

Optimal Capital Allocation

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Abstract

This paper starts with primitive assumptions on consumer preferences and then derives prices consistent with a social optimum within an insurance company and the capital allocation implied therein. The implied allocation “adds up” to the total capital of the firm (a result echoing findings in the congestion pricing literature—where optimal tolls exactly cover the rental cost of the highway). The allocation follows each consumer’s share of recoveries in states where the insurer defaults, weighted by the severity (in terms of consumer welfare impact) of the default. However, the paper goes on to argue that the economic approach employed supports a broader conception of allocation—beyond that based on the marginal impact of each consumer’s risk: Specifically, it argues that allocation based on relative consumer valuations of all units of capital—both marginal and inframarginal—may yield more stable and equitable assignment of cost responsibility.

*I thank Michael Suher for research assistance. Errors are mine. The views expressed in this article are those of the author and do not necessarily reflect the position of the Federal Reserve Bank of New York or the Federal Reserve System.

1 Introduction

This paper studies the allocation of insurance company assets to policies for pricing purposes. It approaches the problem by identifying the socially optimal allocation of coverage across consumers and the optimal level of overall capitalization for the insurance company. It then shows that the decentralized implementation of this optimum features a price per unit of coverage that implies an allocation of capital per unit of coverage to each consumer that “adds up”—in the sense that the sum over all consumers of allocated capital (the product of the consumer’s total coverage amount and the consumer’s capital allocation per unit of coverage) equals the total capital of the firm.

This result mirrors a finding in the transportation economics literature, where optimally set congestion tolls for drivers exactly add up to the rental cost of the highway (Keeler and Small [4]). In the insurance case, the optimal “toll” penalizes consumers for the external effects (on other consumers) associated with their purchase of a marginal unit of coverage. The toll exactly covers the cost of the assets that are needed to offset the impact of the coverage purchase at the margin; and the tolls in aggregate cover the total rental cost of the assets. Expanding the asset base of the insurance company

is analogous to expanding the width of the highway.

On one hand, this paper provides an economic rationale for capital allocation based on the “marginal analysis”—as developed in a literature starting with Myers and Read [12] (and subsequently generalized by Kalkbrenner [2] and Mildenhall [11]). In this paper, a social welfare target replaces the risk measure target of the literature above. When it is assumed (as we do in this paper) that consumers care about the distribution of their own recoveries rather than an average financial target for the company, the resulting allocation “adds up” but is not derived from any commonly used risk measure.

Instead, capital is allocated according to each consumer’s share of value-adjusted recoveries in states where the company fails—with the “value adjustment” reflecting differences in consumers’ marginal utility of wealth in the various states of default. That is, recoveries are valued according to the severity of default: Recoveries in states where consumers are receiving very little per dollar claimed will be weighted more heavily than those where consumers are receiving more. The intuition behind this allocation rule hinges on the fact that, at the margin, additional capital affects consumer welfare only in states where the company is defaulting; likewise, increases in a consumer’s coverage affect others only to the extent that the consumer is a rival

claimant in states of default. Hence, the allocation rule—derived from the marginal social cost of coverage at the optimum—is implied by an exercise in Pigouvian taxation, where prices are derived to make consumers feel the full social cost of their actions. Alternatively, one could view the allocation rule as the result of an internal market for contingent claims on the company in states where it defaults.

Beyond this, the paper explores a drawback associated with allocating capital based on marginal analysis. Since the entire capital of the insurer is allocated based on how the *marginal* unit of capital is used in defraying risk externalities, the *inframarginal* units of capital are not necessarily allocated in proportion to how they are valued by the policyholders. The resulting allocation of capital costs, though they may be consistent with a social optimum in some circumstances, may yield an uneven distribution of consumer surplus.

The paper shows how economic theory supports circumvention of this drawback, after one realizes that the restrictions implied by marginal analysis apply only to the marginal units of coverage. Inframarginal units of coverage can be assigned different allocations of capital, so long as the “adding up” principle is observed and the associated pricing functions induce consumers

to purchase the socially optimal levels of coverage.

The rest of this paper is organized as follows. Section 2 presents formal analysis of the social planning problem and the decentralized implementation of the social optimum, and it connects the prices used in the implementation with allocations of the underlying capital. Section 3 introduces a simple example to illustrate the potentially inequitable allocation of capital that can result from marginal approaches to allocation. Section 4 suggests a more general approach with the potential to remedy inequities associated with marginal allocation methods. Section 5 concludes.

2 Optimal Capitalization & Cost Allocation

In this section, we study the joint problems of how to set the capital level for an insurance company and how to allocate the costs of capital to policyholders. The capital costs considered here are frictional costs (e.g., a tax on capital, as in Froot and Stein [1]). We show that the two problems are intimately linked and characterize the solution.

