

Catastrophe Insurance and Optimal Investment

Sheen Liu and Gene Lai

Department of Finance, Insurance and Real Estate

Washington State University *

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1 Introduction

In the insurance market, the insurers who provide catastrophe insurance face with the risk of rare, but huge catastrophe claims. The introduction of catastrophe related securities into the marketplace provide the insurers the instruments to hedge some of the catastrophe risks they are facing. A catastrophe related security is tied to the pre-specified catastrophe claims, while independent to other claims. It is the ties between the catastrophe related securities and the catastrophe insurance claims that make the catastrophe related securities special to the insurers. The catastrophe securities and the insurance claims are not just correlated each other but simply bounded together for those claims specified in the catastrophe securities. However, the insurers cannot trade away the risk of those claims that are not covered by the catastrophe securities, because the catastrophe related securities do not cover every possible claim. In other words, the catastrophe insurance market is incomplete. Therefore, this partial tie characterizes the relation between the catastrophe securities and the claims. The partial tie poses a different optimal investment problem, what is the best policy to participate in the catastrophe security market, for a catastrophe insurer.

In this paper we study the optimal investment policy for the insurers in the business

*Sheen Liu is at Vancouver and Gene Lai is at Pullman campus.

of catastrophe insurance. The insurer investment problems have been investigated by Browne (1994), Frolova, Kabanov, and Pergamenshchikov (2002), Gaier and Grandits (2002), and Gaier, Grandits, and Schachermayer (2003) in the special case of a geometric Brownian motion as an investment process. In the case of compound Poisson as insurance claim process and the geometric Brownian motion as the investment process the problem is investigated by Hipp and Plum (2000), Kalashnikov and Norberg (2002), and Paulsen (2002). Emmer and Klüppelberg (2004) investigate the case where both insurance claims and the market security follow Lévy process. In those studies the investment securities and the insurance claims are assumed either independent or correlated, while statistic correlation is inadequate to model the relation between the catastrophe related securities and the catastrophe insurance claims. In this study, we model this partially tied relation between the catastrophe securities and the catastrophe claims. The catastrophe claims are rare events. Brownian motion is not a good model for rare events. In this study, both the catastrophe claims and the catastrophe related securities are driven by compound Poisson process.

We first adopt martingale approach based on duality arguments to the optimal investment problem to get a general solution for the optimization problem. Then we combine the martingale approach with the Hamilton-Jacobi-Bellman (HJB) equation of dynamic programming to get the solution for constant coefficients. The HJB approach of dynamic programming has been widely used in the portfolio optimization problems in insurance and actuarial literature. Since the HJB approach depends on an explicit evaluation of the value function to yield the optimal control, the problem specific techniques have to be used to solve for the value function. Those problem specific techniques are rather difficult to find. The martingale method facilitates finding general solutions of the optimization problems Karatzas, Lehoczky, and Shreve (1987), Cox and Huang (1989), Cvitanic and Karatzas (1992), Kramkov and Schachermayer (1999)). Besides, the martingale method can handle “non Markovian” process and non-constant coefficients.

The objection to the martingale approach to insurance is that insurance market is incomplete. The insurance policies are not traded on the market and are not hedgable. Thus, there is no unique martingale measure for the idiosyncratic risks of insurance poli-

cies. This is also true for our case, in which the catastrophe insurance is not completely hedgable. One might be surprised that the martingale approach provides a unique solution to the optimal investment policy in our case. In addition, the unique solution from the martingale approach, unlike the solution from the HJB approach that often applies only to a specific utility function, applies to all well-behaved utility functions.

The contributions of this study to the insurance literature is twofold. We develop a model that captures the relation between the catastrophe related securities and the catastrophe insurance claims. This model is simple and manageable that explicit solutions are derived. Second, we show that the martingale approach provides a unique optimal investment policy for this special incomplete market case. The paper is organized as follows. Section 2 describes catastrophe insurance claim model and the catastrophe related security model. Section 3 proves the existence of the optimal investment policy by using the martingale approach. Section 4 introduces the utility function and the convex dual of the utility function, then identifies the terminal wealth by using the duality arguments. Section 5 combines the martingale approach with the HJB equation to yield the optimal solution to the HJB equation. Section 6 discusses the implication of our results to the insurers' optimal investment policy.

2 Catastrophe insurance and securities

2.1 Net claims of catastrophe insurance

We model the claim process $C(t)$ of catastrophe insurance as a compound Poisson process, where the claims take one of finitely many possible nonzero values $b_i, 1 \leq i \leq L$. Then net claim process is given by Cramér-Lundberg model

$$ct - C(t) \tag{1}$$

where c is the premium income per unit time and $C(t)$ is the total claims. Let B_i represents the claims process and has an i.i.d distribution, then

$$C(t) = \sum_{i=1}^{N(t)} B_i \tag{2}$$

where $N(t)$ is the number of claims up to time t . The Poisson process $\{N(t)\}$ has intensity λ , which is a nonnegative process given on a probability space $(\Omega, \mathcal{F}_t, P)$. Intensity λ is \mathcal{F}_0 measurable and satisfies

$$\int_0^T \lambda(t) dt < \infty \quad P - \text{a.s.} \quad t \geq 0 \quad (3)$$

Let probabilities of the claim size b_1, b_2, \dots, b_L be $p(b_1), p(b_2), \dots, p(b_L)$ which are positive numbers that sum to 1. And let $N_m(t)$ be a Poisson process and denote the number of claims in $C(t)$ of size b_m up to and including time t , then the claims process can be decomposed into L independent Poisson processes $(N_m(t), m = 1, 2, \dots, L)$ as

$$C(t) = \sum_{m=1}^L b_m N_m(t) \quad (4)$$

where each N_m has intensity $\lambda p(b_m)$. It is worth to note that a general compound Poisson process can be approximated by isolated clumps of events as in (4). There have been many successful applications of this approach (Arratia, Goldstein, and Gordon (1990)). This model has the same spirit of peak-over-threshold (POT) method, while it discretized the estimation of tail or a quantile, based on the extreme value of a compound Poisson process (Embrechts, Klüppelberg, and Mikosch, 1997).

