

Individual vs. Aggregate Preferences: The Case of a Small Fish in a Big Pond

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Abstract

We show that there can be significant difference between risk preferences of individual investors and aggregate preferences toward risk in the economy. To demonstrate this, we investigate aggregate properties of an economy where all investors have convex utility functions corresponding to risk seeking behavior. In the case of risk seeking individual agents with identical initial endowments, assuming a budget constraint, and facing perfect competition the aggregate economy is risk neutral. In the case of risk seeking individuals with different initial endowments, we show that if there exists a continuum of wealth classes the economy in the aggregate will exhibit risk averse behavior. Thus, an economy consisting of risk seeking agents can lead to an aggregate economy that is risk averse. We prove that the converse is also true. For an economy that in the aggregate exhibits risk aversion we can construct an economy of all risk seeking agents that in the aggregate produces the given risk averse indifference curve. That is, an economy demanding a risk premium can be formed from individuals who do not demand such compensation. In the context of the equity premium puzzle our analysis shows that for an aggregate economy to display a high degree of risk aversion and to demand a relatively high risk premium individuals do not need to have implausibly high risk aversion.

JEL Classification: G10, D11, D80.

Keywords: Risk aversion, risk seeking, investor sentiment, risk premium.

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Abstract

We show that there can be significant difference between risk preferences of individual investors and aggregate preferences toward risk in the economy. To demonstrate this, we investigate aggregate properties of an economy where all investors have convex utility functions corresponding to risk seeking behavior. In the case of risk seeking individual agents with identical initial endowments, assuming a budget constraint, and facing perfect competition the aggregate economy is risk neutral. In the case of risk seeking individuals with different initial endowments, we show that if there exists a continuum of wealth classes the economy in the aggregate will exhibit risk averse behavior. Thus, an economy consisting of risk seeking agents can lead to an aggregate economy that is risk averse. We prove that the converse is also true. For an economy that in the aggregate exhibits risk aversion we can construct an economy of all risk seeking agents that in the aggregate produces the given risk averse indifference curve. That is, an economy demanding a risk premium can be formed from individuals who do not demand such compensation. In the context of the equity premium puzzle our analysis shows that for an aggregate economy to display a high degree of risk aversion and to demand a relatively high risk premium individuals do not need to have implausibly high risk aversion.

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1 Introduction

Many asset pricing models are based on the idea that a representative agent prices all assets in an economy. This representative agent is the one price setter and is the one whose optimal portfolio is the aggregate market. Theory has tended to model the representative agent using simple, differentiable utility functions such as the power utility or the exponential utility. Models using these simple utility functions, however, have not been able to describe such statistical characteristics as the means and variances of asset returns and this has led to a large literature on puzzles and anomalies.¹

In response to these puzzles, many researchers have sought to better understand how individuals make decisions in hopes of explaining the pricing puzzles. The psychology literature has revealed that individuals are averse to losses, tend to be overconfident, under- or overreact to different news events, choose consumption based on habit, and exhibit other behavioral patterns. Kahneman and Tversky (1979) observed that individuals underweight uncertain outcomes and overweight certain outcomes. As a result, individuals are risk averse over certain gains and risk seeking over certain losses. Friedman and Savage (1948), and subsequently Markowitz (1952), observed that individuals buy both lottery tickets and insurance. They argue that agents must be risk seeking over a range of gains and risk averse over a range of losses.²

These individual behavioral biases along with other observations concerning individuals have leaped into the asset pricing research. Jarrow (1988) shows that the Arbitrage Pricing Theory holds even when agents do not have preferences that can be represented by risk-averse expected utility functions. Jarrow and Zhao (2006) study optimal portfolios of investors with downside

¹See Constantinides (2002) and Hirshleifer (2001) for a review.

²See Brunk (1981), Gregory (1980), Hakansson (1970), and Kwang (1965) for studies of Friedman and Savage utility function.

loss-averse preferences. Shefrin and Statman (2000) develop a positive behavioral portfolio theory (BPT) where investors choose portfolios by considering expected wealth, desire for security and potential, aspiration levels, and probabilities of achieving aspiration levels. The optimal portfolios of BPT investors resemble combinations of bonds and lottery tickets, consistent with Friedman and Savage's observation. Bakshi and Chen (1996) study, both theoretically and empirically, an economy where investors acquire wealth not only for its implied consumption, but for the resulting social status. When investors care about relative social status, propensity to consume and risk taking behavior depend on social standards, and this has an impact on asset prices. Levy and Levy (2002) find strong empirical support for Markowitz utility function in their experimental study. Post and Levy (2005), using stochastic dominance criteria, find that the Markowitz utility function (risk aversion of losses and risk seeking over gains) captures the cross-sectional pattern of stock returns. Constantinides (1990) shows how habit formation helps resolve the equity premium puzzle. Goetzmann and Massa (2002) identify a group of index fund investors who systematically invest in the fund after volatility increases. Coval and Shumway (2005) find evidence that proprietary traders are loss averse (risk averse over gains and risk seeking over losses). Many others have found evidence of risk seeking behavior in aggregate prices (Jackwerth 2000; Levy and Levy 2001, 2002; Rosenberg and Engle 2002; Kliger and Levy 2002; Bakshi and Wu 2006; Green and Rydqvist 1997; Florentsen and Rydqvist 2002) and the evidence spans time, several countries and many different empirical methods.

Two challenges arise. The first concerns the failure of classical models to explain the equity premium. Mehra and Prescott (1988) find that the risk averse agent must have a very large risk aversion coefficient in order to explain the large equity premium observed in the data. The agent must be very risk averse and demand a large compensation for risk. What if individual agents actually do have a Kahneman-Tversky or a Markowitz utility function and are risk seeking over some ranges of wealth? If this is the case, then the equity premium puzzle may be a much larger problem than originally thought.

The second challenge is in regard to the connection between individual behavior and aggregate behavior. The behavioral finance literature describes the behavioral patterns of individuals and how their financial decisions are affected by their behavior. Kahneman and Tversky's prospect theory is about individual behavior. Friedman and Savage (1948) argue that individuals gamble

and buy insurance. Individuals may be loss averse, overconfident, and form consumption habits. What is unclear is how individual behavior affects aggregate prices. On the other hand, many have found evidence of unusual behavior in aggregate prices. For example, Post and Levy (2005) find the Markowitz utility function best explains the cross section of returns. Nonetheless, can we necessarily imply anything about individual behavior from these findings? By observing behavior in aggregate prices, can we conclude that individuals have the same behavior?

We show in this paper that the representative agent of the economy can be very different from the individuals who make up the economy. First, individual behavior does not necessarily lead to the same aggregate behavior. Second, aggregate behavior does not necessarily imply anything concerning the behavior of individuals. Further, individual risk seeking behavior does not necessarily challenge a positive equity premium. We consider an economy comprised solely of risk seeking agents who have utility functions defined over current consumption and consumption at a future date. Surprisingly, the aggregate economy is risk averse. We do not believe or suggest that all individuals are actually risk seeking. We use this extreme assumption in order to contrast individual and aggregate behavior.

Beginning with the case of risk seeking agents facing a budget constraint, we prove that although individual agents are risk seekers, under perfect competition and identical initial endowments, the aggregate economy is risk neutral. Next, we consider the case of risk seeking agents with different initial endowments. If there exists a closed and bounded continuum of wealth classes then the economy's representative agent will exhibit risk averse behavior.

Finally, we prove that for any aggregate economy that demands a risk premium, there exists a collection of risk seeking individuals with a specifically chosen wealth distribution that exactly replicates the given risk averse economy. This finding is relevant for the studies that aim at using the data on aggregate behavior (aggregate consumption, market returns, market volatility) to draw conclusions about individual behavior. An example of such research question is whether we can necessarily say that since we observe a large positive market risk premium that all (or most) agents in the economy are highly averse to risk. The answer to this is no. To show this, we prove that an economy exhibiting aggregate risk averse behavior can be supported by risk-seeking individuals. Though it may be tempting to make statements concerning individual behavior based on aggregate data, the analysis in this paper shows that the relationship between groups and individuals is not

clear or direct. An economy demanding a risk premium can be formed from individuals who do not demand such compensation. Our analysis also implies that for an aggregate economy to display a high degree of risk aversion and to demand a relatively high risk premium, individuals *do not* need to have implausibly high risk aversion.