For transparency, we start by ignoring any issues that might arise due to the correlation of insurance losses with returns in the broader asset markets,

and the interaction between insurance markets and securities markets. We then move on to introduce consumer and insurer participation in securities markets, and extend the results accordingly.

2.1 Allocation without Securities Markets

Formally, consider an insurance company with N consumers, with consumer i facing a loss of size L_i . To characterize the possible states of the world, we define a vector \mathbf{s} of length N , with all the elements taking a value of zero or one: $\mathbf{s}(i) = 1$ indicates that consumer i experienced a loss, while $\mathbf{s}(i) = 0$ indicates that she did not. Let Ω be the set of all possible states of the world.

The following set definitions are useful. First, define the set of states where consumer i experiences a loss as:

$$\Omega^i = \{\mathbf{s} \in \Omega : \mathbf{s}(i) = 1\}.$$

Next, define the set of states where at least one consumer has a loss as:

$$\Omega_1 \equiv \cup_{i=1}^N \Omega^i.$$

Finally, define the set of consumers who lose in state \mathbf{s} as:

$$\Gamma(\mathbf{s}) = \{i : \mathbf{s}(i) = 1\}.$$

The social planner must determine the optimal level of assets A for the company, as well as the levels of insurance coverage for the consumers, with the coverage indemnification level for consumer i denoted as I_i . If a consumer experiences a loss, she will claim payment to the extent of the amount of insurance coverage. If the total of claims is less than company assets, all claims are paid in full. If not, all claimants are paid at the same rate per dollar of coverage. Thus, we may define the consumer's recovery in state $\mathbf{s} \in \Omega$ as:

$$R_i^{\mathbf{s}} \equiv \min \left\{ I_i, \frac{A}{\sum_{j \in \Gamma(\mathbf{s})} I_j} I_i \right\} \text{ for } \mathbf{s} \in \Omega^i$$

$$R_i^{\mathbf{s}} = 0 \text{ for } \mathbf{s} \notin \Omega^i$$

The expected value of recoveries for the i -th consumer is thus given by:

$$E_i \equiv \sum_{\mathbf{s} \in \Omega_1} \Pr(\mathbf{s}) R_i^{\mathbf{s}}.$$

There is a frictional cost—including agency, taxes, and monitoring costs—associated with holding assets in the company. Note that this cost has nothing to do with a risk-reward trade-off (for example, based on the “beta” of insurance liabilities): To remain focused on the main idea, we simplify by assuming that insurance risk is uncorrelated with returns in other asset classes. In what follows, we represent the cost as a “tax” on assets:

$$\tau A \tag{1}$$

It is also common in the literature to represent frictional costs as a tax on equity capital, as in:

$$\tau \left(A - \sum_{\mathbf{s} \in \Omega} \Pr(\mathbf{s}) \sum_{i \in \Gamma(\mathbf{s})} \min \left\{ I_i, \frac{A}{\sum_{j \in \Gamma(\mathbf{s})} I_j} I_i \right\} \right) \tag{2}$$

The latter case is treated in the Appendix.

If we denote the premium paid by the consumer i as P_i , consumer utility may be expressed as:

$$V_i(A, W_i - P_i, I_1, \dots, I_N) \equiv \sum_{\mathbf{s} \notin \Omega^i} \Pr(\mathbf{s}) U_i(W_i - P_i) + \sum_{\mathbf{s} \in \Omega^i} \Pr(\mathbf{s}) U_i(W_i - P_i - L + R_i^{\mathbf{s}}), \quad (3)$$

The social planner then solves (where we have ignored Pareto weights for notational simplicity):

$$\max_{A, \{I_i\}, \{P_i\}} V_i, \quad (4)$$

subject to total premiums covering the sum of total expected claims payments and frictional costs:

$$\sum P_i = \sum E_i + \tau A. \quad (5)$$

Analysis of the problem is complicated by the fact that the objective function and constraint are not smooth—they are only piecewise differentiable. For expositional purposes, we will ignore this complication and confine its proper treatment to the Appendix. The first order optimality conditions (assuming a solution in a smooth part of the function) imply the following:

$$[I_i]: \frac{\partial V_i}{\partial I_i} - \frac{\partial V_i}{\partial W} \left(\frac{\partial E_i}{\partial I_i} + \sum_{j \neq i} \frac{\partial E_j}{\partial I_i} \right) + \sum_{j \neq i} \frac{\partial V_j}{\partial I_i} = 0 \quad (6)$$

$$[A]: \sum_k \frac{\partial V_k}{\partial A} - \frac{\partial V_i}{\partial W} \left(\sum_k \frac{\partial E_k}{\partial A} + \tau \right) = 0 \quad (7)$$

$$[P_i]: \frac{\partial V_i}{\partial W} - \frac{\partial V_j}{\partial W} = 0, \quad (8)$$

where $\frac{\partial V_i}{\partial W}$ is the marginal utility of wealth (the partial derivative with respect to the function's second argument). At this point, we may take two different approaches to interpreting the premiums associated with the optimum.

