A Portfolio Model of Insurance Demand

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Abstract: Insurance can be viewed as one of many assets in an investor's portfolio, and the analysis of the demand for insurance can be carried out within a portfolio model. Doing this makes available a number of well known results and techniques from portfolio analysis, and also leads to the investigation of questions different from those posed in usual insurance demand analysis. One such question and its answer provide conditions under which increases in wealth lead to increases in the purchase of insurance.

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1. Introduction

Insurance is an asset whose return is random, and this asset is just one of the many risky assets held by the insured. Therefore it is possible, and appropriate, to analyze the demand for insurance in the same manner that one examines the demand for any risky asset held in a portfolio of such assets. Analyzing the demand for insurance from this perspective has several advantages. First, well known results and techniques from portfolio theory can be used to obtain properties of the demand for insurance. Second, this perspective leads to the investigation of questions that are different from those typically examined using the standard approach. Finally, the different perspective calls into question assumptions made in the standard model, and suggests alternatives. Under one alternative set of assumptions, wealth increases lead to an increase, rather than a decrease, in the purchase of insurance. This is the case even though the insured is decreasing absolute risk averse.

The standard model of insurance demand is formulated to represent the decision to insure against the loss that can occur from holding a specific asset. The notation, assumptions and comparative static questions are all chosen for the purpose of analyzing this particular decision. The purchaser of insurance is assumed to already possess the asset subject to loss and also to possess some amount of the riskless asset. The only decision that is made is choosing how much of the riskless asset to exchange for insurance. The questions asked and the *ceteris paribus* assumptions made when conducting comparative static analysis in this model follow directly from this perspective on the insurance purchase decision.

The portfolio representation of the insurance purchase decision has a somewhat different perspective. In this representation, the purchaser of insurance is assumed to hold an equilibrium portfolio of risky assets, which include assets subject to loss, and insurance assets indemnifying a portion of those losses. In this setting, comparative static analysis proceeds by asking how various parameter changes affect the allocation of wealth to the insurance asset. This portfolio representation focuses less on the decision to indemnify a particular loss, and more on the overall decision concerning the quantity of the insurance asset to hold in a portfolio of risky assets.

The derivations in the Appendix verify that in the simple versions of these two models considered here, the two perspectives lead to the same formal model, and therefore cannot lead to results that contradict one another. The different perspectives can, however, cause the researcher to pose different questions, and to impose different assumptions when addressing those questions. These different questions and assumptions, and how they contribute to the analysis of the demand for insurance is the subject of this investigation.

2. The Model

The model of insurance demand investigated in this analysis uses standard portfolio notation. The decision maker is assumed to maximize expected utility from random outcome z, where z is the return on a portfolio composed of two assets. The two asset case is chosen to keep the model simple, and to allow an exact match with the usual model of insurance demand. This restriction also has the advantage that there is only one decision variable in the model, another simplifying feature. The two assets in the portfolio are each risky. One is referred to as the insurance asset and its return is denoted i. The other, is referred to as the risky asset and its return is denoted r. This risky asset is itself a portfolio of assets, including at least the asset subject to loss and the riskless asset. The two random return variables are each assumed to have nonnegative support.

Assume that $z = W((1 - \alpha)\cdot r + \alpha\cdot i)$, where W is total wealth, and α is the proportion of wealth allocated to the insurance asset. This proportional version of the portfolio model is used because the majority of the analysis in the literature makes that assumption. It is also assumed that the decision variable α is selected so as to maximize expected utility from z, with the first and second order conditions for this maximization holding. That is, $Eu'(z)W(i - r) = 0$, and $Eu''(z)W^2(i - r)^2 < 0$ are assumed.

Even a quick perusal of the literature attempting to carry out comparative static analysis for portfolios consisting of two or more risky assets makes it clear that without the imposition of significant additional structure, few comparative static findings can be expected.[1] The assumptions that are imposed in the standard formulation of the insurance demand model are used to suggest the additional structure typically imposed in the analysis of insurance demand. Each of the assumptions imposed here are implied by assumptions made in the standard model.

The first assumption made is that the returns from the two risky assets depend on a common source of randomness. In the model posed here, that common random variable is denoted x, and it is assumed that $r = r(x)$ and $i = i(x)$. In addition to the returns depending on a common source of randomness, it is also assumed that the return from the insurance asset is nondecreasing in x, while that from the risky asset is nonincreasing. This assumption allows x to be referred to as the loss variable, terminology frequently used in the standard model, and this assumption can be thought of as one of those that characterize an insurance demand model.

