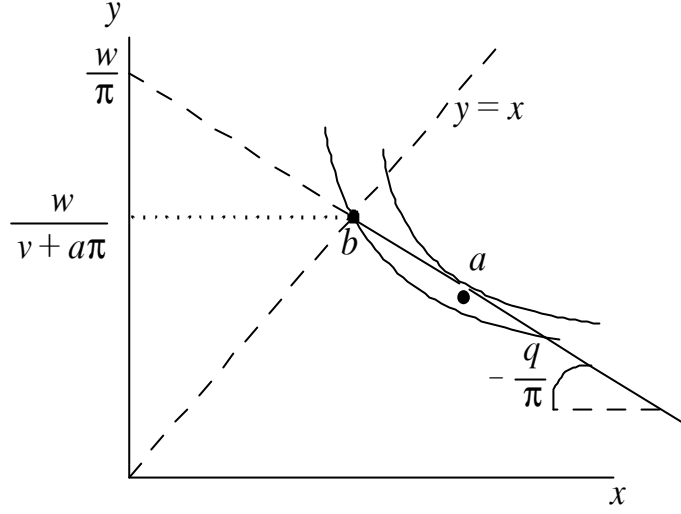


Figure 1:

Proposition 1 *In the optimal solution,*

$$\begin{aligned} \pi &\leq \frac{pv}{1-pa} \implies c^* = ax^* \\ \pi &> \frac{pv}{1-pa} \implies c^* < ax^* \end{aligned}$$

Proof. For an interior solution, the optimum is given by the simultaneous solution to (4) and (5). Firstly, consider the case $(1-p)\pi \leq pq$. In this case the right-hand-side of (5) is less than or equal to 1. This would indicate that $y^* \geq x^*$, but since the second constraint does not allow y to go above x , we get the corner solution in which $y^* = x^*$, that is $c^* = ax^*$. Recalling that $q = v - \pi(1-a)$, it turns out that $(1-p)\pi \leq pq$ reorders directly to $\pi \leq \frac{pv}{1-pa}$, the first statement of the proposition is proved.

Secondly, consider $(1-p)\pi > pq$, that is $\pi > \frac{pv}{1-pa}$. In this case the right-hand-side of (5) is greater than 1, which can only imply $y^* < x^*$, that is $c^* < ax^*$. ■

Proposition 1 is illustrated in figure 1 in state contingent consumption space. The straight line is the budget constraint with slope $-\frac{a}{\pi}$, and the indifference curves have slope $-\frac{(1-p)u'(x)}{pu'(y)}$. Point a , at tangency, is the optimal solution, where $y < x$: partial insurance is purchased. This is preferred to point b on the certainty line, where $x = y = \frac{w}{v+a\pi}$. The proposition is interesting since it clearly shows how the insurance decision depends on the investment opportunity faced by the individual. For this reason, the individual does not necessarily purchase full insurance at the

actuarially fair premium from the insurer's point of view.

3.2 The actuarially fair premium

Clearly, in our model, as in any model of the demand for insurance, the actuarially fair premium is a central concept. An actuarially fair premium is defined as the premium which preserves the expected value of an agent's terminal wealth, independently of the amount of coverage that is purchased. In the standard model of insurance demand, where the risk is exogenous and the wealth is invested in the riskless asset, the actuarially fair premium π is equal to the probability of loss, p .¹⁴ This actuarial premium is the same for the insured and the insurer. For the latter, an actuarially fair premium sets his expected profit equal to 0, independently of the amount of coverage that is provided. As the insurer receives a certain premium income of πc , invested in the riskless asset, and suffers an expected cost corresponding to the expected indemnity, pc , his expected profit is simply $c(\pi - p)$, which is equal to 0 for all possible levels of c only when $\pi = p$. In that model, it is shown that the insured purchases full coverage only if $\pi \leq p$. Thus, the individual is fully covered only if the insurer earns a non-positive expected profit on the contract.

Now consider the actuarially fair premium from the point of view of our insured individual. As was shown above, whatever are the optimal values of the objective variables, the budget constraint saturates in the optimum, that is in the solution to the problem we have $w = vx^* + \pi c^*$, which can be written as $x^* = \frac{w - \pi c^*}{v}$. Given this, the individual's expected terminal wealth is given by $(1-p) \left(\frac{w - \pi c^*}{v} \right) + p \left((1-a) \left(\frac{w - \pi c^*}{v} \right) + c^* \right) = \left(\frac{w - \pi c^*}{v} \right) (1 - pa) + pc^*$. Now, if this expected value is constant independently of the value of c^* , then it must be true that its derivative with respect to c^* is 0, that is, $\frac{\pi(1-pa)}{v} = p$. Defining π_{ind}^a to be the premium that achieves this, we have the result that from the client's point of view, the fair premium is

$$\pi_{ind}^a = \frac{pv}{1 - pa}$$

This may be larger than (equal to, smaller than) p depending on whether $v > (=, <) 1 - pa$.

In particular, assuming a positive expected return for the productive risky asset, or project, we

¹⁴ In that model, the individual's expected terminal wealth under an insurance policy that pays out c in the event of a loss L , against payment of a premium πc , is given by $Ew_1 = w_0 - \pi c - pL + pc$. This is constant for any level of c only if $\pi = p$.

have $v < (1 - p) + p(1 - a) = 1 - pa$, and so $\pi_{ind}^a < p$.

Note that this actuarially fair premium, for our insured individual, is the critical premium that appears in Proposition 1. The fact that the individual will demand full coverage whenever offered a premium that is not greater than her actuarially fair premium, and partial coverage otherwise, is perfectly in line with the same result in the standard model of insurance. However, the difference is that this actuarially fair premium is related to the investment opportunity faced by the individual. The source of this difference is the introduction of an explicit budget constraint in the insured's problem. This makes the opportunity cost of insurance visible. Here, the insured's full insurance premium, $\frac{pv}{1-pa}$, is simply the expected value of one unit of insurance indemnity, p , divided by the expected return on the insured asset, $\frac{1-pa}{v}$. It is the present value of the insurance indemnity, in terms of the risky asset – the asset used for investment.

The actuarially fair premium may not be the same – in general, it will not be the same – as the insurer's actuarially fair premium. We do not model the insurer's problem here, but we know from the literature on optimal insurance supply – see, for example, D'Arcy and Doherty (1988), or Cummins and Phillips (2000) – that the “fair” (actuarial) premium, defined as the premium which preserves the present expected value of the insurer, takes into account the insurer's investment activity and his balance sheet constraint. For one policy, it is given by the expected indemnity divided by the equilibrium expected return on the insurer's investment portfolio. If we denote this expected return by ER , the “fair” premium for a promised payment of 1 unit of insurance indemnity with probability p is $\pi_{ins}^a = \frac{p}{ER}$. This premium would coincide with the individual's full insurance premium, π_{ind}^a , only if the insurer were to invest all of his equity and premium reserve in the same risky asset used by the insured for her investment. Prudential considerations will prevent the insurer from following such an investment rule. In general, π_{ind}^a will be larger than (equal to, smaller than) π_{ins}^a as $\frac{1-pa}{v}$ is smaller than (equal to, larger than) ER , the equilibrium expected return on the insurer's diversified portfolio of assets. Thus, when investment opportunities under a budget constraint are taken into account in the individual's insurance decision, it is possible for the individual to insure fully at a premium returning a positive expected profit to the insurer. In addition, a nice link is created between the theory of optimal insurance demand and fair rating

considerations in the theory of insurance supply.

4 Comparative statics

Of particular interest to us are the comparative statics of the solution vector as the parameters of the system are altered. Perhaps the most interesting of these comparative statics exercises is the effect of changes in w on the optimal values of x and c , but we shall also consider changes in other parameters, concretely the effect of a change in the unit price of insurance (π), in the fraction of risky asset that is at risk (a), in the probability of the loss state (p), and a change in risk aversion (captured by the curvature of the utility function, u). For this entire section, we assume that the solution is interior, that is, we assume $\pi > \frac{pv}{1-pa}$.

4.1 Normality of insurance

We begin by considering the normality of the goods in our model. Recall, that in the traditional model (where the amount of risky good is held constant), under DARA insurance is inferior. Insurance is also inferior under DARA in the models of Meyer and Ormiston (1995) and Eeckhoudt, Meyer and Ormiston (1997).¹⁵ Meyer and Meyer (2004) find that insurance is normal, even under DARA, if relative risk aversion is non-decreasing. In our model we get:

Proposition 2 $\frac{\partial c^*}{\partial w} \geq 0$ as $R_a(x^*) \geq (1-a)R_a(y^*)$, and $\frac{\partial x^*}{\partial w} > 0$ always.