The first, which we pursue below, is to calculate the per unit price of coverage associated with a decentralized implementation of the social optimum: This allows us to decompose the premium into several components, including a risk penalty derived from the externalities associated with the marginal unit of coverage—with the latter penalty having an interpretation as a capital cost allocation. This approach essentially mirrors the calculation and interpretation of the optimal toll in the transportation economics literature.

The second, which we pursue in Section 4, is to interpret the premiums

as cost shares and to identify restrictions on the cost sharing functions that allow individual choice to coincide with the social optimum. This approach follows from Lindahlian analysis of the problem of public good production, the generalization of which has shown—among other things—that equilibrium cost shares may be non-linear and may not be unique.

It is useful at this point to decompose the states of the world into those in which the company defaults (Ω_D) and those in which it is solvent (Ω_Z). Accordingly, we define:

$$\Omega_D = \left\{ \mathbf{s} \in \Omega : \sum_{j \in \Gamma(\mathbf{s})} I_j \geq \mathbf{A} \right\}$$

$$\Omega_Z = \left\{ \mathbf{s} \in \Omega : \sum_{j \in \Gamma(\mathbf{s})} I_j < \mathbf{A} \right\}$$

and extend the notation to decompose the expected loss for each consumer into that associated with states of default and that associated with solvent states:

$$E_i^D \equiv \sum_{\mathbf{s} \in \Omega_D} \Pr(\mathbf{s}) R_i^{\mathbf{s}},$$

$$E_i^Z \equiv \sum_{\mathbf{s} \in \Omega_Z} \Pr(\mathbf{s}) R_i^{\mathbf{s}}.$$

With this notation in hand, we may simplify the optimality condition for the choice of coverage for consumer i , (6), to:

$$\frac{\partial V_i}{\partial I_i} - \frac{\partial V_i}{\partial W} \frac{\partial E_i^Z}{\partial I_i} + \sum_{j \neq i} \frac{\partial V_j}{\partial I_i} = 0. \quad (9)$$

A decentralized implementation of the optimum will feature prices that induce consumers to choose the socially optimal level of coverage. Such prices will force the consumer to “see” the full effects of her coverage choices. We categorize the effects associated with an extra unit of coverage as follows. First, the consumer adds an extra unit of recovery in states of the world where the insurance company is solvent. Second, we identify two effects associated with an extra unit of coverage on recoveries in states of the world where the company defaults—a “first order” effect and a “second order” effect. The “first order” effect is to increase the recovery of the consumer in question according to the prevailing rate of payment. To illustrate, for some $\mathbf{s} \in \Omega_D \cap \Omega^i$, the first order effect of an additional unit of coverage in that state is to increase the consumer’s recovery by:

$$\frac{A}{\sum_{j \in \Gamma(\mathbf{s})} I_j}$$

The “second order” effect is to reduce the recovery rate per dollar claimed for all consumers making a claim in that state, with the rate paid per dollar claimed dropping by:

$$-\frac{A}{\left(\sum_{j \in \Gamma(\mathbf{s})} I_j\right)^2}$$

Note that the “first order” and “second order” effects net out to zero in terms of dollar impact.

$$\frac{A}{\sum_{j \in \Gamma(\mathbf{s})} I_j} - \frac{A}{\left(\sum_{j \in \Gamma(\mathbf{s})} I_j\right)^2} * \sum_{j \in \Gamma(\mathbf{s})} I_j = 0$$

Intuitively, the assets available for payment in any default state are not affected by the increase in coverage: In the end, the coverage increase only affects how the assets are split among claimants.

To proceed, we import a convention used in the transportation economics literature on congestion externalities (e.g., Keeler and Small [4]). In this literature, the consumer is assumed to ignore her own contribution to the

overall level of congestion. The analogy here is to assume that the consumer ignores the “second order” effect of increasing coverage on her own level of recovery in default states. That is, the consumer neglects to incorporate the marginal reduction in the recovery rate in default states that is caused by her marginal increase in coverage. As in the transportation literature, this assumption seems reasonable when the consumer is small in relation to the pool. Translated into mathematics, the “first order” effect of a marginal increase in coverage on the consumer’s recovery in the default state will dominate the second order effect (i.e., $\frac{A}{\sum_{j \in \Gamma(\mathbf{s})} I_j} \gg \frac{A}{(\sum_{j \in \Gamma(\mathbf{s})} I_j)^2} I_i$) if $\frac{I_i}{\sum_{j \in \Gamma(\mathbf{s})} I_j}$ is small. However, the assumption is not innocuous: As we shall see, it is required for the capital allocations implied by a decentralized implementation of the social optimum to “add up” across consumers to the total capital of the insurance company.