A simple example helps gain intuition. Consider a large economy of identical risk-seeking individuals with identical endowments across two states. Agents' preferences across the two states are described by the time separable utility function $E[U(C_1) + U(C_2)]$ where $C_1 = C_2 = C$ is the aggregate consumption, $U(\cdot)$ is the agents' risk-seeking utility function and $E[\cdot]$ denotes the expectation operator. In this simple setting, prices of the two states are equal. Therefore, half the population will enter a bet that pays 1 in state one and zero in state two. The other half of the population will take the opposite bet. Interestingly, though each individual is risk-seeking, the economy behaves as if it is risk-neutral.

The remainder of the paper is organized as follows. Section 2 develops the aggregation results. Subsection 2.2 studies the case of identical agents. Subsection 2.3 analyzes the case when agents have different initial wealth. Section 3 shows that every risk averse (in the aggregate) economy can be obtained by aggregating risk seeking individual agents. Section 4 concludes. All proofs are in the Appendix.

2 The Model of Utility Aggregation

2.1 Convex Utility

In a typical model, an agent solves the expected utility maximization problem,

$$\max \mathbb{E} \left[u(C_0, \tilde{C}_1) \right],$$

subject to a budget constraint, where the utility of consumption is assumed concave and additively separable:

$$u(C_0, C_1) = U_0(C_0) + U_1(C_1).$$

In this paper we depart from the concavity (risk aversion) assumption made about the utility function of individual agents in the economy. Several landmark papers argue that individual utility

functions have a convex region. We take this assumption to the extreme by studying aggregate properties of an economy where *all individuals* have identical *convex* additively separable utility of consumption:

$$u(x_i, y_i) = U_0(x_i) + U_1(y_i),$$

where x_i represents agent i 's consumption at time 0 and may be referred to as the “ X consumption good,” and y_i represents agent i 's (uncertain) consumption at time 1 and may be referred to as the “ Y consumption good.” We assume that $u(x_i, y_i)$ is twice continuously differentiable and increasing in all arguments. Without loss of generality, we work with indifference curves. Individual agents are assumed to have convex utility functions, which correspond to *concave* indifference curves (Figure 1). We will first show that when all agents are identical that the aggregate economy is characterized by *linear* indifference curves, which correspond to risk neutral preference. We will then show that when all agents have the same utility but are heterogeneous in wealth that the aggregate economy is characterized by *convex* indifference curves.

2.2 Identical Agents

We begin with an economy with N identical agents all with the same utility function $u(x_i, y_i)$ and all having the same initial level of wealth. Each agent has a strictly risk seeking utility function that corresponds to strictly concave indifference curves. The economy is initially endowed with a fixed quantity of Y , $Y_{\max} > 0$, where Y_{\max} is the maximum amount of the Y good that the economy can produce. None of the X consumption good is available in this initial case. Since all agents are the same then it is clear that all agents will be initially endowed with an equal quantity of Y . In addition, all agents will be equally happy since they have the same utility, $u(x_i, y_i) = k$, brought about by having the same initial allocation of the Y good.

Suppose, instead, there is a positive amount of the X consumption good, $\bar{X} > 0$, available for allocation. We again want to determine the efficient allocation of both the X and the Y goods. By efficient we mean for a fixed quantity of the X good, the economy should produce the minimum amount of Y necessary so that all agents have utility equal to k as in the previous case. Since the X good is now available, less Y is needed in the economy. For each $\bar{X} > 0$, we want to find the minimum Y required so that all agents utility remains at k and all goods are completely allocated.

By finding all such (X, Y) pairs we are essentially tracing the aggregate economy's indifference curve.³ To do this, we must solve the optimization problem:

$$\begin{aligned}
 & \text{Min}_{x_1, \dots, x_N} && Y = \sum_{i=1}^N \Psi(x_i; k) \\
 & \text{subject to} && \sum_{i=1}^N x_i = \bar{X} \\
 & \text{and} && x_i \geq 0, y_i \geq 0 \text{ for all } i.
 \end{aligned} \tag{1}$$

The capitalized Y and X is used to denote aggregate quantities of the two goods whereas the lower case x and y are used to describe the quantity of the two goods held by individual agents. By the Implicit Function Theorem, the equation $u(x_i, y_i) = k$ defines the implicit function, $y_i = \Psi(x_i; k)$. We do assume no short selling in our problem. However, our results hold for any finite short selling constraint since we can redefine the coordinate axis through a change of variable. This is a reasonable constraint since no agent can realistically short an infinite amount even if he does choose to “bet the farm”. What is important, is to rule out *infinite* borrowing, not borrowing *per se*.

In order solve Problem 1, we must identify the quantity of X and Y allocated to each agent that maintains each agents' utility at k . In determining the optimal allocation we must consider several possibilities. The X good could be equally distributed among all agents or the X good could be allocate unequally among all agents. It turns out that the second alternative is the most efficient. To see why, consider Figure 1. Figure 1 shows an indifference curve for a risk seeking agent. A single agent can at most hold y_{\max} of Y and x_{\max} of X and he is indifferent between holding y_{\max} and holding x_{\max} . The agent is allocated less Y for each unit of X he receives so as to stay on the k^{th} indifference curve. But, the marginal rate of substitution increases with the each unit of X received. Suppose he receives x_A . The slope of the tangent line indicates the trade off between X and Y . Notice that the slope is very small indicating that the next unit of X does not greatly decrease his allocation of Y . However, suppose he receives x_B . Now, the slope of the tangent is much steeper indicating that each additional unit of X greatly decreases his allocation of Y . What we learn from this picture is that the economy is better off allocating a large quantity of

³As previously stated, the term “aggregate indifference curve” refers to the indifference curve for the aggregate economy, which gives the rate at which the total amount of consumption in one period must increase when the aggregate amount of consumption in the other period falls, in order to keep utility levels of all agents unchanged.

X to one individual since the trade off between the goods is high. The trade off is at its maximum when the agent is allocated x_{\max} . So, the economy is required to produce the least amount of Y when all the X good is allocated to only a few agents so that these agents hold none of the Y good. One agent will likely hold a remainder of X in the case that there is not enough X for the agent to receive the entire x_{\max} . The remaining agents will hold all Y and no X . All agents will either specialize in the X or the Y good but one agent will be forced to hold some of both (the remainder of X and enough Y to yield a utility of k). This leads us to the formal statement of Proposition 1.

Proposition 1 (Efficient Allocation) *Let $\bar{X} = (n - 1)x_{\max} + r$, for $n = 1, 2, \dots, N$ and $r \in (0, x_{\max}]$ be the quantity of X available and define x_{\max} as such value of x that solves: $u(x, 0) = k$. Then x_{\max} is the maximum amount of X each individual agent can hold and $y_{\max} = \Psi(0; k)$ is the maximum amount of Y the agent can hold. The solution to (1) is*

$$(x_i, y_i) = \begin{cases} (x_{\max}, 0) & \text{for } i = 1 \text{ to } n - 1 \\ (r, \Psi(r; k)) & \text{for } i = n \\ (0, y_{\max}) & \text{for } i = n + 1 \text{ to } N. \end{cases}$$

As previously mentioned, the no short sale constraint can be exchanged with a finite short sale constraint. In this case we would see a similar allocation. Some agents would short the maximum amount permitted of the X good and hold more than 100% of Y . The remaining agents would short the maximum quantity of Y and hold more than 100% of X . One agent would be allocated some of both goods.