Proof. Differentiating the budget constraint (4) with respect to w conditional on being at a solution gives

$$q \frac{\partial x^*}{\partial w} = 1 - \pi \frac{\partial y^*}{\partial w} \quad (6)$$

Secondly, differentiating the tangency condition (5) with respect to w conditional on being at a solution, and simplifying gives

$$R_a(y^*) \frac{\partial y^*}{\partial w} = R_a(x^*) \frac{\partial x^*}{\partial w} \quad (7)$$

¹⁵ Actually, in Eeckhoudt, Meyer and Ormiston (1997), DARA preferences leads to the conclusion that it is very likely that insurance is inferior (concretely, either insurance is inferior, or the risky asset is normal, or both).

where $R_a(x)$ is the Arrow-Pratt measure of absolute risk aversion. Combining these two equations, we get

$$\frac{\partial x^*}{\partial w} = \frac{R_a(y^*)}{qR_a(y^*) + \pi R_a(x^*)} \quad (8)$$

$$\frac{\partial y^*}{\partial w} = \frac{R_a(x^*)}{qR_a(y^*) + \pi R_a(x^*)} \quad (9)$$

Clearly, from (8), it always happens that the optimal purchase of the risky good is increasing in wealth in this model.

Finally, since $\frac{\partial c^*}{\partial w} = \frac{\partial y^*}{\partial w} - (1-a)\frac{\partial x^*}{\partial w}$, we have

$$\frac{\partial c^*}{\partial w} = \frac{R_a(x^*) - (1-a)R_a(y^*)}{qR_a(y^*) + \pi R_a(x^*)} \quad (10)$$

Thus,

$$\frac{\partial c^*}{\partial w} \geq 0 \text{ as } R_a(x^*) \geq (1-a)R_a(y^*) \quad (11)$$

■

The necessary and sufficient condition for normality/inferiority of insurance indicated in this proposition, $R_a(x^*) \geq (1-a)R_a(y^*)$, can be written in several different interesting ways. These can in turn be used to provide for other sufficient conditions. Firstly, simple but direct re-ordering of (11) reveals that it is equivalent to

$$\frac{\partial c^*}{\partial w} \geq 0 \text{ as } a \geq \frac{R_a(y^*) - R_a(x^*)}{R_a(y^*)} \quad (12)$$

That is, insurance is normal if the fraction of the risky asset that is at risk is greater than the relative change in absolute risk aversion from state of nature 2 to state of nature 1.

Secondly, note that if we multiply condition (11) by x^* , and then sum $c^*R_a(y^*)$ to each side, it reads

$$\frac{\partial c^*}{\partial w} \geq 0 \text{ as } R_r(x^*) + c^*R_a(y^*) \geq R_r(y^*)$$

where $R_r(\cdot)$ is the Arrow-Pratt measure of relative risk aversion. This reorders to¹⁶

$$\frac{\partial c^*}{\partial w} \geq 0 \text{ as } c^*R_a(y^*) \geq R_r(y^*) - R_r(x^*) \quad (13)$$

¹⁶ $cR_a(y)$ is related to, but not equal to, the measure of partial risk aversion introduced by Menezes and Hanson (1970). Their measure, defined as $tR_a(w+t)$, is based on the case where an exogenous risk t is added to riskless initial wealth w .

Thirdly, directly from (13), dividing both sides by $y^*R_a(y^*)$ we get

$$\frac{\partial c^*}{\partial w} \geq 0 \quad \text{as} \quad \frac{c^*}{y^*} \geq \frac{R_r(y^*) - R_r(x^*)}{R_r(y^*)} \quad (14)$$

where the right-hand-side is just the relative change in relative risk aversion over the two states of nature.

Sufficient, but not necessary, conditions for normality of insurance can be obtained directly from any of these equivalent expressions. Three such conditions are:

Corollary 1 *If $a = 1$, then insurance is always a normal good.*

Proof. The proof is immediate from (12), the right-hand-side of which is strictly less than 1. Also, directly from (10) the same result can be found. ■

This result is expected – when the loss state has a total loss, state 2 consumption is simply the insurance indemnity, and it is well known that under separable utility, all state contingent consumptions are normal.

Corollary 2 *Insurance is a normal good if the utility function displays non-decreasing absolute risk aversion.*

Proof. Since (by assumption) we are at an internal solution, we have $y^* < x^*$. Thus, under non-decreasing absolute risk aversion, we have $R_a(x^*) \geq R_a(y^*)$, and so (with $a > 0$) it is always true that $R_a(x^*) > (1 - a)R_a(y^*)$. ■

This result is also to be expected, it also holds in the traditional model of insurance.

Corollary 3 *Insurance is a normal good if the utility function displays DARA and non-decreasing relative risk aversion (assuming some insurance is purchased).*

Proof. Directly from (13), since the left-hand-side is strictly positive when some insurance is purchased, we get the result that a sufficient condition for insurance to be a normal good is that the right-hand-side is negative, which holds if $R'_r \geq 0$ (since at an internal solution $y^* < x^*$). ■

This third corollary mirrors the result of Meyer and Meyer (2004), however normality of insurance occurs for a wider range of utility types than those displaying non-decreasing relative risk aversion. If absolute risk aversion is high enough, there is room for the true condition, (13), to be satisfied, even under decreasing relative risk aversion.

It is interesting to note that out of all the models mentioned, only in Meyer and Meyer (2004) and here is non-decreasing relative risk aversion a sufficient condition for normality of insurance. This may seem strange since the normality of insurance is determined by changes in the purchased *amount* of insurance, when wealth changes. It is naturally linked to absolute risk aversion, which governs the *amount* of risky assets in the overall portfolio. Thus, in the standard model and in Meyer and Ormiston (1995), a wealth increase means an exogenous increase in safe wealth, the risky portion of wealth being given. Under DARA, this will necessarily lead to a counterbalancing reduction in the optimal amount of insurance to yield an increase in the amount at risk. In contrast, relative risk aversion governs the optimal *proportion* of riskless assets in the overall portfolio. It is not directly linked to the normality of insurance.

However, Meyer and Meyer is a model that allows for all three assets to be present (risky asset, risk free asset, and insurance), but that holds the relative proportions of the investment portfolio that is held as risky and riskless assets constant. Although our model has been described as simply one in which the loss state is partial, clearly such an assumption is exactly equivalent to the model of Meyer and Meyer, as that part of the risky asset that is not lost is clearly a riskless part of the investment. The reason why relative risk aversion drives the normality of insurance in these models is that there is a fixed *proportional* relationship between the risky (uninsured) investment and the riskless investment. In Meyer and Meyer, this is imposed by the composite commodity theorem, here it is imposed by a partial loss state. In the IARA and CARA cases, leading to increasing relative risk aversion (IRRA), the amount dedicated to the risky investment does not increase when wealth increases. This implies that the amount allocated to the riskless investment cannot increase either. All of the wealth increase (more than that in the IARA case) must then be allocated to the insured risky asset: the demand for insurance increases. Under DARA, some additional wealth is allocated to the risky (uninsured) asset. But then the constant proportionality with the riskless investment implies that some of the additional wealth must be allocated to the riskless investment. Under IRRA and CRRA, this implies further that the demand for insurance must increase (as the share of insured assets in the portfolio must increase or remain constant). Under decreasing relative risk aversion (DRRA), this is not necessarily the case. To obtain the

desired more risky allocation, given that the relationship between risky and riskless assets in the composite portfolio has remained the same, it may be necessary to decrease the amount invested in insured assets.

As any normal good is ordinary, if the condition for normality of insurance is satisfied, then insurance is ordinary. For example, in our model if we assume $a = 1$, clearly insurance is a normal good from Corollary 1. However, even if insurance is inferior, it still may turn out to be ordinary. Also, from Corollary 3, it turns out that insurance is an ordinary good if relative risk aversion is non-decreasing, *whatever its value*.

4.2 Ordinarity of insurance

We now go on to consider the ordinarity of insurance.

Proposition 3 $\frac{\partial c^*}{\partial \pi} \gtrless 0$ as $c^* [(1-a)R_a(y^*) - R_a(x^*)] \gtrless \frac{a^2}{\pi q}$.

Proof. The proposition can be proved directly from the Slutsky equation;

$$\frac{\partial c^*}{\partial \pi} = \frac{\partial c^h}{\partial \pi} - c^* \frac{\partial c^*}{\partial w} \quad (15)$$

where c^h is the Hicksian demand for insurance (that is, the demand that eventuates when the cost of (c, x) is minimised subject to utility being no less than some minimum).