With this assumption in place, we seek a pricing function P_i^* satisfying:

$$\frac{\partial P_i^*}{\partial I_i} = \frac{\partial E_i^Z}{\partial I_i} - \frac{\sum_{j \neq i} \frac{\partial V_j}{\partial I_i} - \sum_{\mathbf{s} \in \Omega_D \cap \Omega^i} \Pr(\mathbf{s}) \left(\mu_i^{\mathbf{s}} \frac{A}{(\sum_{j \in \Gamma(\mathbf{s})} I_j)^2} I_i \right)}{\frac{\partial V_i}{\partial W}}$$

where $\mu_i^{\mathbf{s}} = \frac{\partial U_i(W_i - P_i - L + R_i^{\mathbf{s}})}{\partial W_i}$, the marginal utility of income for consumer i

in state \mathbf{s} . After some algebra, using (7), we obtain:

$$\frac{\partial P_i^*}{\partial I_i} = \sum_{\mathbf{s} \in \Omega_Z \cap \Omega^i} \Pr(\mathbf{s}) + \phi_i A \left(\sum_{\mathbf{s} \in \Omega_D} \Pr(\mathbf{s}) + \tau \right) \quad (10)$$

with

$$\phi_i = \frac{\sum_{\mathbf{s} \in \Omega_D \cap \Omega^i} \Pr(\mathbf{s}) \sum_{k \in \Gamma(\mathbf{s})} \mu_k^{\mathbf{s}} \left(\frac{I_k}{\sum_{j \in \Gamma(\mathbf{s})} I_j} \right) \left(\frac{1}{\sum_{j \in \Gamma(\mathbf{s})} I_j} \right)}{\sum_{\mathbf{s} \in \Omega_D} \Pr(\mathbf{s}) \sum_{k \in \Gamma(\mathbf{s})} \mu_k^{\mathbf{s}} \left(\frac{I_k}{\sum_{j \in \Gamma(\mathbf{s})} I_j} \right)} \quad (11)$$

The first term on the right-hand side of (10) represents the marginal increase in the expected value of claims paid to consumer i in solvent states, while the second represents a risk penalty—in proportion to consumer i 's marginal effect on aggregate consumer recoveries in states of default. The latter may also be interpreted as a per coverage unit *allocation* of costs relating to 1) all claims paid in states of default and 2) frictional costs of capital. To see this, start by observing that:

$$\sum_i \phi_i I_i = 1.$$

Next, multiply (10) by I_i to obtain:

$$\frac{\partial P_i^*}{\partial I_i} I_i = \left(\sum_{\mathbf{s} \in \Omega_Z \cap \Omega^i} \Pr(\mathbf{s}) \right) I_i + (\phi_i I_i) \left[\left(\sum_{\mathbf{s} \in \Omega_D} \Pr(\mathbf{s}) + \tau \right) A \right] \quad (12)$$

Thus, if each consumer is charged a total variable premium based on the socially efficient price for the marginal unit of coverage—that is, if all units of the consumer’s coverage are priced at the cost of the marginal unit—then the variable premiums for all consumers will exactly cover expected losses and the frictional costs. This echoes the result in Keeler and Small [4], where the sum of the tolls equals the cost of the highway. There are at least three noteworthy aspects of this finding.

First, both frictional costs *and* total expected claims payments in default states are allocated according to $\phi_i I_i$. This factor allocates cost according to the marginal valuation of assets in states of default. For example, those consumers who make claims in default states where the average claimant’s marginal utility of consumption is relatively high will tend to get higher allocations of cost—both of frictional cost, and of expected claims in default states. Thus, the pricing function will always assign the expected value of claims made in solvent states according to the consumers who make the

claims. The same is not true, however, of claims made in states of default: The underlying mathematics are in fact assigning responsibility for the expected value of claims made in default states *and* frictional capital costs in a similar manner.

Second, there are at least two cases where the allocation rule will simplify into an allocation of claims costs (in default states) and frictional costs based on the straight expected value of claims in states of default. One case is where there is only one default state—or, if there is more than one, the same claimants are involved in every default state. The second is where the marginal utilities of consumption are the same for each claimant and are the same across all states of default. In either of these cases, the allocation rule simply follows each consumer’s contribution to the expected value of claims made on the marginal dollar of assets. However, the general allocation rule differs this simple decomposition because the individual contributions are weighted according to the marginal utilities of consumption within that default state. The reason for this is that a straight expected value is not necessarily a good measure of the social cost of a claim in states of default, despite our assumption of “zero beta” insurance risk: Within the internal company “market,” some states of default are more painful than others (in

the sense of being more severe and thus being associated with higher marginal valuations placed on company assets) and must be penalized accordingly. In this sense, contingent claims to assets in a particular state of the world receive a “state price” reflecting the value of those assets to claimants in that state of the world.