Finally, it is assumed that the return and decision variables are such that the portfolio return, $z(x)$, is nonincreasing in x. That is, the return variables and the decision variable are restricted so that the decision maker's holding of the insurance asset cannot lead to states of the world where larger losses are preferred to smaller ones. This assumption is also one that plays a prominent role in models of insurance demand, and is one that is not typically made when analyzing the demand for such assets as put options.

The comparative static analysis conducted here confirms that these assumptions are the most important implications of the several assumptions made in the standard model of insurance demand, and are sufficient for the majority of the analysis that is conducted there. The appendix presents the standard model of insurance demand, transforms this model into portfolio notation and then indicates how the various assumptions imposed there imply the assumptions made here.

3. Wealth Effects

Probably the most limiting comparative static finding in the standard model of insurance demand is one that is often stated using terminology from consumer demand analysis. This finding claims that insurance is an inferior good under decreasing absolute risk aversion (DARA). While it has been argued by Meyer and Meyer (2004) that this is a misuse of the word "inferior", it is correct to say that in the standard insurance demand model, DARA does imply that an increase in wealth leads to a decrease in the optimal quantity of insurance. Since DARA is an often made and well supported restriction on risk preferences, this finding is disturbing. The finding is also a strong one, not only is the proportion of wealth allocated to insurance reduced as wealth increases, but the absolute quantity is reduced as well.

The following theorem indicates something quite different. It shows that increases in wealth can lead to either increases or decreases in the proportion of wealth allocated to insurance, even assuming DARA. First the theorem and proof are presented, and then the manner is which this result differs from the standard one is discussed in detail.

Theorem 1: An increase in W increases (decreases) the proportion of wealth allocated to the insurance asset when relative risk aversion is increasing (decreasing).

Proof: The first order condition defining the optimal value for α, the proportion of wealth allocated to the insurance asset, is $Eu'(z)W[i(x) - r(x)] = 0$. Therefore $\partial\alpha/\partial W$ has the same sign as $Eu''(z)[i(x) - r(x)][(1 - \alpha)\cdot r(x) + \alpha\cdot i(x)] = -(1/W)ER_R(z)u'(z)[i(x) - r(x)]$. This expression is nonnegative (nonpositive) if the relative risk aversion measure, $R_R(z)$, is nonincreasing (nondecreasing) in x. Since z(x) is nonincreasing in x, the assumption that $R_R(z)$ is nondecreasing (nonincreasing) in z implies that $\partial\alpha/\partial W \geq 0$ (≤ 0), the stated result.

First, notice that the proof of this theorem follows exactly the same steps as the proof of the similar result in the single risky and single riskless asset portfolio model. Because of the assumptions made earlier, this two risky asset model is very similar in structure to the one risky and one riskless asset model, with insurance taking on a role similar to that played by the riskless asset in that model. Specifically, the assumptions that both z(x) and (i(x) – r(x)) depend on a single random variable, and that each are monotonic in x imply this similarity. Of course, the single risky and riskless asset portfolio model has been the subject of extensive analysis so many results from that work are available here because of the special assumptions related to the analysis of insurance demand.

Recall that the set of increasing relative risk averse decision makers referred to in Theorem 1 includes decision makers who display DARA. Therefore, Theorem 1 appears to contradict the statement made earlier that under DARA insurance is an inferior good. Because this portfolio representation and the standard one are equivalent representations of the same decision, however, the one result cannot contradict the other, and an explanation of the apparent contradiction is in order.

The reason the two findings appear to be contradictory, yet are not, is that a different and unstated *ceteris paribus* assumption is made when demonstrating each. The proof of Theorem 1 assumes that the returns to the two risky assets, i(x) and r(x), are not changed when wealth changes, a natural assumption to make when doing comparative static analysis in a portfolio model. In the standard model of insurance demand, on the other hand, the natural assumption is also made. There it is assumed that the quantity of the asset whose potential losses are being insured is not changed as wealth changes. This latter assumption implies that the return to the portion of wealth not allocated to insurance must change, and must change in a particular way as wealth is increased. Specifically, the *ceteris paribus* assumption in the standard model implies that increases in wealth reduce the riskiness of r(x), and can change its mean as well. The appendix presents the details of this.