2.2 Catastrophe related securities

Different models of catastrophe reinsurance and catastrophe bonds have been proposed (Burnecki, Kukla, and Taylor, 2005). For example, the catastrophe bonds are modeled as options on the underlying insurance claims. Options can be replicated by a risk free bond and underlying securities, for example, by portfolio delta-gamma approach (Duffie and Pan, 2001). Therefore, it is justified to replace options with simple underlying assets and a risk-free bond. In this study, we assume there exist the catastrophe related securities that are tied to the catastrophe claims in a simple way. The claims follow a compound Poisson process and each claim takes one of finitely many possible nonzero values. The catastrophe related securities are driven by the same compound Poisson process, while the securities are affected by the catastrophe claims only if the sizes of claims take a certain values. Specifically, we assume that a catastrophe related security i is tied to a

certain set of claims b_i . Only when the claims fall into the set, the investors who hold security i lose a certain amount of the value of the security for each unit of the security they are holding. We assume that the security return drop by η_i , or η_i dollar for every dollar invested in the security is paid to the claim.

Insurance market is incomplete. Insurers cannot completely hedge their catastrophe insurance risk by using the catastrophe related securities. To model this incompleteness of the market, we assume there are only K catastrophe related securities available on the market and are tied to the claim size $b_i, i = 1, 2, \dots, K$. There are L possible claim size where L is strictly greater than K . In other words, the catastrophe related securities are issued for a smaller set of claims. For example, let the catastrophe insurance claim be ordered as $b_1 > b_2 > \dots > b_L$. If the claim size is $(b_i)_{i=1,2,\dots,K}$, then some catastrophe related securities drops in their value and the losses are paid to the claims. However, if the claim size is $(b_i)_{i=K+1,K+2,\dots,L}$, the insurance companies have to bear the losses.

The price of K independent catastrophe related securities follows the stochastic differential equation

$$dS_i(t) = S_i(t-) \left[\left(a_i + \sum_{j=1}^K \lambda_i \eta_{ij} \right) dt - \sum_{j=1}^K \eta_{ij}(t) dN_j(t) \right], \quad j = 1, 2, \dots, K \quad (5)$$

where a_i is the mean rate of return and (η_{ij}) is the jump size tied to catastrophe claims. There are K securities driven by K independent Poisson processes and matrix (η_{ij}) is nonsingular. Thus we can transform the securities into K groups (portfolios) of the securities so that each group is driven by only one independent Poisson process. We define them as the basic securities and we use the basic securities in this paper to simplify notations. The basic securities follow the following stochastic equation

$$dS_i(t) = S_i(t-) [(\alpha_i + \lambda_i \beta_i) du - \beta_i dN_i(t)], \quad j = 1, 2, \dots, K \quad (6)$$

where α_i is the mean rate of return and β_i is the jump size of i^{th} basic security. Besides the basic securities we assume there is a riskless security which price follows the differential equation

$$dS_0(t) = rS_0(t)dt, \quad S_0(t) = s_0, \quad 0 \leq t \leq T \quad (7)$$

where r is the short-rate process.

We assume there is a set of equivalent martingale measures and the set is not empty. The equivalent probability measure Q is absolute continuous with respect to the empirical probability measure P with the strictly positive and bounded density, such that all security prices are martingales with respect to Q (Jensen (1999), Björk (1997)). For fixed $(t, x) \in [0, T] \times (0, \infty)$ we define the probability measure Q by $Z(s) = E_t^P \left(\frac{dQ}{dP} \right)$ and $E_P(Z(s)) = 1$ for finite $0 \leq t \leq T$. Thus, Z is Radon-Nikodym density martingale. We assume that there exists a Q -measurable process $\mu = (\mu_1, \mu_2, \dots, \mu_K)^T$ with $1 > \mu_m > 0$ satisfying the integrability conditions of

$$\int_0^T \mu_m(t) dt < \infty \quad P - \text{a.s.} \quad t \geq 0 \quad (8)$$

Then Z is given by (Brémaud, 1981)

$$Z(s) = \prod_{j=1}^K Z_m(s) \quad (9)$$

and

$$Z_m(s) = \begin{cases} \exp[(1 - \mu_m)\lambda s], & \text{if } s < \tau_m(1) \\ (\mu_m)^{N_m(s,t)} \exp[(1 - \mu_m)\lambda s], & \text{if } 0 \leq \tau_m(1) \leq s \end{cases} \quad (10)$$

where, for $t \leq \tau(m) \leq s$, $\tau_j(m)$ is the m^{th} jump of N_m in $[t, s]$, $N_m(s, t)$ is the number of jumps of Poisson process i in $[t, s]$. Under the equivalent martingale measure Q the jump intensity $\tilde{\lambda}_i$ relates to the empirical measure P by $\tilde{\lambda}_i = \lambda_i \mu_i$. Define a martingale $d\tilde{M}_i = dN_i - \mu_i \lambda_i dt$, we can rewrite the price of a catastrophe related security as

$$dS_i(t) = S_i(t-) (\alpha_i - \beta_i \lambda_i (\mu_i - 1)) dt - \beta_i d\tilde{M}(t) \quad (11)$$