Consider the Hicksian problem defined in the space (x, y) :

$$\min_{x,y} qx + \pi y \quad \text{subject to} \quad pu(y) + (1-p)u(x) \geq \bar{u}$$

The solution, (x^h, y^h) is given by the simultaneous solution to the equation

$$pu(y) + (1-p)u(x) = \bar{u} \quad (16)$$

and the corresponding tangency condition

$$\frac{u'(x^h)}{u'(y^h)} = \frac{pq}{(1-p)\pi} \quad (17)$$

Differentiating the utility constraint with respect to π yields

$$pu'(y^h) \frac{\partial y^h}{\partial \pi} + (1-p)u'(x^h) \frac{\partial x^h}{\partial \pi} = 0$$

Dividing this by $u'(y^h)$, and using (17) give us the result that at the Hicksian solution we have

$$p \frac{\partial y^h}{\partial \pi} + (1-p) \left(\frac{pq}{(1-p)\pi} \right) \frac{\partial x^h}{\partial \pi} = 0$$

that is

$$\frac{\partial y^h}{\partial \pi} = -\frac{q}{\pi} \frac{\partial x^h}{\partial \pi} \quad (18)$$

Secondly, differentiating the tangency condition (17) with respect to π , and recalling that $q = v - \pi(1-a)$ we get

$$\begin{aligned} \frac{u''(x^h)u'(y^h)\frac{\partial x^h}{\partial \pi} - u'(x^h)u''(y^h)\frac{\partial y^h}{\partial \pi}}{u'(y^h)^2} &= \frac{-p(1-a)(1-p)\pi - p(v - \pi(1-a))(1-p)}{\pi^2(1-p)^2} \\ &= \frac{-pv}{\pi^2(1-p)} \end{aligned}$$

However, multiplying by the marginal utility of state 2 consumption and simplifying, we get

$$u''(x^h)\frac{\partial x^h}{\partial \pi} + u'(x^h)R_a(y^h)\frac{\partial y^h}{\partial \pi} = -\frac{u'(y^h)vp}{\pi^2(1-p)}$$

Using the tangency condition itself, we have

$$\begin{aligned} u''(x^h)\frac{\partial x^h}{\partial \pi} + u'(x^h)R_a(y^h)\frac{\partial y^h}{\partial \pi} &= -\frac{u'(x^h)(1-p)\pi}{pq} \frac{vp}{\pi^2(1-p)} \\ &= -u'(x^h)\frac{v}{\pi q} \end{aligned}$$

And then dividing by the negative of marginal utility in state 1 yields

$$R_a(x^h)\frac{\partial x^h}{\partial \pi} - R_a(y^h)\frac{\partial y^h}{\partial \pi} = \frac{v}{\pi q} \quad (19)$$

The simultaneous solution to (18) and (19) is given by

$$\begin{aligned} \frac{\partial x^h}{\partial \pi} &= \frac{1}{D^h} \frac{v}{q} \\ \frac{\partial y^h}{\partial \pi} &= -\frac{1}{D^h} \frac{v}{\pi} \end{aligned}$$

where $D^h \equiv \pi R_a(x^h) + q R_a(y^h)$. Finally, from the equation that defines y , we have $\frac{\partial c^h}{\partial \pi} = \frac{\partial y^*}{\partial \pi} - (1-a)\frac{\partial x^h}{\partial \pi}$. Thus

$$\frac{\partial c^h}{\partial \pi} = \frac{1}{D^h} \left(-\frac{v}{\pi} - (1-a)\frac{v}{q} \right)$$

Using the fact that $q = v - \pi(1-a)$, this simplifies to

$$\frac{\partial c^h}{\partial \pi} = -\frac{1}{D^h} \frac{v^2}{\pi q}$$

Substituting this into the Slutsky equation, and using (10) we get

$$\begin{aligned}\frac{\partial c^*}{\partial \pi} &= -\frac{1}{D^h} \frac{v^2}{\pi q} - c^* \frac{R_a(x^*) - (1-a)R_a(y^*)}{qR_a(y^*) + \pi R_a(x^*)} \\ &= \frac{1}{D} \left[c^* ((1-a)R_a(y^*) - R_a(x^*)) - \frac{v^2}{\pi q} \right]\end{aligned}\quad (20)$$

where $qR_a(y^*) + \pi R_a(x^*) \equiv D$, and $D = D^h$ since the solution to the Hicksian problem and the solution to the utility maximisation problem coincide for the appropriate level of initial wealth. Since $D > 0$, the proposition follows directly from (20). ■

Proposition 3 provides a link between the necessary and sufficient conditions for inferiority of insurance (Proposition 2) and the necessary and sufficient conditions to get insurance as a Giffen good. The proposition confirms that the value of $(1-a)R_a(y^*)$ is critical. If it is larger than $R_a(x^*)$, insurance is inferior. If it is even larger (larger than $R_a(x^*) + \frac{v^2}{q\pi c^*}$), insurance is a Giffen good. As a corollary, when the proportion of the investment subject to risk, a , is high enough, insurance is a normal good.

Again, other sufficient (but not necessary) conditions can be obtained. We shall now mention two possibilities:

Corollary 4 *If insurance is an inferior good, it is still ordinary if $\frac{a}{q} \geq R_r(y^*)$.*

Proof. The full sufficient and necessary condition for ordinarity of insurance is (from Proposition 3)

$$c^* ((1-a)R_a(y^*) - R_a(x^*)) < \frac{v^2}{\pi q} = \frac{v}{q} \frac{v}{\pi}$$

However, since $c^* R_a(x^*) > 0$, it is sufficient that

$$c^* (1-a)R_a(y^*) < \frac{v}{q} \frac{v}{\pi}$$

Dividing by $(1-a)$, gives the condition

$$c^* R_a(y^*) < \frac{v}{q} \frac{v}{(1-a)\pi} \quad (21)$$

Since $v - \pi(1-a) > 0$, we have $\frac{v}{\pi(1-a)} > 1$, and so $\frac{a}{q} \frac{v}{(1-a)\pi} > \frac{a}{q}$. And finally, since $y^* > c^*$, we have $R_r(y^*) = y^* R_a(y^*) > c^* R_a(y^*)$. Thus, over all, so long as $R_r(y^*) \leq \frac{a}{q}$, we have

$$c^* R_a(y^*) < R_r(y^*) \leq \frac{a}{q} < \frac{v}{q} \frac{v}{(1-a)\pi}$$

■

It is interesting to note that this particular sufficient condition for ordinarity of insurance also conditions $\frac{\partial x^*}{\partial \pi}$ to be positive. To see this, simply note that the Slutsky equation for the effect of an increase in π on x^* is given by

$$\frac{\partial x^*}{\partial \pi} = \frac{\partial x^h}{\partial \pi} - c^* \frac{\partial x^*}{\partial w}$$

Using the fact that (noted in the proof of Proposition 3) $\frac{\partial x^h}{\partial \pi} = \frac{1}{D^h} \frac{v}{q}$, equation (8), and the definition of D , we have

$$\frac{\partial x^*}{\partial \pi} = \frac{1}{D} \left[\frac{v}{q} - c^* R_a(y^*) \right]$$

Thus, $R_r(y^*) \leq \frac{v}{q}$ implies that $\frac{\partial x^*}{\partial \pi} > 0$ and $\frac{\partial c^*}{\partial \pi} < 0$ simultaneously. In terms of classical demand theory, insurance and the insurable risky asset would be “net substitutes”. This is worth noting, as it is often thought that insurance, and the item being insured are complementary.¹⁷

The sufficient condition in Corollary 4 is also interesting since it clearly shows that insurance can be ordinary when relative risk aversion is greater than 1. Since the term $\frac{v}{q} = \frac{v}{v-\pi(1-a)}$ is clearly greater than 1, there is room for relative risk aversion to be greater than 1 while still allowing insurance to be ordinary. This is important, since much of the existing literature (see Meyer and Ormiston (1995), Eeckhoudt, Meyer and Ormiston (1997), and Meyer and Meyer (2004)) provide sufficient conditions for insurance to be ordinary that rely on relative risk aversion being less than 1, something that is not particularly likely given empirical estimates.¹⁸

It is interesting to consider exactly how high relative risk aversion can get before insurance becomes Giffen. Take the following example; say the probability of the loss state and the proportion of the asset at risk are both one half, that is $p = a = 0.5$. Finally, assume $\pi = 0.75$. Now there are four conditions that the price of the risky good must satisfy: for an internal solution $\pi > \frac{pv}{1-pa}$

¹⁷ Meyer and Meyer (2004) justify making the amount of risk endogenous by stating “In a standard consumer setting, making this assumption [that the risk is exogenous] is similar to determining the demand for hotdog buns, allowing the quantity of soda to vary, but holding the number of hotdogs fixed.” The connotation that perhaps insurance and the good insured are complementary is clear. It is a fact in this model that insurance would not exist if no risky asset was purchased (or no risky investment undertaken). In this sense, insurance and the risky asset are complements. In spite of this, clearly they may react in opposite direction to an increase in the price of insurance, and thus be “net substitutes” in the sense of demand theory.