Finally, although “adding up” property for total variable premiums is convenient, it is important to remember that the restriction imposed by the foregoing analysis is a *marginal* one—concerning the price of the marginal unit of coverage, $\frac{\partial P_i^*}{\partial I_i}$, at the optimum. Taking this reasoning further, the allocation of capital implied by the marginal restriction is actually an allocation of the *marginal* unit of capital—a description of how capital is consumed at the margin in the defrayment of the “congestion” externalities associated with the marginal units of consumer purchases. We have not pinned down the allocation of inframarginal units of capital. They may indeed be allocated in a manner different from the marginal unit, with the optimal allocation depending on the structure of preferences and the distribution of wealth—among other things. In fact, the total premium paid can be decomposed into a fixed cost F_i and a variable component as in:

$$P_i^* = F_i + \frac{\partial P_i^*}{\partial I_i} I_i.$$

The “adding up” property requires that $\sum F_i = 0$, so the fixed components amount to transfers of cost shares among the consumers. In Section 3, we attempt to illustrate the pitfalls of marginal allocation and motivate more general allocation methods.

2.2 Allocation with Securities Markets

In the Appendix (Section 6.3), we introduce securities markets. The social planner now purchases state contingent payoffs for each individual (out of individual wealth) and for the insurance company (out of insurance company assets). The complication is that markets are incomplete in the sense that securities markets cannot be used to fully hedge insurance risks. As in Lakdawalla and Zanjani [5], we extend the arbitrage-free state prices associated with security market equilibrium to price insurance risks by assuming that payoffs made on insurance contracts are valued according to the prevailing security market state.

This generalized model adds notational complexity but does not change

the basic flavor of the results, with the main difference concerning what's being allocated. Appropriate (extended) state prices are applied to claims recoveries, with the result being that the fair financial value of claims recoveries in states of default plus frictional costs associated with capital end up being allocated to consumer.

The basis for allocation, however, is essentially the same as before. Consumer recoveries of assets in states of default are weighted by the marginal social utility of assets in those states, with each consumer's share of both the fair financial value of claims in states of default and frictional capital costs being determined by her value-adjusted share of total recoveries in states of default.

3 An Example

Myers and Read [12] recognized that marginal allocations could be unstable (see pages 566-8) before concluding that they were "reasonably robust" for diversified companies. In this section, we examine situations in which marginal allocations are not robust to small changes, with the objective of motivating allocation methods that are 1) robust to small changes and 2)

consistent with cost sharing equilibrium as defined above.

Start by considering the case of two consumers with constant absolute risk aversion, with coefficient of absolute risk aversion being one. Each consumer has wealth of 5. Consumer A has a 1% probability of losing 10, while Consumer B has a 10% chance of losing the same amount—with the consumers' risks being independent of each other. The frictional cost of assets is 10%. Our example shuts down wealth effects by using constant absolute risk aversion (CARA) utility. The (numerically calculated) solution features an asset level of 10.55 and coverage levels of 9.90 and 9.88 for Consumer A and Consumer B, respectively. (With CARA utility, the foregoing aspects of the solution are independent of the Pareto weights (and associated premiums) assigned to the consumers.)

In this example, allocation is driven by the state of the world where both consumers lose—which is the only one in which the company defaults. Since the coverage levels are roughly equivalent, the consumers receive roughly equal shares of responsibility for frictional costs via application of (13)—despite having vastly different loss probabilities. The potential inequity in the distribution of consumer surplus is revealed in Figure 1, which shows how each consumer's willingness to pay for incremental amounts of assets

changes with the asset level. Because of Consumer B's higher probability of loss, she tends to value inframarginal asset units (at levels below her coverage amount) more highly than does Consumer A.

This point is made even more forcefully if we add a third consumer (Consumer C) whose risk exhibits perfect negative correlation with the risks of the other consumers (in the sense that Consumer C loses only when the other consumers do not lose). Suppose Consumer C has the same wealth and risk aversion as the other consumers, and a 10% chance of losing 10.

Consumer C gets a free ride in this example because he has no bearing on company default. Hence, the aspects of the solution detailed above do not change, and Consumer C receives full insurance. Moreover, application of (13) allocates no assets to Consumer C, since he is never a rival claimant in a default scenario. The potential inequity in this allocation is revealed in Figure 2 (which presents essentially the same information as Figure 1, with the addition of Consumer C). As can be seen in the figure, Consumer C values many of the inframarginal asset units more highly than his compatriots, before dropping off to zero valuation by the time the marginal unit is reached.