The discounted price, $\frac{S_i(t)}{S_0(t)}$, is a martingale under Q . That is

$$\alpha_i - \beta_i (\mu_i - 1) \lambda_i = r \quad (12)$$

In (12) μ_i is uniquely determined for $i = 1, 2, \dots, K$. However, there are no securities $\{i\}, i = K + 1, K + 2, \dots, L$ that are related to claims $(b_i)_{i=K+1, K+2, \dots, L}$ and (12) is not available. Those $(\mu_i)_{i=K+1, K+2, \dots, L}$, appear in the net claim process

$$cdt - \sum_{i=1}^L b_i \mu_i \lambda_i dt - \sum_{i=1}^L b_i d\tilde{M}(t)$$

Under Q , the net claim process must be a martingale, i.e.,

$$c = \sum_{i=1}^L b_i \mu_i \lambda_i \quad (13)$$

In (13), besides $(b_i)_{i=1,2,\dots,K}$ that are uniquely determined by (12) (13), there $L - K$ unknowns, $(\mu_i)_{i=K+1,K+2,\dots,L}$. Therefore, multiple sets of μ 's can satisfy (13) and multiple equivalent martingale measures exist.

The safety loading factor θ is defined by

$$c = (1 + \theta) \sum_{m=1}^L b_m \lambda_m \quad (14)$$

Comparing (14) to (13) we have the explicit relation between μ and the safety loading factor θ

$$\theta = \frac{\sum_{m=1}^L b_m (\mu_m - 1) \lambda_m}{\sum_{m=1}^L b_m \lambda_m} - 1 \quad (15)$$

Since $(\mu_i)_{i=K+1,K+2,\dots,L}$ are not determined uniquely by (12), the safety loading factor cannot uniquely determined by (15). It is uniquely determined only if $K = L$, i.e., if the market is complete. Delbaen and Haezendonk (1989) argue that, in a sufficiently liquid insurance market, classical insurance-premium principles can be reinterpreted in an arbitrage-free pricing setup. The introduction of CAT-futures by CBOT in 1992, together with the new generation of PCS-options aims at offering such liquidity (Embrechts, 1996). It is still controversial surrounding the arbitrage free argument for insurance (Albrecht, 1992). In this study, we do assume $L > K$. Thus, the market is incomplete and multiple equivalent martingale measures exist.

In the next section, we put the insurance claim process and the security market together and summarize the model. Then we define the insurer's optimal investment problem.

3 Insurers' optimal investment policy

The insurance company under consideration is a property-casualty insurance company. Insurance companies act as price takers and trading takes place continuously at equilibrium prices. Insurance companies can borrow and lend at a instantaneous risk-free rate

r . The insurance policy issued by an insurance company is not traded on the market and the premium income is given.

The net claim process is driven by L -variate continuous time Poisson processes. The L -variate Poisson process $\{N(t)_i, i = 1, 2, \dots, L\}^T$ drives the catastrophe insurance claim counting process and $\{b_i, i = 1, 2, \dots, L\}^T$ are the claim sizes for the compound Poisson processes. There exist K basic securities, where $K < L$, defined in the previous section and one riskless asset. They are traded continuously on the fixed time-horizon $[0, T]$. The K risky assets driven by $\{N_i(t), i = 1, 2, \dots, K\}$ are the catastrophe related securities. A probability space $(\Omega, \mathcal{F}_t, P)$. \mathcal{F}_t is the natural filtration generated by the process $\{N_i(t), i = 1, 2, \dots, L\}$ and \mathcal{F}_o contains all P -null sets of \mathbb{F} .

Since the insurance risk securitization of CATs shows no correlation with equities or corporate bonds, we do not include equities in our model. It is a straight forward extension to include equities by combining the results from this study and from previous studies on equity investments. One would expect to find that the optimal policy is a linear combination of the two results because the independence between catastrophe events and equities. The equities available in the market provide only further diversification, not hedging, of risks that the insurers bear.

An insurer invests ¹ in the catastrophe related security under an investment policy w , where $w = \{w, w_2, \dots, w_K\}^T$ represents the investment in the catastrophe related securities $(S_i)_{i=1,2,\dots,K}$. We denote the company's wealth at time 0 by x . The investment policy w is an admissible adapted control process, and $\int_0^T w_i(t)^2 dt < \infty, i = 1, 2, \dots, K$, a.s. The wealth of an insurance company is given by a dynamic portfolio, which is an L -dimensional stochastic process

$$dX(t) = \left\{ X(t-) \sum_{i=1}^K [w_i(\alpha_i - r) + w_i\beta_i\lambda_i + r] + c \right\} dt - \sum_{i=1}^K [X(t-)w_i\beta_i + b_i] dN_i(t) - \sum_{i=K+1}^L b_i dN_i(t) \quad (16)$$

The choice for $w_i(t)$ is unconstrained because $(1 - \sum_{i=1}^K w_i(t))$ of the wealth is invested in riskless security.

¹Issuing a catastrophe security is equivalent to shorting the security.