¹⁸ There is a large (and growing) literature dedicated to estimating the true size of relative risk aversion. In almost all cases, we get numbers that are almost constant, and located between about 1 and up to about 10 (see, for example, Mehra and Prescott (1985), Szpiro (1986), and Levy (1994), all of whom find that relative risk aversion is approximately constant, and is usually valued above 1).

which now reads $v < 1.125$, for a positive “net” price of state 1 consumption $v > \pi(1 - a) = 0.375$, for no riskless asset to be demanded $v + a\pi \leq 1$ which now reads $v \leq 0.625$, and for the risky asset to have a non-negative return $v \leq 1 - pa = 0.75$. Clearly, the first and the fourth conditions are redundant, since they are dominated by the third, and so all that must occur is $0.375 < v \leq 0.625$. Now, since $\frac{v}{v - \pi(1 - a)}$ is decreasing in v , it reaches its greatest value close to $v = 0.375$, and its lowest value at $v = 0.625$. That is, the condition is satisfied by all cases in which relative risk aversion is no greater than the value of $\frac{0.625}{0.625 - 0.75 \times 0.5} = 2.5$. If we use $v = 0.4$, the limit value on relative risk aversion is 16, while at $v = 0.5$, relative risk aversion can go as high as 4 and still satisfy the condition. And we really should recall that we are only discussing a sufficient condition here; the greater is the difference between c^* and y^* (that is, the greater is x^* and the smaller is a), relative risk aversion can go even higher and we may still satisfy the condition expressed in Proposition 3.

Nevertheless, it is also very easy to see that our model also satisfies the sufficient condition that is so frequently mentioned in the earlier literature:

Corollary 5 *If insurance is an inferior good, it is still ordinary if relative risk aversion is not greater than 1.*

Proof. Clearly, if relative risk aversion is uniformly less than 1, then $R_r(y^*) < 1$, which necessarily satisfies the condition in Corollary 4. ■

From Corollary 5, relative risk aversion larger than 1 is a necessary condition for insurance to be a Giffen good, a result already obtained by Hoy and Robson (1981).¹⁹

Indeed, and as a summary, if insurance is inferior, $0 < c^* R_a(y^*) < R_r(y^*) - R_r(x^*)$, but it turns out that $c^* R_a(y^*) < \frac{z}{q}$, which from (21) is a weaker (but with a less intuitive meaning) sufficient condition for ordinarity than that stated in Corollary 4, and taking 1 and $\frac{z}{q}$ as benchmarks for the value of relative risk aversion, three cases are possible:

1. Low and decreasing relative risk aversion; $0 < c^* R_a(y^*) < R_r(y^*) - R_r(x^*) < R_r(y^*) < 1 < \frac{z}{q}$.

¹⁹ See also Briys, Dionne and Eeckhoudt (1989).

2. High and decreasing relative risk aversion; $0 < c^* R_a(y^*) < R_r(y^*) - R_r(x^*) < R_r(y^*)$ and $1 < R_r(y^*) \leq \frac{v}{q}$.
3. Very high and decreasing relative risk aversion; $0 < c^* R_a(y^*) < R_r(y^*) - R_r(x^*) < R_r(y^*)$, $c^* R_a(y^*) < \frac{v}{q}$, and $1 < \frac{v}{q} < R_r(y^*)$.

4.3 Additional comparative statics results

The rest of the propositions consider aspects of the demand for insurance that, while still interesting and important, are not so critical as the normality and ordinarity of insurance. Most of the following results have been considered previously in the literature, and most of them are perfectly in line with intuition. All proofs of these results are given in Appendix B.

Firstly, if the proportion of the risky asset that may be lost increases, we get the result that less of it will be purchased, and it will be more heavily insured:

Proposition 4 $\frac{\partial x^*}{\partial a} < 0 < \frac{\partial c^*}{\partial a}$

Secondly, we consider how the optimal solution is affected by an increase in risk aversion. Following Pratt (1964), an increase in risk aversion can be studied by substituting the utility function $u(\cdot)$ by $h(u(\cdot))$ where h is a strictly increasing and strictly concave function from \mathbb{R}^1 to \mathbb{R}^1 .

Proposition 5 *An increase in risk aversion will increase the amount of coverage and decrease the amount of risky good purchased.*

The fact that an increase in risk aversion leads to both less risk being purchased and more insurance being purchased is, of course, entirely natural and in accordance with logic. It states that an increase in risk aversion will be accommodated by reducing the amount of risk undertaken, but that this reduction in risk will be split between the two possible manners in which it can be done – purchasing less risk to start out with, and insuring this risk more heavily – rather than relying on a single risk reduction method.

Next, we can consider how a change in the probability density defining the loss lottery will affect the optimal choice. Here, to begin with we do this holding the premium constant. That is, we consider the effect of an increase in the probability of loss when the insurer does not alter

the premium as a result. Later, we consider the situation in which the premium is also altered endogenously with the probability.

In the traditional model of wealth lotteries with fully endowed risk, an increase in the probability of loss which is not accompanied by an increase in the premium, and where partial coverage is optimal, will always lead to an increase in the amount of coverage purchased (see Mossin (1968)). As the next proposition shows, the same is true in the current model, although here we are also able to ascertain what happens to the optimal purchase of the risky asset.

Proposition 6 $\left. \frac{\partial c^*}{\partial p} \right|_{\pi} > 0 > \left. \frac{\partial x^*}{\partial p} \right|_{\pi}$.

The result that insurance coverage increases and the purchase of the risky asset decreases after an increase in the probability of the adverse state is very easy to see graphically (figure 1). Since an increase in p has the effect of making the indifference curves flatter at all points in (y, x) space, it will not alter the solution when the coverage constraint saturates, and it must move the solution to the left along the budget constraint when the coverage constraint does not saturate. But if y increases and x decreases, we know that c must increase.

However, as has been argued previously, it is not reasonable to assume that if the probability of the loss state increases, the premium will not increase also (Jang and Hadar (1995)). In the traditional model of insurance under such conditions the sign of the effect on insurance coverage is indeterminate, although the sign of the change does depend on the slope of absolute risk aversion. Concretely, in order to get a determinate effect, absolute risk aversion cannot be decreasing. In the present model, things get very complicated very quickly, so we shall only present an analysis pertaining to the special case of a full loss in state 2, that is $a = 1$.

Proposition 7 *Assume that $a = 1$, and that the insurer increases the premium π in response to an increase in the probability of loss, p ; that is $\frac{\partial \pi}{\partial p} > 0$. Then if relative risk aversion is not less than 1, an increase in p will reduce the value of x^* . On the other hand, the value of c^* will increase, decrease or not change as $(1 - p) \left(1 + \left(\frac{\pi c^*}{v x^*} \right) R_r(x^*) \right) \frac{\partial \pi}{\partial p}$ is less than, greater than, or equal to $\frac{\pi}{p}$.*

One interesting point to note about this result is that, in the traditional model the sign of the effect of an increase in the probability of loss on insurance coverage depends on the slope of absolute risk aversion, but in the present model (as for some of the previous results) it depends

on the size of risk aversion (in this case, of relative risk aversion). However, we can say even more about the effect of an increase in the probability of the loss state on the optimal insurance purchase.