Now that the efficient allocation for each agent has been determined, we can turn our attention to aggregating the demand under perfect competition. This is found by aggregating the demands from Proposition 1. For a given quantity of X , we write the total quantity of Y required by the economy as

$$Y_N(X) = \Psi(r; k) + \left(N - 1 - \text{int} \left[\frac{X}{x_{\max}} \right] \right) y_{\max}, \quad \text{for } X \in [0, X_{\max}].$$

There are two types of agents who hold Y . There are $N - n - 1$ agents who specialize in Y , and there is one agent who holds some of X and Y (the n^{th} agent). This equation sums together the quantity of Y required by the n^{th} agent and the quantity of Y required by those holding y_{\max} (y_{\max} times the number of agents).

To determine the demand under perfect competition, we consider the limit of the aggregate demand function as the number of agents, N , increase to infinity while the supply of Y and X is held constant. By taking the limit in this fashion we keep the focus on the increasing competitiveness of the economy. Perfect competition is modeled by increasing the number of agents in such a way that the influence of each individual participant becomes negligible. This notion of perfect competition is consistent with Aumann (1964) who formally argues that a mathematical model of perfect competition should contain infinitely many participants.

The previous proposition establishes that all but one agent, agent n , will be endowed with either all Y or all X . Hence, all but one agent specialize in either X or Y . As the number of agents increase to infinity, agent n 's contribution to the economy becomes negligible. Therefore, in perfect competition, all agents are at a corner with the exception of the one negligible agent - a set of measure zero. The indifference curve that describes this economy's behavior is a straight line connecting the two points $(X_{\max}, 0)$ and $(0, Y_{\max})$.

Figures 2 and 3 illustrate these arguments. Figure 2 shows the aggregate indifference curve for the economy with two agents. Suppose X_A of X is available. This is a small quantity of X . From Proposition 1, we know that it is most efficient to allocate X to one agent until that agent holds x_{\max} . But there is not enough X for agent 1 to specialize in X . Therefore, agent 1 is allocated all of X_A and enough Y to keep him on the k^{th} indifference curve, while agent 2 holds all Y . This yields the aggregate indifference point indicated by point A . In this case, the shape of the aggregate indifference curve depends only on the first agent since he is the only one with an interior position. This holds true until enough X is available for agent 1 to specialize in X , X_B . Now, agent 1 holds all X and agent 2 holds all Y . If more X is available, then the curvature of agent 2's indifference curve will define the shape of the aggregate indifference curve. Agent 1 will no longer matter since he does not hold an interior position.

Figure 3 shows the case of five agents. Recall that X_{\max} and Y_{\max} are held constant. Therefore, it takes less X for an agent to specialize in X . We now see five "lumps" representing the five agents. It is still the case that the curvature of the aggregate indifference curve is controlled by the one agent holding the interior position. But with five agents, the "lumpiness" is much smaller than in the two agent case. As the number of agents increase, the lumpy aggregate indifference curve will converge to the straight dotted line. That is, the contribution of the one individual with the

interior position becomes less significant in comparison to the many agents specializing in either X or Y .

Proposition 2 formally establishes this claim. The details of the proof are provided in the appendix.

Proposition 2 *For fixed integer N^* , as $z \rightarrow \infty$, Y_z converges to $Y_\infty(X) = -\frac{X_{\max}}{Y_{\max}}X + N^*\Psi(0; k)$ uniformly on $[0, X_{\max}]$ where $X_{\max} = N^*\Psi^{-1}(0; k)$, and $Y_{\max} = N^*\Psi(0; k)$.*

In Proposition 2, we start with an economy with a fixed number of agents, N^* , and this economy achieves some initial level of individual utility. Then we show that as the economy approaches perfect competition, the aggregate indifference curves become straight lines. This proposition states that although individual agents have concave indifference curves, the aggregate economy, under perfect competition and identical initial wealth endowments, has a convex (linear) indifference curve. Though individual agents are risk-seeking, the aggregate economy behaves as if it is risk-neutral.

2.3 Agents with Different Initial Wealth

We now consider the case where the agents all have the same convex utility function, but initial wealth is different across agents. Each agent will be initially endowed with a different quantity of Y .

We begin with a simple case. Suppose there are $N_1 + N_2$ agents divided into two wealth classes. There are N_1 type 1 agents (the poor) who have initial wealth less than the N_2 type 2 agents (the rich). We first consider the case where the economy produces Y_{\max} and no X is available. Wealth is divided in such a way that the efficient allocation of Y_{\max} between the two types causes the type 1 agent to be on the indifference curve k_1 and the type 2 agent to be on the k_2 indifference curve where $k_1 < k_2$. The Y good is initially allocated efficiently. Hence, all poor agents are allocated the same quantity and all the rich agents are allocated the same quantity but the rich are given more than the poor agents. We now ask the same question posed previously. If $X \in (0, X_{\max}]$ is available for allocation, what is the minimum amount of the Y good required in the economy in order for the poor to always be on the k_1^{th} indifference curve and for the rich to remain on the k_2^{th} indifference curve? We are searching for all the efficient (X, Y) allocations that maintain all

agents' level of utility.

This question is exactly the same as the previous problem except that agents are no longer identical. As before, due to the concavity of the indifference curves, the efficient allocation will be a corner solution. In the previous problem, we allocated all the X good to a set of agents so that all agents held all X or all Y and only one agent held a combination of both. This procedure resulted in a correct allocation because the rate at which each agent was willing to substitute Y for X increased with X . Each agent was willing to pay more, in units of the Y good, for each additional increment of the X good. The same is true in this case except we must be careful in selecting which type, the rich or the poor, should be allocate the X first. That is, which type, the rich or the poor, has the greatest rate of substitution for each increment of X ? The answer to this question depends on the concavity of the indifference curve. If $\left| \frac{y_{1,\max}}{x_{1,\max}} \right| > \left| \frac{y_{2,\max}}{x_{2,\max}} \right|$ then the type 1 agent's (the poor) rate of substitution is greater and they should be given the X good first (see Figure 4). If $\left| \frac{y_{1,\max}}{x_{1,\max}} \right| < \left| \frac{y_{2,\max}}{x_{2,\max}} \right|$ then the rich are willing to pay more for the X good and should be given X first. If, however, $\left| \frac{y_{1,\max}}{x_{1,\max}} \right| = \left| \frac{y_{2,\max}}{x_{2,\max}} \right|$ then it does not matter which type receives the X good first and the problem reduces to the previous identical agent case. For the remainder of this section we consider the case $\left| \frac{y_{1,\max}}{x_{1,\max}} \right| > \left| \frac{y_{2,\max}}{x_{2,\max}} \right|$. An example of a simple convex utility function that satisfies this condition is $u(x_i, y_i) = \alpha x_i^2 + \beta y_i^4$, $\alpha > 0$, $\beta > 0$. This is, of course, not the only utility function that satisfies this condition. Though not discussed here, all results hold for the case $\frac{y_{1,\max}}{x_{1,\max}} < \frac{y_{2,\max}}{x_{2,\max}}$.

Without loss of generality, we order the agents such that $i = 1, \dots, N_1$ are type 1 and $i = N_1 + 1, \dots, N_1 + N_2$ are type 2. All agents have identical twice continuously differentiable, monotonically increasing convex utility function. Type 1 agents achieve utility level k_1 and type 2 agents achieve utility level k_2 . By the Implicit Function Theorem, the equation $u(x_i, y_i) = k$ defines the implicit function, $y_i = \Psi(x_i; k)$. The problem to be solved is

$$\begin{aligned} \text{Min}_{x_1, \dots, x_{N_1+N_2}} \quad & Y = \sum_{i=1}^{N_1} \Psi(x_i; k_1) + \sum_{i=N_1+1}^{N_1+N_2} \Psi(x_i; k_2) \\ \text{subject to} \quad & \sum_{i=1}^{N_1+N_2} x_i = \bar{X} \\ \text{and} \quad & x_i \geq 0, y_i \geq 0 \text{ for all } i. \end{aligned} \tag{2}$$

This problem is similar but slightly more difficult than the previous problem. However, we can apply the intuition learned from the previous case. As before, for a fixed quantity of X , we want to

produce the minimum amount of Y necessary to keep the type 1 agents on the k_1 indifference curve and the type 2 agents on the k_2 indifference curve. Because the marginal rate of substitution is greatest for the poor agents (by assumption), allocating the X good to the poor first will decrease the amount of Y needed in the economy more than if the X good was first allocated to the rich.