Under an equivalent martingale measure, using (12) and (13) we rewrite the wealth of an insurance company as

$$dX(t) = X(t)rdt - \sum_{i=1}^K [X(t-)w_i\beta_i + b_i] d\tilde{M}_i(t) - \sum_{i=K+1}^L b_i d\tilde{M}_i(t)$$

The discounted wealth process is governed by the following equation

$$\frac{X(t)}{S_0(t)} = x - \sum_{i=1}^K \int_0^t \frac{1}{S_0(0,u)} [X(u-)w_i\beta_i + b_i] d\tilde{M}_i(u) - \sum_{i=K+1}^L \int_0^t \frac{b_i(u)}{S_0(u)} d\tilde{M}_i(u) \quad (17)$$

We combine discounting and changing measure and define the state price density process by

$$H(s) = \frac{Z(s)}{S_0(s)} \quad (18)$$

Let ξ be the terminal wealth at time T . To reflect the fact that bankruptcy law is enforced, the insurer's investment policy needs to satisfy the budget constraint (Cox and Huang (1989)). The budget constraint requires that, under an equivalent martingale measure, the discounted terminal wealth does not exceed the initial investment, i.e.

$$E[H(T)\xi] \leq x \quad (19)$$

If a portfolio processes results in a positive wealth process, then we say the portfolio *admissible*. Formally, given $x \geq 0$ a portfolio process $\{w\}$ is *admissible* of X , and write $\{w\} \in \mathcal{A}(t, x)$, if the wealth process $X(s)$ satisfies

$$X(s) \geq 0, \quad 0 \leq s \leq T \quad \text{a.s.}$$

Suppose that an insurer is interested in maximizing the utility from her terminal wealth at time T . Let the utility function be $U(x)$ and let $J(x)$ denote the maximal utility attained from the initial wealth x . Then the insurer's optimization problem is

$$J(x) = \sup_w E[U(\xi)] \quad (20)$$

$$\text{subject to } E[H(T)X(T)] \leq x \quad (21)$$

In the current optimization problem, the portfolio w and the terminal wealth $X(T)$ need to be determined. If this can be done in separate steps rather than simultaneously, then the problem in each step is simplified. The following theorem shows that it is indeed possible.

Proposition 1. *Let $x > 0$ be the endowment at time t for an insurer and be given. Let ξ be the terminal wealth at time T be a \mathcal{F}_T -measurable random variable such that the budget constraint is satisfied with equality*

$$E[H(T)X(T)] = x, \quad x > 0 \quad (22)$$

Then there exists a portfolio process $\{w\}$ such that $\{w\}$ is admissible at x and $\xi = \{X(T) \mid X(t) = 0\}$.

Proof. The sketch of the proof is as follows. Define $\Psi(s)$ the expected discounted terminal wealth at time T given information at time s . Then $\Psi(s)$ is martingale under P . By the representation theorem we can identify a predictable process which integrals with respect to the Poisson process is $\Psi(s)$. Compare this predictable process to the martingale in (16) we can construct the portfolio process to make the two martingales equal. Define the positive martingale

$$\Psi(s) \triangleq E[H(T)\xi \mid \mathcal{F}_s], \quad 0 \leq s \leq T$$

$\Psi(s)$ is finite and, thus, is square integrable. By the predictable representation theorem every square integrable random variable $F \in L^2$ has a representation of the form (Brémaud (1981))

$$F = E[F] + \sum_{j=1}^L \int_0^T \phi_j(s) dM_j(s) \quad (23)$$

where $\phi_j, 1 \leq j \leq L$ are predictable processes such that $\int_0^T E[\phi_j^2(s)] ds < \infty$. In particular,

$$\Psi(s) = x + \sum_{j=1}^L \int_0^s \phi_j(u) dM_j(u), \quad 0 \leq s \leq T \quad (24)$$

Define a nonnegative process $X(\cdot)$ by

$$\begin{aligned} \frac{X(s)}{S_0(s)} &\triangleq \frac{1}{Z(s)} E[H(T)\xi \mid \mathcal{F}(s)] \\ &= \frac{1}{Z(s)} \Psi(s) \end{aligned} \quad (25)$$

so that $X(0) = \Psi(0) = x$. $X(\cdot)$ is the time s expected value of the discounted terminal wealth. With the Radon-Nikodym martingale

$$Z(s) = 1 + \sum_{j=1}^L \int_0^s Z(t, u-) (\mu_j(u) - 1) dM_j(u),$$

Itô's rule implies

$$\begin{aligned} d\left(\frac{X(s)}{S_0(s)}\right) &= d\frac{\Psi(s)}{Z(s)} \\ &= \frac{1}{Z(s-)}d\Psi(s)^c - \frac{\Psi(s-)}{Z^2(s-)}dZ(s)^c + \frac{\Psi(s)}{Z(s)} - \frac{\Psi(s-)}{Z(s-)} \end{aligned} \quad (26)$$

$$= -\frac{1}{Z(s-)}\sum_{j=1}^L\phi_j\lambda_j ds + \frac{\Psi(s-)}{Z(s-)}\sum_{j=1}^L(\mu_j - 1)\lambda_j ds + \frac{\Psi(s)}{Z(s)} - \frac{\Psi(s-)}{Z(s-)} \quad (27)$$

For the last two terms we have

$$\frac{\Psi(s)}{Z(s)} - \frac{\Psi(s-)}{Z(s-)} = \sum_{j=1}^L \left[\frac{\Psi(s) - \mu_j \Psi(s-)}{\mu_j Z(s-)} \right] dN_j(s) \quad (28)$$

$$= \sum_{j=1}^L \left[\frac{\Psi(s) - \Psi(s-) + \Psi(s-) - \mu_j \Psi(s-)}{\mu_j Z(s-)} \right] dN_j(s) \quad (29)$$

$$= \sum_{j=1}^L \left[\frac{\phi(s-) - (\mu_j - 1)\Psi(s-)}{\mu_j Z(s-)} \right] dN_j(s) \quad (30)$$