The two models of insurance demand are equivalent models. Therefore, in principle both of these results concerning the effect of an increase in W on the demand for insurance can be demonstrated in either model. For each of the findings, however, it is easier and more natural to impose the appropriate *ceteris paribus* assumption and derive the result in the one setting than in the other. It is also the case that each finding presents different and relevant information concerning the demand for insurance. Theorem 1 indicates what happens to the demand for insurance when wealth increases assuming that the return on the risky portion of the portfolio is the same as before the wealth increase. On the other hand, the statement that DARA implies that insurance is inferior, indicates how the demand for insurance for a specific asset changes as wealth increases, assuming that the quantity of that specific asset is unaffected by the additional wealth.

Meyer and Meyer examine these two alternative *ceteris paribus* assumptions in the context of the composite commodity theorem, and argue that the term "inferior" as defined in consumer demand analysis is not appropriately used in the standard claim that insurance is an inferior good under DARA, but that consumer demand terminology is appropriate to use under the assumptions of the portfolio model. That is, when the quantity of the specific asset is held fixed, the *ceteris paribus* assumptions of consumer demand analysis are violated, while when the return to the risky asset is held fixed these assumptions are maintained. Thus, Theorem 1 indicates that insurance can be inferior or normal, using those terms as they are commonly defined in consumer demand analysis.

As an added point, it is more likely that data exist to test Theorem 1 with its *ceteris paribus* assumption, than to test the standard result. Cross section data concerning the quantity of insurance purchased and how this quantity varies with wealth can be used for testing Theorem 1, while information detailing how the level of insurance for a specific asset changes as wealth increases is less likely to be available.[2]

This section has derived a different result concerning the effect of changing wealth on the demand for insurance. The derivation illustrates the type of result that can be obtained from examining insurance demand in a portfolio setting, and also points out the importance of the *ceteris paribus* assumption in doing comparative static analysis. The remainder of the paper continues the use of the portfolio formulation to obtain other such results. The next section asks how changes in either i(x) or r(x) affect the demand for insurance.

3. Stochastic Dominant Changes in i(x) or r(x).

The discussion in this section is primarily concerned with the effect on α of changing the random return to either the risky asset or the insurance asset, although the effect of such changes on expected utility is discussed as well. Many of the results presented come directly from theorems previously demonstrated in a general portfolio setting and do not require formal proof. Before discussing the results themselves, however, the *ceteris paribus* assumption imposed in the analysis must be clarified. It is assumed that when a change is made in the return to one risky asset, this is accomplished without changing the return to the other risky asset. For instance, a First Degree Stochastic (FSD) improvement to i(x) occurs while holding r(x) fixed. Consideration of the effect of changing random variable x is ruled out by this assumption since such changes alter both i(x) and r(x) simultaneously. To repeat, the *ceteris paribus* assumption is that the random variable x is unchanged and that only one of the two return variables, i(x) or r(x) is changed.

Two main methods of changing i(x) and r(x) are considered. In the first, a random variable ε is added to the return variable and in the second, the return itself is transformed deterministically by adding a function k(x) to that describing how the return depends on x. The first type of change that is considered in the analysis below is an FSD improvement to either i(x) or r(x). Theorem 2 is a special case of a more general portfolio theorem presented by Meyer and Ormiston (1994).

<u>Theorem 2:</u> When random variable ε whose support is nonnegative, or $k(x) \geq 0$ for all x, is added to i(x) [r(x)] , keeping r(x) [i(x)] fixed, then α increases [decreases] when relative risk aversion is less than or equal to one.

The portfolio model analyzed by Meyer and Ormiston is one with two risky assets which are not necessarily independent. They show that when all conditional distribution functions for one of the two assets improves in a FSD sense, then relative risk aversion less than or equal to one is sufficient to imply that an increased portion of wealth is allocated to the asset whose return has improved. Under the *ceteris paribus* assumption here, the conditional cumulative distribution function for i(x) or r(x) will often be degenerate. Even so, FSD improvements in these conditional cumulative distribution functions occur when either a positive constant term that can depend on x, or a stochastic term whose support is nonnegative is added to the return, so the Meyer and Ormiston result applies.

These FSD improvements of i(x) can result from a number of changes in the standard model. One such change is when indemnification increase for some levels of loss, does not decrease for any loss level, and the price of insurance is held fixed. Another is when the price of insurance is reduced while holding indemnification fixed. Theorem 2 indicates that if relative risk aversion is less than one, such improvements lead to increased purchase of insurance. This finding concerning the change in the indemnification function can easily be demonstrated in the standard model, while that concerning a price decrease cannot since changing price in the standard model also changes the return to risky asset. FSD improvements to the return to the risky asset can occur in the standard model because of an increased return to the riskless asset.