Corollary 6 *Assume that $a = 1$ and that relative risk aversion is constant and equal to 1 (i.e. logarithmic utility), then $\frac{\partial c^*}{\partial p} \gtrless 0$ as $\frac{\partial \pi}{\partial p} \lesseqgtr \frac{\pi}{p}$.*

In order to say more, we need to be more precise about exactly how the premium depends upon p . However, a great deal of the economics of insurance is based upon the assumption that the insurance premium is calculated as the expected value of a unit of coverage multiplied by a constant loading factor that is not less than 1. That is, the premium is set such that $\pi = kp$, where $k \geq 1$ is the loading factor. For this case, we get;

Corollary 7 *Assume that $a = 1$, and that insurance is priced according to a loading factor, $\pi = kp$, where $k \geq 1$, then an increase in the probability of loss will increase (decrease, not affect) the optimal purchase of insurance if relative risk aversion is uniformly less than (greater than, equal to) 1.*

A special, but obvious, case occurs for insurance that is sold in a perfectly competitive market, i.e. the loading factor is equal to 1, and Corollary 7 applies. Since it is far more reasonable to expect that relative risk aversion is always greater than 1, for the case of a premium based on a loading factor, when the insurer does respond to an increase in the probability of loss by increasing the premium, it turns out that the optimal purchase of insurance is likely to be decreasing in the probability of loss. This result compares with that of Jang and Hadar (1995), who consider the same problem in the traditional model (where the size of the risk is not a choice variable). In that paper, they show that for the most realistic case of decreasing absolute risk aversion, the effect of an increase in the probability of loss on the insurance decision cannot be signed. Here, the sign depends upon the size of relative risk aversion as it compares to 1, and for the most reasonable case of relative risk aversion uniformly greater than 1, the result is that the effect on insurance is the opposite to the case when the insurer does not increase the premium in response to the increase in probability. What this implies is that, when the probability of loss increases, and the premium increases as well, then the effect of the premium increase on the demand for insurance is greater than the effect of the probability increase when relative risk aversion is greater than 1.

5 Comparison with the Literature

The model studied in this paper introduces two extensions to the traditional literature on the demand for insurance. On the one hand, the analysis has been carried out under the assumption that the decision regarding how much risk to undertake has been made simultaneously with the decision on how much insurance to purchase. On the other hand, this extension requires that we introduce a budget constraint explicitly into the analysis. The first of these extensions has been considered previously, but the model here returns certain results that are quite different.

There are three principal papers that have considered the effect of making the insurable risk an endogenously determined variable – Meyer and Ormiston (1995), Eeckhoudt, Meyer and Ormiston (1997), and Meyer and Meyer (2004). From here on, we shall refer to these papers as MO, EMO and MM respectively. The current paper can be seen to be a simplified version of each of these models, but where the simplification is different in each case. Firstly, as compared to MO, we have the same basic set-up but with a discrete loss distribution, and we allow our individual's endowment to be in the risk-free asset (the endowment in MO is in risky asset), although we restrict her ability to borrow this asset. These simplifications allow us to arrive at many reasonable comparative statics effects that cannot be found in the more complex model of MO. As far as EMO goes, both there and in our model the individual's initial endowment is in the risk-free asset and purchases of insurance and the risky asset are financed from this endowment. However, here our discrete setting has implied that we have had to impose a restriction upon borrowing of the risk-free asset that is not needed in the EMO paper. Once again, our simpler setting gives us a corresponding increase in clarity of comparative statics results.

Finally, our model is also a simplified version of MM, where we only consider two states of nature. As is made clear in MM, x can be interpreted to be an investment portfolio of which a proportion $1 - a$ is in riskless assets and a proportion a is in risky assets. Holding a constant is then identical to the composite commodity theorem which is the theme of MM.

The fundamental differences between the existing literature and the model presented here, as far as the most interesting results are concerned, can be summed up as follows:

1. Relationship between full coverage and premium

- (a) Standard model; full coverage if and only if insurer earns expected profit of 0.
- (b) MO, EMO and MM do not discuss this issue.
- (c) Here; partial coverage is possible when the insurer earns an expected profit of 0, and full coverage is possible with a positive expected profit for the insurer, depending on the relationship between the expected returns on the insured's and the insurer's portfolios.

2. Normality of insurance

- (a) Standard model; insurance is inferior if DARA and risk held constant (Mossin).
- (b) MO; insurance is inferior if DARA.
- (c) EMO; either insurance is inferior, or risky asset is normal, or both, if DARA.
- (d) MM; insurance is normal if relative risk aversion is non-decreasing.
- (e) Here; insurance is normal if relative risk aversion is non-decreasing. With decreasing relative risk aversion, insurance is still normal if the proportion of the risky asset that is at risk is not less than the proportional change in absolute risk aversion over the two states of nature.

3. Increase in insurance premium

- (a) Standard model; insurance may be Giffen if DARA.
- (b) MO; insurance is ordinary if DARA and IRRA.
- (c) EMO; insurance is ordinary if DARA and relative risk aversion not greater than 1.
- (d) MM; insurance is ordinary if relative risk aversion not greater than 1.
- (e) Here; insurance is ordinary under various circumstances: if relative risk aversion is non-decreasing; or if it is uniformly less than 1; or if its value is not greater than the ratio of the unit price of the risky asset to the net price of consumption in the no-loss state.

4. FSD increase in size of loss (premium constant)

- (a) Standard model; insurance increases.
- (b) MO; insurance increases if relative risk aversion not greater than 1.
- (c) EMO; either insurance increases, or risky asset decreases, or both, if DARA and IRRA and relative risk aversion not greater than 1.
- (d) MM; insurance increases if relative risk aversion not greater than 1.
- (e) Here; insurance increases and risky asset decreases, independently of risk aversion characteristics.

5. FSD increase in size of loss (premium increases)

- (a) Standard model; indeterminate if DARA, and insurance decreases if CARA or IARA (Jang and Hadar, 1995).
- (b) MO, EMO and MM do not discuss this issue.
- (c) Here; assuming the loss state has a full loss, and that the insurance premium is a constant loading factor times the probability of the loss state, then insurance increases (decreases, remains constant) as relative risk aversion is uniformly less than (greater than, equal to) 1.

In order to fully appreciate this information, recall the principle differences between the models; the standard model allows no choice of risky asset. MO, EMO, and MM as well as the current model all allow for a simultaneous choice of risk and insurance, but only here have we considered the effects for partial or full coverage. The exogenous nature of the loss in the standard model leads directly to the result there that if risk aversion is decreasing in wealth, insurance is inferior and may be Giffen. In MO, the risky portion of wealth is also exogenously given, but since the insurance premium is paid for by selling part of that risk, the final risk situation is endogenous. However, in that model the individual is not given the opportunity to pay for insurance out of risk free wealth. The MO model continues to characterize insurance as inferior if risk aversion is

decreasing in wealth. It is also curious to note that in the MO model (as well as in both the EMO and the MM models) several of the results hang on the condition that relative risk aversion be not greater than 1. Since it is rather doubtful that indeed this will be so, the relevant results are all the more weaker for this condition.

The EMO model is the most general of all of the models, allowing simultaneous choice of risk, insurance coverage and safe asset holding, a general loss density, and a non-linear indemnity. Thus, the difference between the EMO model and the current one is simply that here we have assumed a far simpler environment (two states of nature). This assumption carries with it the implication that the insurance indemnity cannot be differentiated from a linear one, and so we have had to impose a restriction on borrowing. On the other hand, however, the simpler environment here has the significant benefit of providing clear-cut comparative statics, where the complex model of EMO cannot.

6 Conclusions

This paper offers a very simple economic theory of insurance, that allows the comparative statics of insurance demand to be easily considered. The principal point of departure of the theory given here and more traditional theory (referred to here as the standard model) is that in this model we allow the decision maker to simultaneously choose the amount of risky asset and the amount of insurance coverage under a budget constraint. This simple, but very natural, extension to the traditional approach yields several interesting results.

The first clear contrast between the model of this paper and the traditional model is the fact that it may be optimal for a risk averse individual to purchase partial coverage even though the premium is equal to the expected value of the loss. This does not happen in the traditional model (see, for example, Mossin (1968)). This difference is easy to explain; it is due entirely to the fact that the individual faces an opportunity cost of insurance purchases. Using funds to purchase insurance means exchanging an expected return of $\frac{1-pa}{v}$ (the expected return on the risky asset) against an expected return of $\frac{p}{\pi}$ (return on the insurance contract). This reduces the expected terminal wealth as soon as $\pi > \frac{vp}{1-pa}$, which is less than p . Thus, if the premium per unit, π ,

is equal to the probability of loss, p , the individual chooses partial coverage. Nevertheless, the standard result of “full coverage at an actuarially fair premium” still holds in our model, provided the “fair” premium is defined as being that which maintains the investor’s expected value of wealth constant.