In the last equality we used (24) that implies

$$\Psi(s) - \Psi(s-) = \phi(s-)$$

Substituting (30) to (27), we have

$$\begin{aligned} d\left(\frac{X(s)}{S_0(s)}\right) &= -\frac{1}{Z(s-)}\sum_{j=1}^L\lambda_j ds + \frac{\Psi(s-)}{Z(s-)}\sum_{j=1}^L(\mu_j - 1)\lambda_j ds \\ &+ \sum_{j=1}^L \left[\frac{\phi(s-) - (\mu_j - 1)\Psi(s-)}{\mu_j Z(s-)} \right] dN_j(s) = \sum_{j=1}^L \left[\frac{\phi(s-) - (\mu_j - 1)\Psi(s-)}{\mu_j Z(s-)} \right] d\tilde{M}_j(s) \end{aligned} \quad (31)$$

Comparison of (31) to (17) suggests the following portfolio process

$$w_i(t)\beta_i X(t-) = -b_i - \frac{\phi(s-) + (1 - \mu_j)\Psi(s-)}{\mu_j H(s-)}, \quad i = 1, 2, \dots, K \quad (32)$$

We conclude that, for given $\{\xi\}$, there exists such a portfolio process that the wealth process $X(\cdot)$ is (17). \square

Proposition 1 not only guarantees the existence of a portfolio process that matches the terminal wealth ξ that satisfies the budget constraint, but also provides the formula for the optimal investment policy (32). Before we can use (32) to find the optimal

investment policy in terms of the coefficients specifying the security process and the claim process, we need to find the process ϕ in (23), which is dependent on utility function. In the following section we introduce utility function and its convex dual. The duality facilitates finding the optimal terminal wealth. At the end of the section, we show by an example how to get the explicit optimal investment policy by combining the results from the duality method and (32).

4 Convex dual of utility function and optimal terminal wealth

An insurer's utility is given by $U(x)$. A utility function is a strictly increasing, strictly concave and C^1 function $U : (0, \infty) \rightarrow [-\infty, \infty)$ with

$$\begin{aligned} U(0) &\triangleq \lim_{x \rightarrow 0} U(x) \geq -\infty \\ U'(\infty) &\triangleq \lim_{x \rightarrow \infty} U'(x) = 0 \end{aligned} \tag{33}$$

The conditions for reasonable asymptotic elasticity are (Kramkov and Schachermayer (1999))

$$\limsup_{x \rightarrow \infty} \frac{xU'(x)}{U(x)} < 1$$

Function U' is strictly decreasing and continuous, mapping $[0, \infty]$ onto $[0, U'_i(0)]$. Thus it has a strictly decreasing and continuous inverse function I , mapping $[0, U'(0)]$ onto $[0, \infty]$. Set $I(y) = 0$ for $U'(0) \leq y \leq \infty$ so that I is defined, finite, and continuous on the extended half-line $(0, \infty]$, and

$$I(U'(x)) = x, \quad 0 < x < \infty \tag{34}$$

$$U'(I_i(y)) = \begin{cases} y & \text{if } 0 < y < U'(0) \\ U'(0) & \text{if } U'(0) \leq y \leq \infty \end{cases} \tag{35}$$

The *convex dual* of U is the function $\tilde{U}(y) \triangleq \sup_{x \in \mathbb{R}} [U(x) - xy]$. Then \tilde{U} , mapping \mathbb{R} onto $(-\infty, \infty)$, is convex, nonincreasing, lower semicontinuous (Karatzas, Lehoczky, and

Shreve, 1987). For $0 < y < U'(0)$, from the definitions of \tilde{U}_i and I_i in (35), \tilde{U}_i reaches its maximum at $x = I(y)$ and we have

$$U(x) - xy \leq U(I(y)) - yI(y) \quad (36)$$

with equality if and only if $x = I(y)$.

We need to identify y in (36). One way to do it is through the analogy between y and the Lagrange multiplier in optimization problems. We consider a stochastic control problem (21), concerning the maximization of the expected utility of terminal wealth. For $x \in \{\mathcal{X}(\infty), \infty\}$, $J(x)$ maximizes $E[U(\xi)]$ over a terminal wealth random variable ξ and under the budget constraint $E[H(t, T)U(\xi)] \leq x$. If $y > 0$ is a Lagrange multiplier that enforces the budget constraint, the problem reduces to the unconstrained maximization problem

$$E^P[U(\xi)] + y \{x - E^P[H(t, T)\xi]\} \quad (37)$$

We rearrange (37) and apply the duality result (36). We have

$$\begin{aligned} & xy + E^P[U(\xi) - yH(t, T)\xi] \\ & \leq xy + E^P[U(I(yH(t, T))) - yH(t, T)I(yH(t, T))] \end{aligned} \quad (38)$$

The terminal wealth reaches maximum if (38) with equality:

$$\xi^* = I(yH(t, T)) \quad (39)$$

Therefore, the wealth is a function of y we can define function $\mathcal{X}(t, y)$ as

$$\mathcal{X}(y) = E_t^P \{H(T)I(yH(T))\} < \infty \quad (40)$$

The function \mathcal{X} is nonincreasing and continuous on $(0, \infty)$, and strictly decreasing on $(0, \infty)$, where $\mathcal{X}(0+) \triangleq \lim_{y \downarrow 0} \mathcal{X}(y) = \infty$, $\mathcal{X}(\infty) \triangleq \lim_{y \rightarrow \infty} \mathcal{X}(y)$. In particular, the function \mathcal{X} restricted to $(0, \infty)$ has a strictly decreasing inverse function $\mathcal{Y} : (\mathcal{X}(\infty), \infty) \xrightarrow{\text{onto}} (0, \infty)$, so that

$$\mathcal{X}(\mathcal{Y}(x)) = x, \quad \forall x \in (\mathcal{X}(\infty), \infty) \quad (41)$$