4 Cost Sharing Equilibria

If insurance company assets are viewed as a public good within the community of company policyholders, perhaps we can exploit techniques developed for more general public goods problems to our case. The problem of optimal public good provision can be addressed, in principle, by charging “personalized prices” to individual consumers that, at the optimum, 1) coincide with marginal valuations of the public good 2) sum up to cover the total cost of provision. This solution is referred to as a Lindahl equilibrium. Mas-Colell and Silvestre [7] introduced the concept of a *cost sharing equilibrium* as a generalization of the Lindahl equilibrium—with the basic idea being that any cost sharing rule is viable if all consumers agree on the optimal level of public good provision.

We adapt their definitions to our application:

Definition 1 A cost share system is a set of functions $g_i(A, I_1, \dots, I_N)$, for $i = 1, \dots, N$ such that 1) $g_i(0, I_1, \dots, I_N) = 0$ for all i , 2) $g_i(0, I_1, \dots, I_N) = 0$ if $I_i = 0$ and 3) $\sum g_i(A, I_1, \dots, I_N) = \sum E_i + \tau A$.

Definition 2 A cost sharing equilibrium is a cost share system, an asset level A^* , indemnity levels I_1^*, \dots, I_N^* , and an allocation of initial wealth $\{W_i\}$ such that: $V_i(A^*, W_i - g_i(A^*, I_1^*, \dots, I_N^*), I_1^*, \dots, I_N^*) \geq V_i(\bar{A}, W_i - g_i(\bar{A}, \bar{I}_1, \dots, \bar{I}_N), \bar{I}_1, \dots, \bar{I}_N)$ for all i and for all $\bar{A}, \bar{I}_1, \dots, \bar{I}_N \geq 0$.

Proposition 1 from Mas-Colell and Silvestre can be adapted to show that any cost sharing equilibrium produces an optimal state.¹

Theorem 1 *A cost sharing equilibrium is Pareto optimal.*

Proof: See Appendix.

Cost sharing equilibria feature an allocation of all insurance costs, including capital costs, to consumers. For our purposes, the important point is that the cost sharing system associated with the equilibrium need not be linear and could allocate inframarginal costs differently than marginal costs. To be sure, marginal allocation restrictions must be observed. At the margin, each consumer must face the cost share implied by (10) for marginal changes in coverage:

$$\frac{\partial g_i(A^*, I_1^*, \dots, I_N^*)}{\partial I_i} = \sum_{\mathbf{s} \in \Omega_Z \cap \Omega^i} \Pr(\mathbf{s}) + \phi_i A \left(\sum_{\mathbf{s} \in \Omega_D} \Pr(\mathbf{s}) + \tau \right) \quad (13)$$

and the cost share associated with marginal changes in assets must follow:

¹However, there is no guarantee here that the equilibrium belongs to the core. The presence of negative externalities among the consumers prevents the equivalence results from the public goods literature (see, e.g., Weber and Weismeth [14]) from applying.

$$\frac{\partial g_i(A^*, I_1^*, \dots, I_N^*)}{\partial A} = \frac{\frac{\partial V_i}{\partial A}}{\sum_k \frac{\partial V_k}{\partial A}} \left(\sum_{\mathbf{s} \in \Omega_D} \text{Pr}(\mathbf{s}) + \tau \right). \quad (14)$$

The latter condition allocates the cost shares at the margin according to the ratio of each consumer’s marginal valuation to the marginal social valuation, the “ratio equilibrium” condition identified by Kaneko [3] and Mas-Colell and Silvestre [8]. It is observed in these works further that this marginal condition on the cost sharing function can serve as the basis for a proportional cost allocation that yields both an equilibrium and a social optimum. However, we are only guaranteed these results if the total cost of production is convex in the amount produced: In this case, total costs:

$$\sum E_i + \tau A$$

are concave in assets, meaning that allocating total cost based on the ratio of individual marginal valuation to social marginal valuation ($\frac{\frac{\partial V_i}{\partial A}}{\sum_k \frac{\partial V_k}{\partial A}}$ at the optimum) will not generally be feasible.

Equation (13), on the other hand, could be used as a basis for allocation.

As noted previously, such an approach “adds up”:

$$\sum_{i=1}^N \frac{\partial g_i(A^*, I_1^*, \dots, I_N^*)}{\partial I_i} I_i = \sum_{i=1}^N E_i + \tau A$$

Furthermore, it is easily deduced that prices per unit of coverage defined by $\frac{\partial g_i(A^*, I_1^*, \dots, I_N^*)}{\partial I_i}$ will induce the consumers to choose the socially optimal levels of coverage when assets are set at A^* and wealth levels are set at:

$$\hat{W}_i = W_i - \left(g_i(A^*, I_1^*, \dots, I_N^*) - \frac{\partial g_i(A^*, I_1^*, \dots, I_N^*)}{\partial I_i} I_i \right)$$

for each consumer (where W_i is the wealth level for consumer i in the cost sharing equilibrium). In this sense, frictional costs can be allocated based on the shape of the equilibrium cost shares in the neighborhood of the optimum, although we are not guaranteed that this will be an equilibrium as defined above.