Since this actuarially fair premium may be lower or higher than the premium that sets the insurer’s expected profit equal to zero, we get our second significant departure from the standard model: the insured may prefer full insurance even though the insurer gets a positive expected profit; she may also prefer partial insurance although the insurer charges an actuarially fair premium, from his point of view. The outcome depends on whether the expected return on the insured’s risky investment is lower or higher than the expected return on the insurer’s asset portfolio. By taking the asset’s expected return into account in the insured’s “fair” premium, our model introduces a new perspective and creates a link between the theory of insurance demand and “fair” premium calculation principles in financial economics and actuarial science. This result is robust to extensions to more general densities. It simply reflects the fact that the insurer does not invest all of the premium income in the insured asset, and so the two face different opportunity costs of funds.

Secondly, the comparative statics of the optimal decision have also been considered. These comparative statics differ from those of the traditional model of insurance due to the assumption that the amount of risk to be undertaken is a choice variable. In the model it turns out that insurance is a normal good at all wealth levels for a wide range of risk aversion characteristics. In particular, even if the utility function displays decreasing absolute risk aversion, an increase in wealth can still increase the amount of insurance. It depends on the rate at which risk aversion decreases, compared to the proportion of the investment that is at risk.

We have also shown that an increase in the premium will decrease the purchase of insurance for an even wider range of utility functions. It turns out that it is not unlikely that an increase in the unit price of insurance will lead to a decrease in the purchase of insurance and an increase in the purchase of the risky asset, thus insurance and risk can be termed “net substitutes”, in terms of traditional demand theory.

We have also shown that the intuitively logical result that an increase in risk aversion will both increase the amount of insurance coverage and decrease the amount of risk purchased also holds in the model. The need to reduce risk is met by two simultaneous movements that achieve the required end – less risk is purchased and it is more heavily insured.

We have also shown that both an increase in the proportion of the risky asset that is at risk and an increase in the probability of loss that is not accompanied by an increase in the premium will have the effect of decreasing the amount of risky good purchased, and increasing the amount of insurance coverage purchased. That is, these types of increase in risk will again be countered by two simultaneous risk reduction strategies – less risk will be purchased, and it will be more heavily insured. The fact that in this setting an increase in the probability of loss increases the purchase of insurance is also found in the standard model. On the other hand, for the special case of a full loss in state 2, if the insurer increases the premium after an increase in the probability of loss, conditional upon relative risk aversion being not less than 1, the optimal purchase of risky good will still decrease, and the effect on the optimal purchase of insurance coverage depends upon the value of absolute risk aversion. However, for the particularly interesting case of an insurer who sets the premium at the probability of loss multiplied by a constant loading factor, we do get a particularly clean result as far as the effect on insurance of an increase in the probability of loss. In contrast to what is found in the standard model (the effect can only be signed if the utility function displays non-decreasing absolute risk aversion, and then it is negative i.e. insurance decreases), we find here that the sign of the effect depends upon the comparison between relative risk aversion and the number 1. Concretely, for the interesting case of relative risk aversion greater than 1, it turns out that (conditional upon a competitive insurance environment) an increase in the probability of loss will decrease the optimal purchase of insurance. That is, the effect of the premium increase outweighs the effect of the increase in probability.

The model studied in the paper suggests several interesting questions. To begin with, when the individual has a need to reduce the risk of his optimal decision due to an increase in risk aversion, we know that risk will be reduced by two simultaneous strategies; x will be reduced and c will be increased. This, however, begs the question of exactly how the split will occur, that is,

how much risk will be reduced by decreasing the purchase of the risky good, and how much will be reduced by increasing insurance coverage?²⁰

Secondly, an interesting problem presents itself if we consider exactly how the risky asset and insurance coverage would indeed be priced if, for example, each was supplied by a monopolist. One can imagine the first order condition for an expected profit maximum by each supplier, and how each of their optimal decisions depends upon the decision of the other through the consumer's demands for each product. Using, for example, a standard Cournot type model, it would be interesting to study this price setting game, making use of the comparative statics that the current model provides.

Thirdly, it would be interesting to remove the linear nature of the indemnity without removing the discrete nature of the model by simply adding more states of nature, thereby allowing the determination of the optimal position in all three assets simultaneously. In such a model, we would also be able to eliminate our restrictions concerning borrowing capacity.

Alternatively, we may want to assume that the safe good is consumption itself (as opposed to terminal wealth). Such an extension opens the door to studying the cross subsidy effects that occur over different consumption goods when the comparative statics are considered. The addition of this new aspect of the model will not alter the result of Proposition 1, since it will not alter the basic difference between the insurer and the consumer as to the understanding of what exactly constitutes a fair premium, but it will doubtless add to our understanding of how real-world insurance purchases are carried out.

References

- Briys, E., G. Dionne and L. Eeckhoudt (1989), "More on Insurance as a Giffen Good", *Journal of Risk and Uncertainty*, **2**; 415-420.
- Chesney, M. and H. Loubergé (1986), "Risk Aversion and the Composition of Wealth in the Demand for Full Insurance Coverage", *Schweizerische Zeitschrift für Volkswirtschaft und Statistik*, 122, 359-370.
- Cleeton, D. L. and B. B. Zellner (1993), "Income, Risk Aversion and the Demand for Insurance", *Southern Economic Journal*, 60, 146-156.

²⁰ The same question for the case of an increase in the probability of loss is answered in the first two equations in the proof of Proposition 5.

- Cummins, J.D. and R. Phillips (2000), "Applications of Financial Pricing Models in Property-Liability Insurance", in G. Dionne (Ed.) *Handbook of Insurance*, Kluwer Academic Publishers, pp. 621-56..
- Cummins, J.D. and O. Mahul (2004), "The Demand for Insurance with an Upper Limit on Coverage", *Journal of Risk and Insurance*, 71, 253-264.
- D'Arcy, S. and N. Doherty (1988), *The Financial Theory of Pricing Property-Liability Insurance Contracts*, Huebner Foundation Monographs.
- Dionne, G. and L. Eeckhoudt (1984), "Insurance and Saving: Some Further Results", *Insurance: Mathematics and Economics*, 3, 101-110.
- Doherty, N. and H. Schlesinger (1983), "Optimal Insurance in Incomplete Markets", *Journal of Political Economy*, 91, 1045-1054.
- Eeckhoudt, L., J. Meyer and M. Ormiston (1997), "The Interaction Between the Demands for Insurance and Insurable Assets", *Journal of Risk and Uncertainty*, 14, 25-39.
- Eeckhoudt, L. and Venezian (1990), "Insurance Purchases When the Level of Investment in a Risky Asset is Endogenously Determined", unpublished manuscript.
- Ehrlich, I. and G. Becker (1972), "Market Insurance, Self Insurance, and Self Protection", *Journal of Political Economy*, 80, 623-48.
- Hirshleifer, J. (1965), "Investment Decision Under Uncertainty: Choice-Theoretic Approaches", *Quarterly Journal of Economics*, LXXIX, 509-36.
- Hirshleifer, J. (1966), "Investment Decision Under Uncertainty: Applications of the State-Preference Approach", *Quarterly Journal of Economics*, LXXX, 252-77.
- Hoy, M. and A.J. Robson (1981), "Insurance as a Giffen Good", *Economics Letters*, 8; 47-51.
- Jang, Y. and J. Hadar (1995), "A Note on Increased Probability of Loss and the Demand for Insurance", *The Geneva Papers on Risk and Insurance Theory*, 20, 213-216.
- Levy, H. (1994), "Absolute and Relative Risk Aversion: An Experimental Study", *Journal of Risk and Uncertainty*, 8, 289-307.
- Mehra, R. and E. Prescott (1985), "The Equity Premium - A Puzzle", *Journal of Monetary Economics*, 15, 145-161.
- Menezes, C.F. and D.L. Hanson (1970), "On The Theory of Risk Aversion", *International Economic Review*, 1(3), 481-487.
- Meyer, D., and J. Meyer (2004), "A More Reasonable Model of Insurance Demand", in Aliprantis, C.D, K.J. Arrow, P. Hammond, F. Kubler, H.M. Wu, and N.C. Yannelis (Eds.), *Assets, Beliefs, and Equilibria in Economic Dynamics. Essays in Honor of Mordecai Kurz*, Studies in Economic Theory (vol. 18), Springer-Verlag, pp. 733-742.
- Meyer, J., and M. Ormiston (1995), "Demand for Insurance in a Portfolio Setting", *The Geneva Papers on Risk and Insurance Theory*, 20, 203-211.
- Mossin, J. (1968), "Aspects of Rational Insurance Purchasing", *Journal of Political Economy*, 76, 553-568.
- Pratt, J. (1964), "Risk Aversion in the Small and in the Large", *Econometrica*, 32, 122-136.
- Szpiro, G. (1986), "Measuring Risk Aversion: An Alternative Approach", *The Review of Economics and Statistics*, LXVIII, N°1.