With function $\mathcal{Y}(x)$, we have the optimal terminal wealth in terms of the initial wealth x

$$\xi^*(T) \triangleq I(\mathcal{Y}(x)H(t, T)) \quad (42)$$

and an explicit expression for $J(x)$

$$J(x) = E[U(I(yH(T)))],$$

$$(t, y) \in [0, T] \times (0, \infty)$$

We need the derivatives of $J(x)$ in the section of HJB equation of dynamic programming later. First, we introduce the convex dual of $J(x)$, which is defined by (see (36))

$$\tilde{J}(y) = \sup_{x \in \mathbb{R}} [J(x) - xy], \quad u \in \mathbb{R} \quad (43)$$

The function $J(x)$ can be recovered from the dual function $\tilde{J}(y)$ by the Legendre transform inversion formula

$$J(x) = \inf_{y \in \mathbb{R}} [\tilde{J}(y) + xy], \quad u \in \mathbb{R} \quad (44)$$

Next, we find the explicit expression of the derivative of $J(x)$ by using the same technique in proving the envelope theorem (Mas-Colell, Whinston, and Green (1995)).

Lemma 1. *If $J(x)$, $\mathcal{X}(y)$ and $\mathcal{Y}(x)$ are finite, then we have*

$$\frac{\partial}{\partial y} \tilde{J}(y) = -\mathcal{X}(y), \quad 0 < y < \infty \quad (45)$$

$$\frac{\partial}{\partial y} J(x) = \mathcal{Y}(x), \quad -\infty < x < \infty \quad (46)$$

Proof. $\tilde{J}(y)$ is well defined on $0 < y < \infty$ where it is finite. $\tilde{J}(y)$ is the value attained by maximizing $F(x, y) = J(x) - xy$ where y is a parameter in the optimization problem. Let $x(\cdot)$ be the maximizer function. Then $\tilde{J}(y) = F(x(y), y)$ for $0 < y < \infty$. Using the chain rule, we have

$$\frac{\partial}{\partial y} \tilde{J}(y) = \frac{\partial}{\partial x} F(x(y), y) x'(y) + \frac{\partial}{\partial y} F(x(y), y), \quad 0 < y < \infty \quad (47)$$

The first order condition to the maximization problem (43) yield

$$\frac{\partial}{\partial x} F(x(y), y) = \frac{\partial}{\partial x} J(x) - y = 0$$

Substituting it into (47) we have

$$\frac{\partial}{\partial y} \tilde{J}(y) = \frac{\partial}{\partial y} F(x(y), y) = -x, \quad 0 < y < \infty$$

Replacing x with $\mathcal{X}(y)$ yields (45).

$J(x)$ is well defined where x is finite. Let $\mathcal{Y}(\cdot)$ be the minimizer function. By definition (44), we have

$$\begin{aligned} \frac{\partial}{\partial x} J(x) &= \frac{\partial}{\partial x} [\tilde{J}(\mathcal{Y}(x)) + x\mathcal{Y}(x)] \\ &= \mathcal{Y}'(x) \left(\frac{\partial}{\partial y} \tilde{J}(y) + x \right) + \mathcal{Y}(x), \quad 0 < y < \infty \end{aligned} \quad (48)$$

The first order condition to the minimization problem (44) yield

$$\frac{\partial}{\partial y} \tilde{J}(y) + x = 0$$

Then we have (46). □

Now we have both the optimal terminal wealth (42) and the optimal portfolio (32), which complete the solution to the optimization problem. Next, we provide an examples for a CRRA utility function.

4.1 Maximizing constant relative risk aversion utility

For a CRRA utility function

$$U(x) = \frac{x^p}{p}$$

where $p < 1$ and $p \neq 0$.²

$$I(y) = x = y^{\frac{1}{p-1}}, 0 < y < \infty$$

and

$$\mathcal{X}(y) = E[H(T)I(yH(T))] = y^{\frac{1}{p-1}} E \left[H(T)^{\frac{p}{p-1}} \right]$$

and

$$\mathcal{Y}(x) = \left(\frac{x}{E \left[H(T)^{\frac{p}{p-1}} \right]} \right)^{p-1}$$

The optimal terminal wealth is given as

$$\xi^* = I(\mathcal{Y}(x)H(T)) = \frac{x}{E \left[H(T)^{\frac{p}{1-p}} \right]} (H(T))^{\frac{1}{p-1}}$$

² $U(x) = \ln x$ if $p = 0$.

and

$$\begin{aligned}
X(t) &= \frac{1}{H(t)} E[H(T)\xi \mid \mathcal{F}(t)] = \frac{x}{H(t)E\left[H(T)^{\frac{p}{1-p}}\right]} E\left[H(T)^{\frac{p}{1-p}} \mid \mathcal{F}(t)\right] \\
&= \exp\left\{\frac{p}{p-1} \sum_{j=1}^L \int_0^t [(1-\mu_j)\lambda_j - r] du + \frac{p}{p-1} \sum_{j=1}^L \sum_{0 < \tau(m) \leq t} \ln \mu_{jm}\right\} \\
&= \Omega(t) \exp\left\{\int_0^t \sum_{j=1}^L \left[\frac{p}{p-1} ((1-\mu_j)\lambda_j - r) - \left(1 - \mu_j^{\frac{p}{p-1}}\right) \lambda_j\right] du\right\}
\end{aligned} \tag{49}$$

where

$$\Omega(t) = \exp\left\{\int_0^t \sum_{j=1}^L \left[(1 - \mu_j^{\frac{p}{p-1}}) \lambda_j du + \sum_{0 < \tau(m) \leq t} \ln \mu_{jm}^{\frac{p}{p-1}}\right]\right\}$$