This leads us to two possibilities: if X is less than what is needed for all the poor agents and if X is more than what is needed for all the poor agents. In the first case, the rich will not receive any of the X good since the X good is first allocated to the poor. Instead, the poor will receive all the X good in the same way as described in Proposition 1. In the second case, all the poor agents specialize in the X good. The remaining X will be allocated to the rich as described in Proposition 1. This allocation is formalized in Proposition 3.

Proposition 3 *Let the number of type 1 and type 2 agents be N_1 and N_2 , respectively. The type 1 agents have an initial utility of k_1 and the type 2 agents have initial utility $k_2 > k_1$ and assume that $\left| \frac{y_{1,\max}}{x_{1,\max}} \right| > \left| \frac{y_{2,\max}}{x_{2,\max}} \right|$ where $x_{i,\max} = \Psi^{-1}(0; k_i)$ and $y_{i,\max} = \Psi(0; k_i)$ for $i=1,2$. If $\bar{X} = (n-1)x_{1,\max} + r \leq N_1x_{1,\max}$, where $n = 1, 2, \dots, N_1$, and $r_1 \in (0, x_{1,\max}]$, then the solution to problem (2) is*

$$(x_{1,i}, y_{1,i}) = \begin{cases} (x_{1,\max}, 0) & \text{for } i = 1 \text{ to } n-1 \\ (r_1, \Psi(r_1; k_1)) & \text{for } i = n \\ (0, y_{1,\max}) & \text{for } i = n+1 \text{ to } N_1 \end{cases}$$

$$(x_{2,i}, y_{2,i}) = (0, y_{2,\max}) \quad \text{for } i = N_1 + 1 \text{ to } N_1 + N_2.$$

If $N_1x_{1,\max} > \bar{X} \geq N_1x_{1,\max} + N_2x_{2,\max}$ and $\bar{X} = N_1x_{1,\max} + (n-1)x_{2,\max} + r_2$ where $r_2 \in (0, x_{2,\max}]$ and then the efficient allocation is

$$(x_{1,i}, y_{1,i}) = (x_{1,\max}, 0) \quad \text{for } i = 1, \dots, N_1$$

$$(x_{2,i}, y_{2,i}) = \begin{cases} (x_{2,\max}, 0) & \text{for } i = N_1 + 1 \text{ to } N_1 + n - 1 \\ (r_2, \Psi(r_2; k_2)) & \text{for } i = N_1 + n \\ (0, y_{2,\max}) & \text{for } i = N_1 + n + 1 \text{ to } N_1 + N_2 \end{cases}$$

Notation $x_{i,\max} = \Psi^{-1}(0; k_i)$ means that $x_{i,\max}$ is the value of x that solves $u(x, 0) = k_i$ for $i=1,2$.

The efficient allocation in Proposition 3 is very similar to the allocation described in Proposition 1. All but one agent holds either all X or all Y . It should not be surprising, then,

that the aggregate indifference curves under perfect competition are very similar to the those described in Proposition 2. Whereas Proposition 2 describes a linear aggregate indifference curve with slope $-\frac{Y_{\max}}{X_{\max}}$, the aggregate indifference curve here is piecewise linear where the first linear segment has slope $-\frac{y_{1,\max}}{x_{1,\max}}$ for $X \in [0, N_1 x_{1,\max})$ and the second piece has slope $-\frac{y_{2,\max}}{x_{2,\max}}$ for $X \in (N_1 x_{1,\max}, N_1 x_{1,\max} + N_2 x_{2,\max}]$. Since, by assumption, $\left| \frac{y_{1,\max}}{x_{1,\max}} \right| > \left| \frac{y_{2,\max}}{x_{2,\max}} \right|$, the two linear segments join to form convex aggregate indifference curves. The aggregate indifference curves, therefore, are continuous and convex but not differentiable at the point where the two linear segments connect (see Figure 5). Such an aggregate indifference curve represents an economy that is risk averse! We now formalize this.

Proposition 4 *The aggregate demand of an economy with N_1 type 1 agents with utility k_1 and N_2 type 2 agents with utility k_2 such that $k_1 < k_2$ is*

$$Y(X) = \begin{cases} -\frac{\Psi(0;k_1)}{\Psi^{-1}(0;k_1)} N_1 \Psi^{-1}(0;k_1) + N_2 \Psi(0;k_2), & X \in [0, N_1 x_{1,\max}] \\ -\frac{\Psi(0;k_2)}{\Psi^{-1}(0;k_2)} (X - N_1 \Psi^{-1}(0;k_1)) + N_2 \Psi(0;k_2), & X \in [N_1 x_{1,\max}, N_1 x_{1,\max} + N_2 x_{2,\max}] \end{cases}$$

where $N_1 x_{1,\max} = N_1 \Psi^{-1}(0;k_1)$ and $N_2 x_{2,\max} = N_2 \Psi^{-1}(0;k_2)$.

We now extend this result to n wealth classes. For simplicity, we assume that all wealth classes have the same number of agents, $N_i = N$ for all i . This is without loss of generality. The agents are ordered by indifference curve, $k_i < k_{i+1}$. Then, for wealth classes $i = 1$ to n , the system of linear equations is:

$$Y_i(X) = -\frac{y_{i,\max}}{x_{i,\max}} [X - x_{i-1}] + y_{i-1} \quad \text{for } X \in [x_{i-1}, x_i], \quad (3)$$

where $x_i = N \sum_{j=1}^i x_{j,\max}$ and $y_i = Y_{\max} - N \sum_{j=1}^i y_{j,\max}$ with $x_0 = 0$ and $y_0 = Y_{\max}$. The maximum amount of X and Y that this economy can hold is $X_{\max} = N \sum_{j=1}^n x_{j,\max}$ and $Y_{\max} = N \sum_{j=1}^n y_{j,\max}$. This system of equations creates a family of continuous, convex, but not differentiable indifference curves. The indifference curves are not differentiable at the points where the linear segments connect. However, the indifference curves can be made smooth by infinitely increasing the number of wealth classes so as to create a continuum of k values over a range k_{\min} to k_{\max} while holding X_{\max} and Y_{\max} constant. That is, for every indifference curve $k \in [k_{\min}, k_{\max}]$, there exists agents with

the appropriate level of wealth that places them on that indifference curve. Adding wealth classes in this way reduces the length of each line segment. If there are an infinite number of wealth classes creating, then the aggregate indifference curve will become smooth. Whereas Figure 5 shows a piecewise linear aggregate indifference curve, Figure 6 shows a much smoother curve that is created through the addition of wealth classes. The following Theorem proves that the resulting aggregate indifference curve will be convex and differentiable. Hence, such an economy will demand a risk premium.

Theorem 5 *Let there exist $[k_{\min}, k_{\max}]$, a continuum of utility levels with $k_{\min} > 0$ and $k_{\max} < \infty$, and for each level of utility $k_i \in [k_{\min}, k_{\max}]$ there exists an infinite number of agents so that the i -th agent type has a demand function described by (3). If, for all $k_i < k_{i+1}$, $\left| \frac{y_{i,\max}}{x_{i,\max}} \right| > \left| \frac{y_{i+1,\max}}{x_{i+1,\max}} \right|$ and fixing the maximum amount of X and Y at X_{\max} and Y_{\max} , respectively, the aggregate indifference curve for the economy is strictly convex and differentiable over $[0, X_{\max}]$. If for all $k_i < k_{i+1}$, $\left| \frac{y_{i,\max}}{x_{i,\max}} \right| = \left| \frac{y_{i+1,\max}}{x_{i+1,\max}} \right|$ then the aggregate indifference curve is convex (linear).*

With two wealth classes, the economy behaves as if it has a piecewise linear, convex indifference curve. This indifference curve has two pieces and is not differentiable at the point where the two linear segments connect. By increasing the number of wealth classes, the number of linear segments increase and the length of each segment decreases. This theorem states that if there exists a closed and bounded continuum of wealth classes then the aggregate indifference curve is strictly convex and differentiable. Any gaps in wealth will result in a non-differentiable kink in the aggregate indifference curve, but the curve will still be convex.