Appendix A.

Proof of Lemma 1:

Take firstly the case $v + a\pi < 1$. In this case the solution will set $z^* = 0$. The reason is that, by purchasing one unit of x together with a units of c (that is, by purchasing a fully insured unit of risky asset), the consumer ensures a unit of terminal wealth. Since $v + a\pi < 1$, this costs less than a unit of terminal wealth held as the safe asset. Thus, any purchase of safe asset is strictly dominated by purchases of fully insured risky asset, i.e. the safe asset will not be purchased.²¹

Secondly, consider the case when $v + a\pi > 1$. In this case the solution will set $c^* = 0$. To see why, imagine the consumer has a portfolio that includes some positive amount of each of the three assets. Given this, she would do well to sell a unit of x and decrease her spending on insurance by a units in order to earn $v + a\pi$ of certain wealth. She would do so until it is no longer possible to continue doing so, which (since $c \leq ax$) occurs when c reaches 0, and so no insurance will ever be purchased.²²

Appendix B

Proof of proposition 4:

The derivative of the tangency condition (5) with respect to a is:

$$\frac{u''(x^*) \frac{\partial x^*}{\partial a} u'(y^*) - u'(x^*) u''(y^*) \frac{\partial y^*}{\partial a}}{u'(y^*)} = u'(y^*) \left(\frac{p}{1-p} \right)$$

Using the tangency condition to substitute for the marginal utility of state 1 consumption on the right-hand-side, and simplifying, we get

$$R_a(y^*) \frac{\partial y^*}{\partial a} - R_a(x^*) \frac{\partial x^*}{\partial a} = \frac{\pi}{(v - \pi(1 - a))} > 0$$

Thus, we have $R_a(y^*) \frac{\partial y^*}{\partial a} > R_a(x^*) \frac{\partial x^*}{\partial a}$, from which we must rule out the possibility that

$$\frac{\partial y^*}{\partial a} < 0 < \frac{\partial x^*}{\partial a}.$$

²¹ Note that, even though a portfolio consisting entirely of fully insured risky assets dominates holding any amount of safe asset, the fully insured portfolio may not be optimal, that is, it may itself be dominated by a portfolio consisting of partially insured risky assets.

²² We must be careful in not attaching an implication for purchases of uninsured units of x . So long as the price of x satisfies $v < 1 - p$, then the expected value of an uninsured unit of risky asset is greater than that of a unit of safe asset. In this case, it is well known that a risk averse consumer will always invest in some amount of x .

On the other hand, differentiating the budget constraint (4) we get

$$0 = \pi \left(\frac{\partial x^*}{\partial a} + \frac{\partial y^*}{\partial a} \right)$$

that is $\frac{\partial x^*}{\partial a} = -\frac{\partial y^*}{\partial a}$. Thus one is negative and the other positive. So we can only have $\frac{\partial y^*}{\partial a} > 0$, and correspondingly $\frac{\partial x^*}{\partial a} < 0$. Finally, since $\frac{\partial y^*}{\partial a} = (1-a)\frac{\partial x^*}{\partial a} + \frac{\partial c^*}{\partial a}$, and since the first term on the right-hand-side is negative, we must have $\frac{\partial c^*}{\partial a} > 0$.

Proof of proposition 5:

At an internal solution, ($\pi > \frac{m}{1-ap}$) the solution for the case of $v(\cdot) \equiv h(u(\cdot))$ is given by the duly adjusted versions of (1) and (2), which taking into account the internal solution assumption (i.e. $\delta_2 = 0$), and denoting the new solution by $(\tilde{x}^*, \tilde{y}^*)$ we have

$$(1-p)h'(u(\tilde{x}^*))u'(\tilde{x}^*) = (v - \pi(1-a))\delta_1$$

and

$$ph'(u(\tilde{y}^*))u'(\tilde{y}^*) = \pi\delta_1$$

Dividing the first of these by the second, and simplifying, we have the tangency condition in the equilibrium expressed in terms of state contingent consumption;

$$\frac{h'(u(\tilde{x}^*))u'(\tilde{x}^*)}{h'(u(\tilde{y}^*))u'(\tilde{y}^*)} = \frac{p(v - \pi(1-a))}{(1-p)\pi}$$

However, from the original problem, we also know that

$$\frac{u'(x^*)}{u'(y^*)} = \frac{p(v - \pi(1-a))}{(1-p)\pi}$$

so we have

$$\frac{h'(u(\tilde{x}^*))u'(\tilde{x}^*)}{h'(u(\tilde{y}^*))u'(\tilde{y}^*)} = \frac{u'(x^*)}{u'(y^*)}$$

Now, since we also know that in the new solution we have partial coverage (Proposition 1), then we also have a lower consumption in state 2 than in state 1, $\tilde{x}^* > \tilde{y}^*$, and since u is an increasing function and h is increasing and concave, it holds that $h'(u(\tilde{y}^*)) > h'(u(\tilde{x}^*))$, so that

$$\frac{u'(\tilde{y}^*)}{u'(\tilde{x}^*)} < \frac{u'(y^*)}{u'(x^*)} \tag{22}$$

Therefore we can definitely rule out the possibility that $\tilde{x}^* = x^*$ and $\tilde{y}^* = y^*$.

Finally, from the fact that in both solutions the budget constraint saturates, we know that $(v - \pi(1 - a))\tilde{x}^* + \pi\tilde{y}^* = (v - \pi(1 - a)x^* + \pi y^*$. Hence, it can never be true that the increase in risk aversion leads to an increase in both choice variables, or to a decrease in both - i.e. one must increase and the other must decrease. However, (22) makes it clear that we can also rule out the possibility that $\tilde{y}^* < y^*$ and $\tilde{x}^* > x^*$ since that would lead to a decrease in marginal utility in the no-loss state and an increase in the marginal utility in the loss state, that is an increase in the ratio of the two. Thus, we can safely conclude that the increase in risk aversion leads to a decrease in state 1 consumption (x), and an increase in state 2 consumption (y).

Finally, since $y = (1 - a)x + c$, and since x decreases the only way y can increase is if c increases.

Proof of proposition 6:

Again, begin with the tangency condition in state contingent consumption space (5). Differentiating with respect to p gives

$$\frac{u''(x^*)\frac{\partial x^*}{\partial p}u'(y^*) - u'(x^*)u''(y^*)\frac{\partial y^*}{\partial p}}{u'(y^*)} = u'(y^*) \left[\frac{(v - \pi(1 - a))(1 - p)\pi + p(v - \pi(1 - a))\pi}{(1 - p)^2\pi^2} \right]$$

The term in brackets on the right-hand-side simplifies immediately so that this equation reads

$$\frac{u''(x^*)\frac{\partial x^*}{\partial p}u'(y^*) - u'(x^*)u''(y^*)\frac{\partial y^*}{\partial p}}{u'(y^*)} = u'(y^*) \left[\frac{(v - \pi(1 - a))}{(1 - p)^2} \right]$$

Using the tangency condition to substitute for the marginal utility of state 1 consumption on the right-hand-side and simplifying, we get

$$R_a(y^*)\frac{\partial y^*}{\partial p} - R_a(x^*)\frac{\partial x^*}{\partial p} = \frac{\pi}{(1 - p)p} > 0$$

Thus we know that

$$R_a(y^*)\frac{\partial y^*}{\partial p} > R_a(x^*)\frac{\partial x^*}{\partial p} \tag{23}$$

However, from the budget constraint (4), we also know that $0 = (v - \pi(1 - a))\frac{\partial x^*}{\partial p} + \pi\frac{\partial y^*}{\partial p}$. But since both of the (net) state contingent prices are positive, this indicates that the two derivatives must have opposite sign. Using (23), the only option is $\frac{\partial y^*}{\partial p} > 0 > \frac{\partial x^*}{\partial p}$. Finally, since $\frac{\partial y^*}{\partial p} = (1 - a)\frac{\partial x^*}{\partial p} + \frac{\partial c^*}{\partial p}$, and since $\frac{\partial x^*}{\partial p} < 0$, we must have $\frac{\partial c^*}{\partial p} > 0$.