Using (49) we have

$$\Psi(t) \triangleq E[H(T)\xi \mid \mathcal{F}(t)] = H(t)X(t)$$

where

$$\begin{aligned}
H(t)X(t) &= \frac{x}{E\left[H(T)^{\frac{p}{p-1}}\right]} E\left[H(T)^{\frac{p}{p-1}} \mid \mathcal{F}(t)\right] \\
&= \frac{x\Omega(t)}{E\left[H(T)^{\frac{p}{p-1}}\right]} \exp\left\{\frac{p}{p-1} \sum_{j=1}^L \int_0^T \left[(1-\mu_j)\lambda_j - \left(1 - \mu_j^{\frac{p}{p-1}}\right) \lambda_j - r\right] du\right\} \\
&= \frac{xd\Omega(t)}{E\left[H(T)^{\frac{p}{p-1}}\right]} \exp\left\{\frac{p}{p-1} \sum_{j=1}^L \int_0^T \left[(1-\mu_j)\lambda_j - \left(1 - \mu_j^{\frac{p}{p-1}}\right) \lambda_j - r\right] du\right\} \\
&= \Psi(t) \sum_{j=1}^L \left(\mu_j^{\frac{p}{p-1}} - 1\right) dM_j(t)
\end{aligned} \tag{50}$$

From (32) we have

$$-\frac{w_j^* \beta_j X(t) - b_j}{S_0(t)} = \Psi(s-) \left[\frac{\left(\mu_j^{\frac{p}{p-1}} - 1\right) - (\mu_j - 1)}{\mu_j Z(s-)} \right]$$

Recall (12), finally we have the optimal portfolio from the martingale approach

$$w_i^* = \frac{1}{\beta_i} \left[1 - \left(\frac{\beta_i \lambda_i}{\alpha_i - r + \beta_i \lambda_i} \right)^{\frac{1}{1-p}} - \frac{b_i}{x} \right] \quad (52)$$

It should be noted we get a unique solution in (52), which one may be surprised to see the result. The market is incomplete, there is no unique equivalent martingale measure and the solutions we got should represent a set of rather than a unique solutions. This set is an interval and is often quite large in general. Then insurers have to consider all the set of expectations $E_Q(\xi)$ of the terminal wealth under any equivalent martingale measure Q . What we have here is a special case, in which the optimal investment policy can be uniquely determined. We will come back to this point in Section 6.

Since insurance market is incomplete and martingale measure is not unique, it is thus common to apply the dynamic programming approach using the HJB equation to find solutions. In the next section, we apply the dynamic programming approach to the same optimization problem. Then we compare the results from the two different methodologies.

5 Solutions of Dynamic Programming

The martingale approach used in the previous section does not require the underlying assets process “Markovian”, even though we implicitly assume constant coefficients in Section (4.1) for simplicity. The HJB equation of dynamic programming does requires the underlying assets to follow Markov process. We specialize in this section the model to the case of constant coefficients. The assumption of constant coefficients permit us to employ “Markovian” method.

Under Definition (33) the value function $J(x)$ is of class C^1 on the set $\{(t, x) \in [0, T) \times \mathbb{R}\}; x > \mathcal{X}(t, \infty)$ and satisfy the boundary conditions

$$J(t, x) = \sup_w E[U(\xi)] \quad (53)$$

$$\text{subject to } \frac{1}{H(t)} E[H(T)X(T)] \leq x \quad (54)$$

$$J(T, x) = U(x) \quad (55)$$

Define a function $f(t, x, w)$

$$f(t, x, w) = \frac{\partial}{\partial t} J(t, x) + \left[\sum_{i=1}^K w_i ((\alpha_i - r) + \beta_i \lambda_i) x + rx + c - \sum_{i=1}^L b_i \lambda_i \right] \frac{\partial}{\partial x} J(t, x) \\ + \sum_{i=1}^K \lambda_i [J(t, (1 + w_i \beta_i) x - b_i) - J(t, x)] + \sum_{i=K+1}^L \lambda_i [J(t, (x - b_i)) - J(t, x)] \quad (56)$$

J satisfies the following HJB equation of dynamic programming:

$$f(t, x, w^*) = \max_{w \in \mathbb{R}^L} f(t, x, w) = 0 \quad (57)$$

with boundary condition

$$J(T, x) = U(x) \quad (58)$$

The first order conditions gives

$$(\alpha_i - r + \beta_i \lambda_i) \frac{\partial}{\partial x} J(t, x) = -\lambda_i \beta_i \frac{\partial}{\partial x} [J(t, (1 - w_i^* \beta_i) x - b_i)] \quad i = 1, 2, \dots, K \quad (59)$$

Recall Lemma 1, we can rewrite the first order condition in terms of \mathcal{Y} for a fix t

$$\alpha_i - r + \beta_i \lambda_i = -\lambda_i \beta_i \frac{\mathcal{Y}(t, (1 - w_i \beta_i) x - b_i)}{\mathcal{Y}(t, x)} \quad i = 1, 2, \dots, K \quad (60)$$

The result of Lemma 1 also coincides with the first derivative of the boundary condition with respect to x for a fix time T in (58)

$$\frac{\partial}{\partial x} J(T, x) = U'(x) = \mathcal{Y}(T, x), \quad 0 \leq x < \infty$$

Next, we show that (60) is the optimal solution to the HJB equation (57).