There is one issue that requires a comment. We have shown that the aggregate economy's utility function is quasiconcave but this does not necessarily lead to a concave utility function. Assuming that the aggregate economy's utility is also time additive (as we assumed with individuals), if

$$u(C_0, C_1) = U_0(C_0) + U_1(C_1)$$

is quasiconcave then at least either U_0 or U_1 must be concave (see Yaari 1977 for proof). If we assume that U_0 and U_1 are from the same family of utility functions (both are power utilities), then it is necessarily true that both must be concave and hence $u(C_0, C_1)$ is concave. However, if we assume that U_0 and U_1 are independent then one of them might not be concave. In this case,

as Gorman (1970) argues, there will be “considerable regions” over which one of the consumption goods will be inferior. In our economy, however, we are giving the economy the choice between consuming today and consuming tomorrow. To consider one of the two consumption goods to be inferior over large regions does not make sense in this setting. Hence, it is reasonable to assume that both U_0 and U_1 are concave causing $u(C_0, C_1)$ to be concave. Therefore, though each individual agent is risk-seeking, the economy behaves as if it is risk-averse! Preferences toward risk are very different at the individual and at the aggregate levels.

3 De-Aggregating the Economy

We have shown that an economy consisting of risk seeking agents can lead to an aggregate economy that is risk averse. This, of itself, is an interesting result since it calls into question how researchers apply individual behavior in an economy where aggregate behavior is observed. But, we can still take this analysis one step further. Since we observe aggregate behavior (aggregate consumption, aggregate prices, etc.), can we necessarily use this data to make statements regarding individual behavior? Can we necessarily say that since we observe a positive risk premium that all (or most) agents in the economy are risk averse? The answer to this is no.

To show this, we simply need to perform the above analysis backwards. We start with an aggregate convex indifference curve, implying that the economy is risk averse. We prove the existence of a risk seeking agents that aggregate to the same risk averse aggregate indifference curve. For any decreasing, convex function $g : [0, X_{\max}] \rightarrow [0, Y_{\max}]$, there exists a distribution of wealth classes that replicates the convex curve. Said differently, any economy exhibiting aggregate risk-averse behavior can be supported by risk-seeking individuals. All that is needed is an appropriate wealth distribution.

Theorem 6 *Consider an economy of risk averse agents with aggregate demand defined by $Y = g(X)$ with the properties $g'(X) < 0$, $g''(X) > 0$, $0 < g'(X) < \infty$ for all $X \in [0, X_{\max}]$, $g(0) = Y_{\max}$ and $g(X_{\max}) = 0$. There exists a distribution of wealth such that an economy of risk seeking agents, with indifference curves $Y = \Psi(X; k_i)$ satisfying the property $\left| \frac{y_{i,\max}}{x_{i,\max}} \right| > \left| \frac{y_{i+1,\max}}{x_{i+1,\max}} \right|$ for $k_i < k_{i+1}$, has the same aggregate demand as the economy of risk averse agents.*

Though it may be tempting to make statements concerning individual behavior based on ag-

gregate data, this analysis shows that the relationship between groups and individuals is not clear or direct. An economy demanding a risk premium can be formed from individuals who do not demand such compensation.

Our results are staggering when contrasted with a more conventional intuition. Suppose we assume that the representative agent of the economy has, for example, a negative exponential utility function. By observing returns on the market we infer that the risk aversion of the representative investor must be some value, RA_A . According to arguments that lead to the equity premium puzzle, judgment may be applied to this value leading us to a conclusion that the level of risk aversion of the representative agent is too high. This puzzle is further deepened if we assume that the individuals that make up the economy also have negative exponential utility. When all individuals have negative exponential utility, the representative investor also has negative exponential utility with risk aversion equal to the wealth-weighted harmonic mean of the individual agents' risk aversion,

$$\frac{1}{RA_A} = \sum_{k=1}^K \frac{W_k}{W_A} RA_k, \quad W_A = \sum_{k=1}^K W_k.$$

Harmonic means are never larger than arithmetic means and are equal only when all individuals are equally risk averse. Therefore, the risk aversion of the representative investor (of the aggregate economy) is less than the wealth-weighted average of the individuals' risk aversion. There are individual agents with risk aversion even higher than that of the representative agent! If the risk aversion of the representative agent is already deemed to be high, then there will be individuals with implausibly high levels of risk aversion. In this framework, the finding of a high risk premium in the aggregate leads to an even more implausible conclusion about the individual agent.

In contrast, our analysis shows that for an aggregate economy to display a high degree of risk aversion and to demand a relatively high risk premium, individuals *do not* need to have implausibly high risk aversion.

Prior to concluding, we should make a final technical remark. The results of the previous two theorems do not require perfect competition within each wealth class. It is sufficient to have an infinite number of wealth classes consisting of only one agent. The assumption of perfect competition within each wealth class is made merely to rely on our previously established results.

4 Conclusion

The relation between risk preferences of individual agents in the economy and the attitude toward risk in the aggregate is an important one in financial economics. Ultimately, it is preferences in the aggregate that determine the relationship between risk and return, and the pricing of risk in the economy. Yet, it is biases of individuals that characterize deviations from canonical fully rational financial models. We show that there may be dramatic, significant differences, in the nature of risk preferences of individual investors and aggregate preferences toward risk in the economy. To demonstrate this, the paper investigates aggregate properties of an economy where all investors have convex utility function. First, it is shown that when all investors have identical wealth, assuming a budget constraint and under perfect competition the aggregate indifference curve is convex (linear). This result shows that the assumptions of perfect competition and the budget constraint that eliminates unbounded borrowing or short selling may be of the same degree of importance as the assumption on the shape of the individual utility function.

We then consider an economy where all agents have convex utility function, but different initial levels of wealth. If there is a finite number of different wealth classes then the aggregate indifference curve is continuous, convex, but not differentiable because it consists of finite number of linear segments. We prove that with a continuum of wealth classes, each being characterized by perfect competition, the aggregate indifference curve is convex and differentiable. The theorem shows that an economy that consists of small (atomistic) *risk-seeking* individual investors in the aggregate is characterized by an indifference curve consistent with *risk aversion*. The result shows how aggregation of individual agents each with a convex utility function can yield an indifference curve consistent with concave utility function.

We then go a step further. We start with an aggregate convex indifference curve (which corresponds to the case of risk aversion). We show that there exists an economy composed of risk seeking individuals and a distribution of wealth such that in the aggregate the economy produces the same risk averse indifference curve as given. This result shows that empirical studies based on aggregate data can potentially be consistent with a wide variety of individual investor behavior specifications, even the ones based on utility functions with convex regions. Therefore, caution must be taken when drawing conclusions about individual behavior based on aggregate data.

We began this paper listing several utility functions that include both risk seeking and risk averse

regions (Friedman and Savage 1948, Markowitz 1952, Kahneman and Tversky 1979, Tversky and Kahneman 1992). Each of these utility functions were created to describe individual behavior. What is still unknown, concerning these functions, is how these functions aggregate. Do the risk seeking regions remain in the aggregate or do they vanish, as they do in this paper? We believe that this paper is an important stepping stone in exploring the aggregate properties of these other utility functions.