For these FSD improvements in the return to an asset, it is easy to show that expected utility is increased for all utility functions that prefer more to less. Thus, these changes both increase expected utility for the insured and also lead to increased demand for the asset when relative risk aversion is less than one. If relative risk aversion is greater than one, expected utility is increased, but allocation to the asset may increase or decrease.

A very similar finding concerning adding random noise to the return to an asset follows as a special case of Meyer and Ormiston's theorem concerning the effect of an increase in the riskiness of the return distribution for an asset on the allocation to the asset.

Theorem 3: When random variable ε, whose mean is zero conditional on the value of x, is added to i(x) [r(x)], keeping r(x) [i(x)] fixed, then α decreases [increases] when relative risk aversion is increasing and less than or equal to one, and absolute risk aversion is decreasing.

Adding such ε to the return to an asset increases the riskiness of each conditional cumulative distribution function for that asset, and therefore reduces demand for that asset. That is, when i(x) or r(x) is made riskier by adding noise, the demand for that asset is reduced whenever relative risk aversion is increasing and less than one, and absolute risk aversion is decreasing. Adding random noise to the return to the risky asset is similar to, but not the same as, including background risk in the model. It is also easy to show that this added noise reduces expected utility for all risk averse decision makers since it increases the riskiness of the portfolio return z(x).

Increasing the riskiness of each of the conditional distribution functions for insurance or the risky asset using deterministic transformations of i(x) or r(x) is not possible when some or all of these these distribution functions are degenerate. Thus, that method of changing the riskiness of those returns is not discussed in Theorem 3. Instead, a different question concerning deterministic transformations of i(x) or r(x) is posed, and this result is formally demonstrated using the same sufficient condition and using techniques common to portfolio analysis of the similar question.

Recall that when one restricts k(x) appropriately, adding k(x) to i(x) (or r(x)) increases the riskiness of that return. Furthermore, the effect of adding k(x) to i(x) can be determined by first replacing i(x) by [i(x) + $\theta \cdot$k(x)] in the expression defining z(x) and then calculating the effect of increasing θ from zero to one. This methodology can be applied to Eu(z) to determine the effect on expected utility, and to the first order expression Eu'(z)W(i(x) - r(x)) to determine the effect on α. Adding such k(x) to i(x) change i(x) keeping r(x) fixed, but do not increase the riskiness of each conditional distribution function for i(x).

For i(x), which is nondecreasing in x, the return is made riskier when k(x) which satisfy E[k(x)] = 0 and $\int_a^y k(x)\,dx \leq 0$ for all y in [a, b] are added to i(x). The interval [a, b] is assumed to contain the support of random variable x. Such a deterministic transformation of i(x) decreases return to insurance when those returns are low, and increases the return when the returns are high, spreading the return to the insurance asset in a mean preserving manner.

Theorem 4: When a deterministic transformation k(x) satisfying E[k(x)] = 0 and $\int_a^y k(x)\,dx \leq 0$ for all y in [a, b] is added to i(x), expected utility does not decrease for risk averse decision makers, and the optimal value for α does not decrease when relative risk aversion is increasing and less than one and absolute risk aversion is decreasing.

Proof: $dEu(z)/d\theta = Eu'(z)\alpha Wk(x)$. Now u'(z) is nondecreasing in x under risk aversion since $u''(z) \leq 0$ and $dz/dx \leq 0$. This is true for all θ in [0, 1] and is sufficient to imply that expected utility is no lower after k(x) is added to i(x) that before. Turning to the first order expression,

$dEu'(z)W(i(x) + \theta\cdot k(x) - r(x))/d\theta = E[u'(z)W\cdot k(x) + u''(z)(i(x) + \theta\cdot k(x) - r(x))\alpha W\cdot k(x)] = WEu'(z)[1 - R_R(z) + R_A(z)r(x)W]\cdot k(x)$. When $u'(z)[1 - R_R(z) + R_A(z)r(x)W]$ is nondecreasing in x for all θ, this is sufficient for the first order to be increased when θ increases from zero to one and hence the optimal α to not decrease. The assumptions on relative and absolute risk aversion in the theorem are sufficient for this.

Notice that these increases in the riskiness of the return to insurance increase expected utility, and lead to an increase in the allocation to the insurance asset, the exact opposite of the effect of adding random noise demonstrated in Theorem 3. This is because the changes in risk caused by these deterministic transformations increases the effectiveness of insurance in reducing the overall riskiness of the portfolio. The effect of such a change on expected utility is well known, with a recent and elegant demonstration given by Gollier and Schlesinger (1996).