Lemma 2. (Verification theorem) *Let $v(t, x)$ be a solution of the dynamic programming equation (57) with the boundary condition (58) such that $v \in C^1(\mathfrak{A})$ where \mathfrak{A} is admissible set of w . Then the solution (61) is optimal.*

Proof. First we prove the concavity of $E[U(\xi)]$. Let x_1, x_2 are two initial equities and $x_i \in [\mathcal{X}(\infty), \infty), i = 1, 2$. And let w_1, w_2 are two investment strategies and $w_i \in \mathfrak{A}$. For $\rho_1, \rho_2 \in (0, 1)$ with $\rho_1 + \rho_2 = 1$, the portfolio pair $(\rho_1 w_1 + \rho_2 w_2)$ is in \mathfrak{A} with initial wealth $x \triangleq \rho_1 x_1 + \rho_2 x_2$ and

$$X(t; x, w) = \rho_1 X(t; x_1, w_1) + \rho_2 X(t; x_2, w_2)$$

Consequently

$$\begin{aligned}
& \rho_1 E [U(X(T; x_1, w_1)) + \rho_2 U(X(T; x_2, w_2))] \\
& \leq E [U(\rho_1 X(t; x_1, w_1) + \rho_2 X(t; x_2, w_2))] \\
& = E [U(X(T; x, w))]
\end{aligned}$$

Let $w \in \mathfrak{A}$ be arbitrary. Apply Itô's Lemma to v and X we obtain

$$\begin{aligned}
v(T, X(T); w) &= v(t, x) + \int_t^T \frac{\partial}{\partial t} v(u, X(u)) du + \frac{\partial}{\partial x} v(u, X(u)) dX^c(u) \\
&+ [v(u, X(u)) - v(u-, X(u-))] (dN - \lambda du) + [v(u, X(u)) - v(u-, X(u-))] \lambda du
\end{aligned}$$

Since v satisfies the HJB equation, we have

$$v(T, X(T); w) \leq v(t, x) + \int_t^T [v(u, X(u)) - v(u-, X(u-))] dM$$

Taking the conditional expectation on both sides yields

$$E[v(T, X(T); w)] \leq v(t, x)$$

and equality is obtained only if $w = w^*$:

$$E[v(T, X(T); w^*)] = v(t, x)$$

□

We have shown the optimal portfolio to the HJB equation. We summarize the result in the following Proposition.

Proposition 2. *The function $J(x, t)$ of (55) satisfies the problem for the HJB equation of dynamic programming (57) with the initial boundary condition (58) and the maximization over $w \in \mathbb{R}^{M+1}$ are achieved by:*

$$\alpha_i - r + \beta_i \lambda_i = \beta_i \lambda_i \frac{\mathcal{Y}(t, (1 - w_i^* \beta_i) x - b_i)}{\mathcal{Y}(t, x)}, \quad i = 1, 2, \dots, M \quad (61)$$

5.1 Maximizing CRRA utility by HJB equation

For constant coefficients, one can see that (61) from HJB equation is easier to use than (32) from martingale approach. As an example, we consider a CRRA utility function and we have (see Section 4.1)

$$\mathcal{Y}(x) = \left(\frac{x}{E \left[H(T)^{\frac{p}{p-1}} \right]} \right)^{p-1}$$

Then we can calculate the optimal portfolio directly from (61).

$$w_i^* = \frac{1}{\beta_i} \left[1 - \left(\frac{\beta_i \lambda_i}{\alpha_i - r + \beta_i \lambda_i} \right)^{\frac{1}{1-p}} - \frac{b_i}{x} \right]$$

which is exactly the same as we got from the martingale approach. As we can see, this policy depends on the wealth level, i.e., the amount of wealth $w_i^* x$ invested in security i is not constant, but a function of wealth level x .

6 Discussions

The optimal investment policy for an insurer with CRRA utility has two components. The first represents the hedging against the claim b_i . The insurer should short the catastrophe related security to offset the insurance claims b_i . The second component (see (52))

$$\left[1 - \left(\frac{\beta_i \lambda_i}{\alpha_i - r + \beta_i \lambda_i} \right)^{\frac{1}{1-p}} \right] x = \left[1 - \left(\frac{1}{\mu_i} \right)^{\frac{1}{1-p}} \right] x$$

is a function of the market risk premium μ_i of the security i . If there is no risk premium, then $\mu_i = 1$. This component is zero and the optimal policy is just to hedge the claim. For a positive risk premium, the insurer's optimal policy is to hedge less than 100%. For a CRRA utility function, this component depends on the insurer's wealth level. The higher level of the wealth an insurer has, the more risk that the insurer should take herself.

We solve the optimal investment policy problem for an insurer by the martingale approach. This martingale approach would have given multiple solutions depending on the measure we pick from a set of equivalent martingale measures. However, for

the current case the optimal policy is uniquely determined. The reason for the unique optimal policy is that the security market itself is complete. The incomplete part of the insurance market is related to the claims that are not hedgeable for insurers. When we ask the question how an insurer should take the advantage of the security market to hedge her claim risk, the part of market that is incomplete is irrelevant to the optimal investment policy. There are multiple martingale measures for the part of insurance claims that cannot be hedged (see Section 2.2). Nonetheless, insurers have no control over the claims that cannot be hedged and, thus, those claims are irrelevant for the optimal investment question. Therefore, to determine the optimal investment policy the martingale approach is a very useful tool even in an incomplete insurance market: it provides not only a general solution to the optimal investment problem, but also a unique solution to the problem.

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