A Appendix

A.1 Utility Aggregation: Identical Agents

Proof of Proposition 1. Efficient Allocation for Identical Agents. The maximum quantity of X that any agent can hold is $x_{\max} : u(x_{\max}, 0) = k$. Notice that each $y_i = \Psi(x_i; k)$ is concave and continuous for all $x_i \in [0, x_{\max}]$. This is so because, by Implicit Function Theorem,

$$\frac{d^2 y_i}{dx_i^2} = -\frac{u_{xx}u_y^2 - 2u_{xy}u_xu_y + u_{yy}u_x^2}{u_y^3} < 0.$$

Note that $u_{xy} = 0$ because we study additively separable utility function. Since the sum of concave functions is concave, the interior optimum of (1) is a maximum and not a minimum. Due to the non-negativity constraints, the set of feasible solutions is compact and therefore, the solution exists and the solution must be located on the boundary of the feasible solution set.

Suppose $\bar{X} = x_{\max}$. Then the only corner solution available is associated with x_{\max} being given to only one agent. Hence, the first agent holds $(x_1, y_1) = (x_{\max}, 0)$ and the remaining $N - 1$ agents hold $(x_i, y_i) = (0, \Psi(0; k))$ for $i = 2, \dots, N$.

Now suppose the economy has $\bar{X} = 2x_{\max}$. We already know that it is better to allocate to the first agent $(x_1, y_1) = (x_{\max}, 0)$ than to divide x_{\max} amongst several. This leaves us with an economy with $N - 1$ agents and only x_{\max} left to distribute. But this is the same as the original problem (except with one less agent). Hence, the second agent is allocated $(x_2, y_2) = (x_{\max}, 0)$. All remaining agents receive $(x_i, y_i) = (0, \Psi(0; k))$ for $i = 3, \dots, N$.

Finally, suppose $\bar{X} = (n - 1)x_{\max} + r$ is available. By the above argument, the first $n - 1$ agents will receive $(x_{\max}, 0)$. The problem is now reduced to allocating r amongst the remaining $N - n + 1$ agents. The corner solution is no longer associated with $(x_{\max}, 0)$ since $r \leq x_{\max}$. The corner is now at the point $(x, y) = (r, \Psi(r; k))$ and one agent will receive all of r while the remaining agents receive none of the endowed \bar{X} .

Therefore, for $\bar{X} = (n - 1)x_{\max} + r$, for $n = 1, \dots, N$, $n - 1$ agents will hold $(x_i, y_i) = (x_{\max}, 0)$, the n^{th} agent will hold $(x_i, y_i) = (r, \Psi(r; k))$, and the remaining $N - n - 1$ agents will hold $(x_i, y_i) = (0, \Psi(0; k))$. This is the only corner solution such that all X_{\max} is allocated and hence, this must be the optimal allocation. This allocation is efficient by construction. ■

Proof of Proposition 2: Aggregate Indifference Curve. To model the increasing size of

the economy. we start with a fixed integer $N^* > 0$ that represents the number of agents. Agents in this initial economy achieve level of utility k_o . The goal is to pass to the limit by increasing the number of agents while holding $Y_{\max} = N^* \cdot \Psi(0; k_o)$ fixed. Holding the endowment fixed while increasing the total number of agents in the economy implies that each agent will be initially endowed with a decreasing quantity of Y . This causes the initial level of utility of each agent to decrease. Instead of taking the limit as N grows without bound, we introduce the variable $z, z > 0$, and write zN^* where z is allowed to grow to infinity. The agents' decreasing utility is described by the function $k : z \rightarrow k(z)$ defined implicitly through the equation

$$Y_{\max} = zN^* \cdot \Psi(0; k(z))$$

where $Y_{\max} = N^* \cdot \Psi(0; k_o)$.

This definition of $k(z)$ assures that as we increase the number of agents (passing to the case of perfect competition by letting $z \rightarrow +\infty$), we do not cause the endowment in the economy to increase without bound; Y_{\max} remains constant for all z . It must be the case that $\frac{dk(z)}{dz} < 0$. Similarly, in the initial economy $x_{\max} = \Psi^{-1}(0; k_o)$, and in general $x_{\max}(z) = \Psi^{-1}(0; k(z))$, and $X_{\max} = N^* \Psi^{-1}(0; k_o)$. In order to fix the amount of X as we increase the number of agents, we introduce function $f(z)$ such that as we write zN^* , $X_{\max} = \frac{X_{\max}(z)}{f(z)}$ stays fixed. The function $f(z)$ counteracts the growth caused by increasing z . Solving for $f(z)$ gives

$$f(z) = \frac{zN^* \Psi^{-1}(0; k(z))}{N^* \Psi^{-1}(0; k_o)} = \frac{z \Psi^{-1}(0; k(z))}{\Psi^{-1}(0; k_o)}.$$

The demand function can now be written as

$$\begin{aligned} Y_z(X) &= \Psi \left(X f(z) - \Psi^{-1}(0; k(z)) \operatorname{int} \left[\frac{X f(z)}{\Psi^{-1}(0; k(z))} \right]; k(z) \right) \\ &\quad + \left(zN^* - 1 - \operatorname{int} \left[\frac{X f(z)}{\Psi^{-1}(0; k(z))} \right] \right) \Psi(0; k(z)) \\ &= \Psi \left(X \frac{z \Psi^{-1}(0; k(z))}{\Psi^{-1}(0; k_o)} - \Psi^{-1}(0; k(z)) \operatorname{int} \left[\frac{zX}{\Psi^{-1}(0; k_o)} \right]; k(z) \right) \\ &\quad + \left(zN^* - 1 - \operatorname{int} \left[\frac{zX}{\Psi^{-1}(0; k_o)} \right] \right) \Psi(0; k(z)) \end{aligned}$$

for $X \in [0, X_{\max}]$.

To prove that as $z \rightarrow \infty$, $Y_z \rightarrow Y_{\infty}$ uniformly, we must show that for any $\epsilon > 0$ there exists a $z^* > 0$ such that for all $z \geq z^*$, $|Y_z - Y_{N^*}| < \epsilon$. So, for any $\epsilon > 0$, we will show that such a z^*

exists. Consider the following:

$$\begin{aligned}
|Y_z - Y_\infty| &= \left| \Psi \left(X \frac{z \Psi^{-1}(0; k(z))}{\Psi^{-1}(0; k_\circ)} - \Psi^{-1}(0; k(z)) \text{int} \left[\frac{zX}{\Psi^{-1}(0; k_\circ)} \right]; k(z) \right) \right. \\
&\quad \left. + \Psi(0; k(z)) \left(zN^* - 1 - \text{int} \left[\frac{zX}{\Psi^{-1}(0; k_\circ)} \right] \right) + \frac{\Psi(0; k_\circ)}{\Psi^{-1}(0; k_\circ)} X - N^* \Psi(0; k_\circ) \right| \\
&= \left| \Psi \left(X \frac{z \Psi^{-1}(0; k(z))}{\Psi^{-1}(0; k_\circ)} - \Psi^{-1}(0; k(z)) \text{int} \left[\frac{zX}{\Psi^{-1}(0; k_\circ)} \right]; k(z) \right) \right. \\
&\quad \left. - \Psi(0; k(z)) - \Psi(0; k(z)) \text{int} \left[\frac{zX}{\Psi^{-1}(0; k_\circ)} \right] + \frac{\Psi(0; k_\circ)}{\Psi^{-1}(0; k_\circ)} X \right|
\end{aligned}$$

The second equality above comes from choosing $k(z)$ so that $zN^*\Psi(0; k(z)) = N^*\Psi(0; k_\circ)$. Now, we can decompose z as $z = I \frac{\Psi^{-1}(0; k_\circ)}{X} + r_e$ where I is any non-negative integer and $r_e \in [0, \frac{\Psi^{-1}(0; k_\circ)}{X}]$.