In general, however, the equilibrium cost shares involve transfers among consumers—leading the shares of total cost to deviate from the allocation implied by the marginal rules. Moreover, there is not necessarily a one-to-one correspondence between social optima and equilibrium cost shares: Although marginal behavior of the cost shares will be unique at a particular optimum, multiple equilibria may exist with different allocations of inframar-

ginal costs.²

5 Summary

Previous work on the economics of capital allocation have relied on *ad hoc* assumptions that either integrated a insurer level financial target (e.g., risk measure) into consumer preferences (Zanjani [15]) or constrained firm behavior by imposing a risk measure target (e.g., Meyers [10]). In this paper, we approach the problem with standard preference assumptions and derive a new marginal allocation rule. This rule is not tied to any specific risk measure, but instead allocates capital based on each consumer’s “value-adjusted” recoveries in states where the insurer defaults.

The paper also considers alternatives to the marginal approach to capital allocation. We note that inframarginal capital units can, in some cases, be allocated differently than the marginal unit—with the resulting allocation having attractive properties with respect to stability and equity. While this seems a promising foundation for producing allocations featuring stable

²It is possible that a Shapley value approach could be applied as a solution concept, although other solutions are possible, and I have not verified that the game here meets the assumptions required. For an example of an *ad hoc* application of Shapley value in apportioning risk loads, see Mango [6].

distribution of consumer surplus, we have not explored any specific rules that would yield such stability while meeting the required conditions for equilibrium. This is left for future research.

Figure 1
Willingness to Pay for Incremental Asset Units at Different Asset Levels

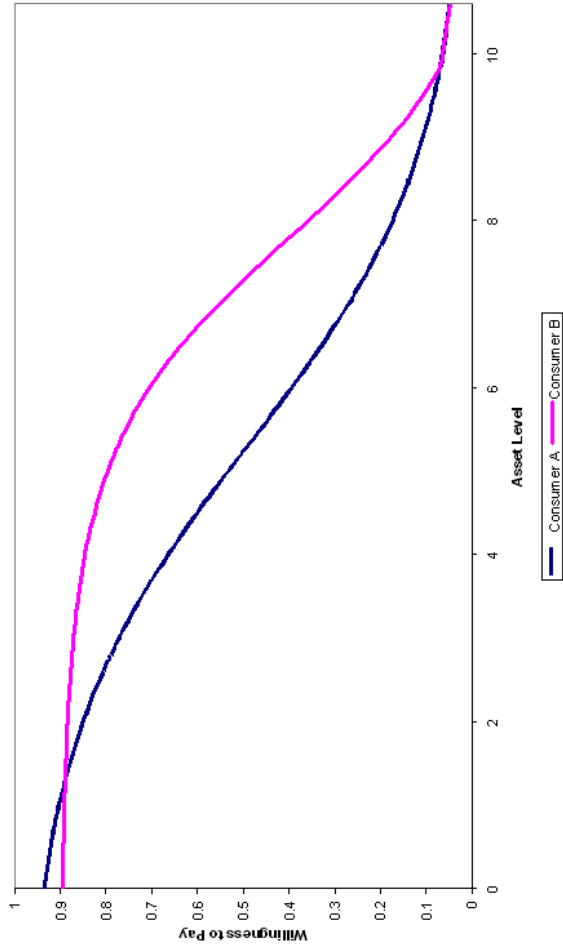
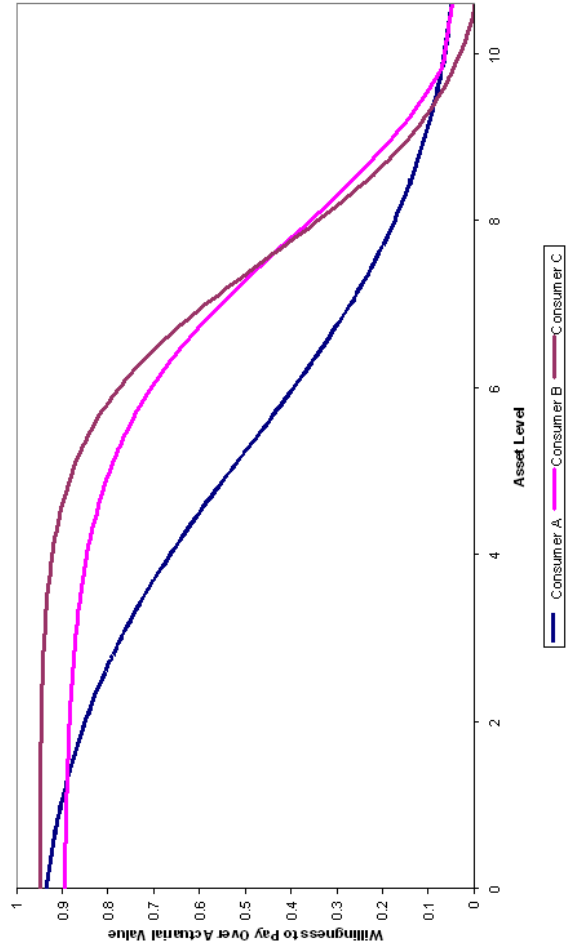