Proof of Proposition 7:

In the proof of proposition 6 it was shown that

$$R_a(y^*) \frac{\partial y^*}{\partial p} - R_a(x^*) \frac{\partial x^*}{\partial p} = \frac{\pi}{(1-p)p}$$

When $a = 1$, we have $y^* = c^*$, so that $R_a(c^*) \frac{\partial c^*}{\partial p} - R_a(x^*) \frac{\partial x^*}{\partial p} = \frac{\pi}{(1-p)p}$. Together with the fact that $0 = v \frac{\partial x^*}{\partial p} + \pi \frac{\partial c^*}{\partial p}$, after some simplification, it can be shown that the effects on the two optimal variables, holding the premium constant are in fact the following;

$$\begin{aligned} \left. \frac{\partial x^*}{\partial p} \right|_{\pi} &= -\frac{\pi}{p(1-p) [\pi R_a(x^*) + v R_a(c^*)]} \\ \left. \frac{\partial c^*}{\partial p} \right|_{\pi} &= \frac{v}{p(1-p) [\pi R_a(x^*) + v R_a(c^*)]} \end{aligned}$$

On the other hand, when $a = 1$, (6) reads $c^* + v \frac{\partial x^*}{\partial \pi} + \pi \frac{\partial c^*}{\partial \pi} = 0$, and (7) reads $R_a(x^*) \frac{\partial x^*}{\partial \pi} - R_a(c^*) \frac{\partial c^*}{\partial \pi} = \frac{1}{\pi}$. The simultaneous solution to these two equations, again after some simplification, yields

$$\begin{aligned} \frac{\partial x^*}{\partial \pi} &= \frac{1 - R_r(c^*)}{\pi R_a(x^*) + v R_a(c^*)} \\ \frac{\partial c^*}{\partial \pi} &= -\frac{1}{\pi} \left[\frac{v + \pi c^* R_a(x^*)}{\pi R_a(x^*) + v R_a(c^*)} \right] \end{aligned}$$

Now, the full effects of an increase in the probability of loss are

$$\begin{aligned} \frac{\partial x^*}{\partial p} &= \left. \frac{\partial x^*}{\partial p} \right|_{\pi} + \frac{\partial x^*}{\partial \pi} \frac{\partial \pi}{\partial p} \\ \frac{\partial c^*}{\partial p} &= \left. \frac{\partial c^*}{\partial p} \right|_{\pi} + \frac{\partial c^*}{\partial \pi} \frac{\partial \pi}{\partial p} \end{aligned}$$

Thus we have

$$\begin{aligned} \frac{\partial x^*}{\partial p} &= -\frac{\pi}{p(1-p) [\pi R_a(x^*) + v R_a(c^*)]} + \frac{1 - R_r(c^*)}{[\pi R_a(x^*) + v R_a(c^*)]} \frac{\partial \pi}{\partial p} \\ \frac{\partial c^*}{\partial p} &= \frac{v}{p(1-p) [\pi R_a(x^*) + v R_a(c^*)]} - \frac{1}{\pi} \frac{v + \pi c^* R_a(x^*)}{[\pi R_a(x^*) + v R_a(c^*)]} \frac{\partial \pi}{\partial p} \end{aligned}$$

Simplifying, we get

$$\frac{\partial x^*}{\partial p} = \frac{1}{[\pi R_a(x^*) + v R_a(c^*)]} \left[-\frac{\pi}{p(1-p)} + (1 - R_r(c^*)) \frac{\partial \pi}{\partial p} \right] \quad (24)$$

$$\frac{\partial c^*}{\partial p} = \frac{1}{[\pi R_a(x^*) + v R_a(c^*)]} \left[\frac{v}{p(1-p)} - \left(\frac{v + \pi c^* R_a(x^*)}{\pi} \right) \frac{\partial \pi}{\partial p} \right] \quad (25)$$

Since both prices and absolute risk aversion are all positive, the signs of each of the effects are the same as the signs of the terms in parenthesis. That is, after a minimal amount of simplification,

we get

$$\begin{aligned} \frac{\partial x^*}{\partial p} &\stackrel{\leq}{\geq} 0 \quad \text{as } (1 - R_r(c^*)) \frac{\partial \pi}{\partial p} \stackrel{\leq}{\geq} \frac{\pi}{p(1-p)} \\ \frac{\partial c^*}{\partial p} &\stackrel{\leq}{\geq} 0 \quad \text{as } \frac{\partial \pi}{\partial p} \stackrel{\leq}{\geq} \frac{v\pi}{p(1-p)(v + \pi c^* R_a(x^*))} \end{aligned}$$

This proves the statement concerning the effect of an increase in p upon the optimal purchase of the risky asset. To get the effect on the optimal purchase of insurance coverage, note that $\pi c^* R_a(x^*) = v \left(\frac{\pi c^*}{v x^*} \right) R_r(x^*)$. Substituting this into the previous expression, and cancelling the common term (v) and carrying out a simple reordering yields the indicated result.

Proof of Corollary 7:

Note that, from the tangency condition at the optimum we know that

$$\frac{pv}{(1-p)\pi} = \frac{u'(x^*)}{u'(c^*)} \Rightarrow \frac{\pi}{v} = \frac{pu'(c^*)}{(1-p)u'(x^*)}$$

so it turns out that

$$\frac{\pi c^*}{v x^*} = \left(\frac{p}{(1-p)} \right) \left(\frac{u'(c^*)c^*}{u'(x^*)x^*} \right)$$

Thus the condition can be written as

$$\frac{\partial c^*}{\partial p} \stackrel{\leq}{\geq} 0 \quad \text{as } (1-p) \left(1 + \left(\frac{p}{(1-p)} \right) \left(\frac{u'(c^*)c^*}{u'(x^*)x^*} \right) R_r(x^*) \right) \frac{\partial \pi}{\partial p} \stackrel{\leq}{\geq} \frac{\pi}{p}$$

that is

$$\frac{\partial c^*}{\partial p} \stackrel{\leq}{\geq} 0 \quad \text{as } \left((1-p) + p \left(\frac{u'(c^*)c^*}{u'(x^*)x^*} \right) R_r(x^*) \right) \frac{\partial \pi}{\partial p} \stackrel{\leq}{\geq} \frac{\pi}{p}$$

Note that the term in parentheses on the right-hand-side of the condition is a convex combination of the number 1 and the number $\left(\frac{u'(c^*)c^*}{u'(x^*)x^*} \right) R_r(x^*)$. However, consider the function $u'(z)z$. Its slope is $u''(z)z + u'(z)$, and so $u'(z)z$ is increasing (decreasing, constant) as $u''(z)z + u'(z) \stackrel{\geq}{\leq} 0$, that is, as $R_r(z) \stackrel{\leq}{\geq} 1$. For the case at hand, i.e. relative risk aversion equal to 1, it turns out that $u'(c^*)c^* = u'(x^*)x^*$, and so we get $\left(\frac{u'(c^*)c^*}{u'(x^*)x^*} \right) R_r(x^*) = 1$. This means that the convex combination term on the right-hand-side of the condition is exactly equal to 1, which gives the required result.

Proof of Corollary 8:

Clearly, when $\pi = kp$, we get $\frac{\partial \pi}{\partial p} = k = \frac{\pi}{p}$. Substituting this into the condition leads to

$$\frac{\partial c^*}{\partial p} \begin{matrix} \geq \\ \leq \end{matrix} 0 \text{ as } \left((1-p) + p \left(\frac{u'(c^*)c^*}{u'(x^*)x^*} \right) R_r(x^*) \right) \begin{matrix} \leq \\ \geq \end{matrix} 1$$

which in turn implies

$$\frac{\partial c^*}{\partial p} \begin{matrix} \geq \\ \leq \end{matrix} 0 \text{ as } \left(\frac{u'(c^*)c^*}{u'(x^*)x^*} \right) R_r(x^*) \begin{matrix} \leq \\ \geq \end{matrix} 1$$

However, it was shown in the previous corollary that $u'(z)z$ is increasing (decreasing, constant) as $u''(z)z + u'(z) \begin{matrix} \geq \\ \leq \end{matrix} 0$, that is, as $R_r(z) \begin{matrix} \leq \\ \geq \end{matrix} 1$. Hence, since we know that for the case at hand $x^* > c^*$, it turns out that

$$\frac{u'(c^*)c^*}{u'(x^*)x^*} \begin{matrix} \leq \\ \geq \end{matrix} 1 \text{ as } R_r(z) \begin{matrix} \leq \\ \geq \end{matrix} 1$$

In short then, if $R_r(z) > 1$, we get $\frac{u'(c^*)c^*}{u'(x^*)x^*} > 1$, and consequently $\left(\frac{u'(c^*)c^*}{u'(x^*)x^*} \right) R_r(x^*) > 1$ which implies that $\frac{\partial c^*}{\partial p} < 0$. An identical reasoning suffices to prove the other two cases.