Decomposing z in this way, we can now write

$$\begin{aligned}
\text{int} \left[\frac{zX}{\Psi^{-1}(0; k_\circ)} \right] &= \text{int} \left[\frac{X}{\Psi^{-1}(0; k_\circ)} \left(I \frac{\Psi^{-1}(0; k_\circ)}{X_\circ} + r_e \right) \right] \\
&= \text{int} [I] + \text{int} \left[\frac{X}{\Psi^{-1}(0; k_\circ)} r_e \right] \\
&= I = (z - r_e) \frac{X}{\Psi^{-1}(0; k_\circ)}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|Y_z - Y_\infty| &= \left| \Psi \left(X \frac{z \Psi^{-1}(0; k(z))}{\Psi^{-1}(0; k_\circ)} - \Psi^{-1}(0; k(z)) (z - r_e) \frac{X}{\Psi^{-1}(0; k_\circ)}; k(z) \right) \right. \\
&\quad \left. - \Psi(0; k(z)) - \Psi(0; k(z)) (z - r_e) \frac{X}{\Psi^{-1}(0; k_\circ)} + \frac{\Psi(0; k_\circ)}{\Psi^{-1}(0; k_\circ)} X \right| \\
&= \left| \Psi \left(X \frac{z \Psi^{-1}(0; k(z))}{\Psi^{-1}(0; k_\circ)} - X \frac{z \Psi^{-1}(0; k(z))}{\Psi^{-1}(0; k_\circ)} - X \frac{r_e}{\Psi^{-1}(0; k_\circ)}; k(z) \right) \right. \\
&\quad \left. - \Psi(0; k(z)) - \frac{z \Psi(0; k(z))}{\Psi^{-1}(0; k_\circ)} X + \frac{r_e}{\Psi^{-1}(0; k_\circ)} X + \frac{\Psi(0; k_\circ)}{\Psi^{-1}(0; k_\circ)} X \right| \\
&= \left| \Psi \left(X \frac{r_e}{\Psi^{-1}(0; k_\circ)}; k(z) \right) - \Psi(0; k(z)) - \frac{\Psi(0; k_\circ)}{\Psi^{-1}(0; k_\circ)} X + \frac{r_e \Psi(0; k(z))}{\Psi^{-1}(0; k_\circ)} X + \frac{\Psi(0; k_\circ)}{\Psi^{-1}(0; k_\circ)} X \right| \\
&= \left| \Psi \left(X \frac{r_e}{\Psi^{-1}(0; k_\circ)}; k(z) \right) - \Psi(0; k(z)) + \frac{r_e \Psi(0; k(z))}{\Psi^{-1}(0; k_\circ)} X \right| \\
&\leq \left| \Psi \left(X \frac{r_e^*}{\Psi^{-1}(0; k_\circ)}; k(z) \right) - \Psi(0; k(z)) + \frac{r_e^* \Psi(0; k(z))}{\Psi^{-1}(0; k_\circ)} X \right| \\
&\leq \left| \Psi \left(X^* \frac{r_e^*}{\Psi^{-1}(0; k_\circ)}; k(z) \right) - \Psi(0; k(z)) + \frac{r_e^* \Psi(0; k(z))}{\Psi^{-1}(0; k_\circ)} X^* \right|
\end{aligned}$$

The third equality again uses the fact that $zN^*\Psi(0; k(z)) = N^*\Psi(0; k_\circ)$. Since r_e is bounded and $\Psi(x_i; k)$ is closed and bounded, then there exists an r_e^* that maximizes $\Psi \left(X_\circ \frac{r_e}{\Psi^{-1}(0; k_\circ)}; k(z) \right) - \Psi(0; k(z)) + \frac{r_e}{\Psi^{-1}(0; k_\circ)} X$. The same argument holds for the existence of X^* that maximizes the expression. Hence, the above inequality holds for r_e^* . Finally, to finish the proof, we note that as $z \rightarrow \infty$, $\Psi(x_i; k(z)) \rightarrow 0$ monotonically. Thus, by the intermediate value theorem there exists a z^* such that for $\epsilon > 0$

$$\left| \Psi \left(X^* \frac{r_e^*}{\Psi^{-1}(0; k_\circ)}; k(z^*) \right) - \Psi(0; k(z^*)) + \frac{r_e^* \Psi(0; k(z^*))}{\Psi^{-1}(0; k_\circ)} X^* \right| = \epsilon.$$

Hence, for all $z > z^*$ and for all X we have

$$\left| \Psi \left(X^* \frac{r_e^*}{\Psi^{-1}(0; k_o)}; k(z) \right) - \Psi(0; k(z)) + \frac{r_e^* \Psi(0; k(z))}{\Psi^{-1}(0; k_o)} X^* \right| < \epsilon.$$

■

A.2 Utility Aggregation: Agents with Different Initial Wealth

Proof of Proposition 3: Two Classes of Agents. The proof for the first part is exactly the same as the proof of Proposition 1 with the exception that we must show that the type 1 agents are allocated \bar{X} before the type 2 agents. As before, each type 1 agent can hold a maximum of $x_{1,\max}$ of X and each type 2 agent can hold a maximum $x = x_{2,\max}$ of X . Since $k_1 < k_2$, then $x_{1,\max} < x_{2,\max}$. Suppose $\bar{X} = \epsilon$ where $\epsilon > 0$ is an arbitrarily small real number. We already know that the efficient allocation is to give ϵ to one agent. According to the objective function, we choose the agent by determining which is willing to give up more of his allocation of Y for ϵ . Said differently, we give ϵ to the agent who has the greatest rate of substitution. Thus, since we have $|y'_{1i}(0)| > |y'_{2i}(0)|$, the first ϵ goes to a type 1 agent.

The inequality $|y'_{1i}(0)| > |y'_{2i}(0)|$ holds for an arbitrary $u(x_i, y_i)$ that satisfies our conditions of differentiability, monotonicity, and convexity. Since utility function is additively separable, $u_x(\cdot)$ is a function of x only, and $u_y(\cdot)$ is a function of y only. Both partial derivatives are positive because $u(x_i, y_i)$ is an increasing function. By the implicit function theorem,

$$\frac{dy_{1i}}{dx} \Big|_{x=0, k=k_1} = -\frac{u_x(x)}{u_y(y)} \text{ and } \frac{dy_{2i}}{dx} \Big|_{x=0, k=k_2} = -\frac{u_x(x)}{u_y(y)}.$$

Since the derivative is evaluated at $x = 0$, and the agents in the two classes have different utility with $k_1 < k_2$, it must be that the only difference is in the allocation of y . Then, the nominators are equal because they are a function of x only. By convexity of $u(x_i, y_i)$, the partial derivative $u_y(y)$ is higher for higher values of y . Then, the denominator is larger in the second ratio and therefore

$$|y'_{1i}(0)| = \left| \frac{dy_{1i}}{dx} \Big|_{x=0, k=k_1} \right| > |y'_{2i}(0)| = \left| \frac{dy_{2i}}{dx} \Big|_{x=0, k=k_2} \right|$$

Now, suppose an additional ϵ is added to the economy. Then, we can give this additional amount to the type 1 agent who just received ϵ or to a type 2 agent. Since each agents' demand function

is concave and decreasing, the rate at which each agent is willing to give Y for each additional increment of X increases,

$$|y'_{1i}(\epsilon)| = \left| \frac{dy_{1i}}{dx} \Big|_{x=\epsilon, k=k_1} \right| > |y'_{1i}(0)| = \left| \frac{dy_{1i}}{dx} \Big|_{x=0, k=k_1} \right| > |y'_{2i}(0)|.$$

The first inequality holds true because $u_x(x)$ is increasing by convexity of $u(x_i, y_i)$. Therefore, the X consumption good will be allocated to the type 1 agents until all agents receives $x_{1,\max}$ or the supply of X has been depleted. If all type 1 agents have $x_{1,\max}$, then the remaining supply of X will be allocated to the type 2 agents one agent at a time with each type 2 agent holding no more than $x_{2,\max}$. ■

Proof of Proposition 4: Aggregate Indifference Curve for Two Classes of Agents.

The proof is identical to the proof of Proposition 2. Let \bar{X} be the amount of the X good available. If $\bar{X} \leq N_1 x_{1,\max}$, then the problem is the same as having only the type 1 agents since the type 2 agents will not receive any \bar{X} . When $\bar{X} > N_1 x_{1,\max}$, all the type 1 agents have $x_{1,\max}$ and so the problem reduces to having only the type 2 agents. ■

Proof of Theorem 5: Convexity of Aggregate Indifference Curve.