Using exactly the same methodology and for the same assumptions on utility one can show that increases in the riskiness of r(x) reduce expected utility and reduce the investment in the risky asset thereby increasing the demand for insurance. For the return to the risky asset to increase in riskiness, however, k(x) must satisfy $E[k(x)] = 0$ and $\int_a^y k(x)\,dx \geq 0$ for all y in [a, b]. The inequality is reversed from that in Theorem 4 because the return r(x) is decreasing rather than increasing in x. Thus, those changes both decrease expected utility and decrease the amount allocated to the asset.

4. Conclusion

This work shows that there is more that can be demonstrated concerning the demand for insurance, and that changing the standard model is one way to begin to do this. Starting with a portfolio model with two risky assets and imposing assumptions so that one of the risky assets is an insurance asset, a few of these new results have been obtained.

Appendix

The standard model of insurance demand is often written as $z = W_0 + M - x + \theta(I(x) - P)$. In this model, W_0 is the value of wealth not held in the form of the asset subject to loss, M is the maximum value of the asset subject to loss and x is the random loss. I(x) is the indemnification function for one unit of insurance, P is the price of insurance, and θ is the number of units of insurance purchased. Transforming this expression into the portfolio notation used in the body of the paper requires two things, a rearranging of terms, and the introduction of notation for a new parameter. The new parameter is denoted V, and V represents the nonrandom value of the asset subject to loss. It must be that case that $0 \leq V \leq M$, and when risk aversion is assumed in the pricing of this risky asset, $V \leq (M - Ex)$ as well. The specific value for V is unimportant, but notation for its magnitude is necessary so that the value of the asset subject to loss can be included as a component of wealth.

Rearrange the terms in the outcome variable $z(x) = W_0 + M - x - \theta P + \theta I(x)$ as follows. $z(x) = (W_0 + V - \theta P)(W_0 + M - x - \theta P)/(W_0 + V - \theta P) + \theta P(I(x))/P$. Now define the return to the risky asset as $r(x) = (W_0 + M - x - \theta P)/(W_0 + V - \theta P)$ which is the return per dollar invested in all assets other than insurance. In this standard model, there are two non-insurance assets, a riskless asset whose return is one, and a risky asset whose return is $(M - x)/V$. The conversion is continued by defining $i(x) = I(x)/P$, where i(x) is the return per dollar invested in the insurance asset. To complete the change in notation, let $W_0 + V$ equal total wealth W and define $\alpha = \theta P/(W_0 + V)$. This implies that $z = W((1 - \alpha)\cdot r + \alpha\cdot i)$.

Typical assumptions concerning the indemnification function are that $0 \leq I(x) \leq x$, and that $0 \leq I'(x) \leq 1$. These are sufficient to imply that i(x) is nonnegative and nondecreasing. Also, it is usually assumed that $0 \leq \theta \leq 1$ and that x has support in [0, M]. Together these assumptions imply that z(x) is nondecreasing for all x and for all allowed values of α. Thus, the two assumptions in the body of the paper hold in this standard model of insurance demand.

Since $r(x) = (W_0 + M - x - \theta P)/(W_0 + V - \theta P)$, the effect of an increase in wealth on r(x) is determined by the sign of $\partial r(x)/\partial W_0 = (V - M + x)/(W_0 + V - \theta P)^2$. Two things follow immediately from this. First, changes in wealth increase the return on the non insurance part of the portfolio when losses are large and decrease return when loses are small. This is a reduction in the riskiness of that portion of the portfolio. Second, the impact is to also reduce the mean when $V < (M - Ex)$. The most important fact to recognize, however is that changes in wealth change the return to non insurance assets in the standard model of insurance demand. Changes in the price of insurance also change r(x). An increase in the price causes a reduction in the amount of wealth invested in the riskless asset, and changes r(x) in manner similar to a wealth reduction of size θ.

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Endnotes

1. Hadar and Russell (1974) and Hadar and Seo (1990) are two examples of this research. For instance, Hadar and Russell show that even first degree stochastic dominant improvements in the return to an asset do not lead to increased allocation to that asset without additional strong assumptions such as independence. Hadar and Seo show that the assumption of independence for the two risky assets allows various results from the single risky asset portfolio to continue to hold.

2. Foncel (2004) gives information on how expenditure on insurance varies with other indicators of risk taking.