We start the proof by choosing some partition of k_i s, for $i = 1$ to n , over the interval $[k_{\min}, k_{\max}]$ such that $k_i < k_{i+1}$. As n increases, we choose a new partition such that $\lim_{n \rightarrow \infty} |k_{i-1} - k_i| = dk_i$, $\lim_{n \rightarrow \infty} k_i = k_{\min}$, $\lim_{n \rightarrow \infty} k_n = k_{\max}$. Define a continuous and differentiable function $f : [0, X_{\max}] \rightarrow [k_{\min}, k_{\max}]$ such that $f(0) = k_{\min}$, $f(X_{\max}) = k_{\max}$, and $f'(X) > 0$ with $0 < f'(X) < \infty$ for all $X \in [0, X_{\max}]$.

Define $k_i = f(x_i)$. Hence, the partition of k_i s is associated with some partition of x_i s on $[0, X_{\max}]$ defined by $f^{-1}(k_i)$ (note that the inverse does exist since f is strictly increasing). By the continuity of $f^{-1}(k)$, since $\lim_{n \rightarrow \infty} |k_{i-1} - k_i| = dk_i$, then $\lim_{n \rightarrow \infty} |f^{-1}(k_{i-1}) - f^{-1}(k_i)| = \lim_{n \rightarrow \infty} |x_{i-1} - x_i| = dx_i$.

For each i , we have

$$Y(X) = -\frac{\Psi(0; f(x_i))}{\Psi^{-1}(0; f(x_i))} [X - x_{i-1}] + y_{i-1} \quad \text{for } X \in [x_{i-1}, x_i].$$

Evaluating this expression at $X = x_i$ and letting $n \rightarrow \infty$ we get

$$dY(x_i) = -\frac{\Psi(0; f(x_i))}{\Psi^{-1}(0; f(x_i))} dx_i \quad \text{for } x_i \in [0, X_{\max}].$$

That is, the partition on X becomes infinitely fine and each Y_i is defined over an infinitely small interval. Thus Y becomes differentiable. Further, convexity comes directly from the assumption

$$\frac{\Psi(0; k_i)}{\Psi^{-1}(0; k_i)} > \frac{\Psi(0; k_{i+1})}{\Psi^{-1}(0; k_{i+1})}$$

for $k_i < k_{i+1}$. As $x \in [0, X_{\max}]$ increases, $f(x)$ increase and $\frac{dY}{dx}$ approaches zero. Therefore, $\frac{dY}{dx} = -\frac{\Psi(0; f(x))}{\Psi^{-1}(0; f(x))}$ decreases at a decreasing rate. Hence, the function, $Y(x)$, is convex and differentiable on the interval $[0, X_{\max}]$. ■

Proof of Theorem 7: Construction of Economy with Risk Seeking Agents. In the proof we will show how to start with an aggregate convex indifference curve and construct wealth distribution in an economy where each individual has a concave indifference curve. Define a partition over $[0, X_{\max}]$ as $x_i = i \frac{X_{\max}}{n}$, for $i = 0$ to n . For each x_i , let $y_i = g(x_i)$ where g is as defined in the theorem. We will show that the function $g(x)$ can be reproduced from a system of concave indifference by carefully choosing a distribution of wealth, k_i s. From (3), we can write

$$Y_i(X) = -\frac{\Psi(0; k_i)}{\Psi^{-1}(0; k_i)} [X - x_{i-1}] + y_{i-1} \quad \text{for } X \in [x_{i-1}, x_i] \ .$$

We choose k_i so that each function $Y_i(X)$ is a chord connecting the points $(x_{i-1}, g(x_{i-1}))$ and $(x_i, g(x_i))$. Hence,

$$-\frac{\Psi(0; k_i)}{\Psi^{-1}(0; k_i)} = \frac{g(x_i) - g(x_{i-1})}{x_i - x_{i-1}} \ .$$

The above statement simply states that choosing k_i carefully, the line segment $Y(x_i)$ can be made to have the same slope as the chord connecting $(x_{i-1}, g(x_{i-1}))$ and $(x_i, g(x_i))$. Such a k_i exists since from the conditions $\frac{\Psi(0; k_i)}{\Psi^{-1}(0; k_i)} > \frac{\Psi(0; k_{i+1})}{\Psi^{-1}(0; k_{i+1})}$ and $0 < g'(X) < \infty$, we can choose a range of k_i s sufficiently large so as to match any slope $\frac{g(x_i) - g(x_{i-1})}{x_i - x_{i-1}}$ for all $x_i \in [0, X_{\max}]$. That is, we can find a k_i sufficiently small (large) so as to make the slope $\frac{\Psi(0; k_i)}{\Psi^{-1}(0; k_i)}$ as steep (shallow) as we need.

Substituting into $Y_i(X)$ gives

$$Y_i(X) = \left(\frac{g(x_i) - g(x_{i-1})}{x_i - x_{i-1}} \right) [X - x_{i-1}] + g(x_{i-1}) \quad \text{for } X \in [x_{i-1}, x_i] \ .$$

Y_i , for $i = 1$ to n , represents a point in a sequence of systems of linear equations. Each system of equations consists of n chords connecting points on $g(x)$.

We now need to show that this choice of k_i causes the sequence of systems to converge to $g(X)$ as $n \rightarrow \infty$. For any i , choose any $\bar{x} \in (x_{i-1}, x_i)$. We can rewrite this point as $\bar{x} = x_{i-1} + r \left(\frac{X_{\max}}{n} \right)$ for $r \in (0, 1)$. For all $\delta > 0$, let $n > n_0 = \frac{X_{\max}}{\delta}$

$$\begin{aligned}
|x_{i-1} - \bar{x}| &= \left| x_{i-1} - x_{i-1} - r \left(\frac{X_{\max}}{n} \right) \right| \\
&= r \left(\frac{X_{\max}}{n} \right) \\
&\leq \left(\frac{X_{\max}}{n} \right) \\
&\leq \left(\frac{X_{\max}}{n_0} \right) \\
&= \delta
\end{aligned}$$

Hence, for all $\delta > 0$, there exists an n_0 such that for all $n > n_0$, $|x_{i-1} - \bar{x}| < \delta$. By the continuity of $g(X)$, for every $\epsilon > 0$, there exists a $\delta(n_0) > 0$ such that if $|x_{i-1} - \bar{x}| < \delta$, then $|g(x_{i-1}) - g(\bar{x})| < \epsilon$. Said differently, for every $\epsilon > 0$, there exists an n_0 such that for all $n > n_0$, $|x_{i-1} - \bar{x}| < \delta(n_0)$, and hence, $|g(x_{i-1}) - g(\bar{x})| < \epsilon$. So, let $\epsilon > 0$ and choose n_0 to satisfies the continuity definition. For all $n > n_0$,

$$\begin{aligned}
|Y_i(\bar{x}) - g(\bar{x})| &= \left| \left(\frac{g(x_i) - g(x_{i-1})}{x_i - x_{i-1}} \right) [\bar{x} - x_{i-1}] + g(x_{i-1}) - g(\bar{x}) \right| \\
&= \left| \left(\frac{g(x_i) - g(x_{i-1})}{x_i - x_{i-1}} \right) [\bar{x} - x_{i-1}] + (g(x_{i-1}) - g(\bar{x})) \right| \\
&\leq \left| \left(\frac{g(x_i) - g(x_{i-1})}{x_i - x_{i-1}} \right) \delta + \epsilon \right| \\
&\leq \epsilon.
\end{aligned}$$

The last inequality is true since $g'(X) < 0$. But, this is true for any i and for any $\bar{x} \in (x_{i-1}, x_i)$. Therefore, for n large enough, each chord, Y_i , can be made arbitrarily close to $g(X)$ and hence, the sequence of systems of linear equations approaches $g(X)$ on $[0, X_{\max}]$ uniformly as the sequence approaches infinity. ■

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