Figure 2
Willingness to Pay for Incremental Asset Units at Different Asset Levels



6 Appendix

For expositional purposes, the main body of the paper 1) sacrifices technical rigor by focusing on solutions in the differentiable regions of the objective function, 2) simplifies by assuming that frictional costs are proportional to assets instead of capital, 3) assumes that insurance risk is unrelated to risks in the securities markets, and 4) omits the proof of Theorem 1. We address each of these issues in turn in the following subsections.

6.1 Analysis of Optima in Non-Differentiable Regions of the Objective Function

As noted earlier, the objective function (4) and constraint (5) are piecewise differentiable with respect to A and I_i ($i = 1, \dots, N$). Hence, the optimality conditions (6) and (7) should technically be rewritten as inequality conditions with right-hand and left-hand derivatives:

$$[I_i]^+ : \frac{\partial^+ V_i}{\partial I_i} - \frac{\partial V_i}{\partial W} \left(\frac{\partial^+ E_i}{\partial I_i} + \sum_{j \neq i} \frac{\partial^+ E_j}{\partial I_i} \right) + \sum_{j \neq i} \frac{\partial^+ V_j}{\partial I_i} \leq 0 \quad (15)$$

$$[I_i]^- : \frac{\partial^- V_i}{\partial I_i} - \frac{\partial V_i}{\partial W} \left(\frac{\partial^- E_i}{\partial I_i} + \sum_{j \neq i} \frac{\partial^- E_j}{\partial I_i} \right) + \sum_{j \neq i} \frac{\partial^- V_j}{\partial I_i} \geq 0 \quad (16)$$

$$[A]^+ : \sum_k \frac{\partial^+ V_k}{\partial A} - \frac{\partial V_i}{\partial W} \left(\sum_k \frac{\partial^+ E_k}{\partial A} + \tau \right) \leq 0 \quad (17)$$

$$[A]^- : \sum_k \frac{\partial^- V_k}{\partial A} - \frac{\partial V_i}{\partial W} \left(\sum_k \frac{\partial^- E_k}{\partial A} + \tau \right) \geq 0 \quad (18)$$

We restrict our attention to pricing functions that are also piecewise differentiable, and start by defining:

$$\Omega_D^0 = \left\{ \mathbf{s} \in \Omega : \sum_{j \in \Gamma(\mathbf{s})} I_j = \mathbf{A} \right\}.$$

Also, using (17) and (18), we have:

$$\frac{\partial V_i}{\partial W} \geq \frac{\sum_k \frac{\partial^+ V_k}{\partial A}}{\sum_k \frac{\partial^+ E_k}{\partial A} + \tau} \quad (19)$$

$$\frac{\partial V_i}{\partial W} \leq \frac{\sum_k \frac{\partial^- V_k}{\partial A}}{\sum_k \frac{\partial^- E_k}{\partial A} + \tau} \quad (20)$$

The marginal condition for the pricing function at the optimum can be written as:

$$\frac{\partial^+ P_i^*}{\partial I_i} = \frac{\partial^+ E_i^Z}{\partial I_i} - \frac{\sum_{j \neq i} \frac{\partial^+ V_j}{\partial I_i} - \sum_{\mathbf{s} \in \Omega_D \cap \Omega^i} \Pr(\mathbf{s}) \left(\frac{\mu_i^{\mathbf{s}} A}{(\sum_{j \in \Gamma(\mathbf{s})} I_j)^2} I_i \right)}{\frac{\partial V_i}{\partial W}},$$

Substituting in from (20) yields

$$\frac{\partial^+ P_i^*}{\partial I_i} \geq \frac{\partial^+ E_i^Z}{\partial I_i} - \frac{\sum_{j \neq i} \frac{\partial^+ V_j}{\partial I_i} - \sum_{\mathbf{s} \in \Omega_D \cap \Omega^i} \Pr(\mathbf{s}) \left(\frac{\mu_i^{\mathbf{s}} A}{(\sum_{j \in \Gamma(\mathbf{s})} I_j)^2} I_i \right)}{\sum_k \frac{\partial^- V_k}{\partial A}} \left(\sum_k \frac{\partial^- E_k}{\partial A} + \tau \right),$$

or

$$\frac{\partial^+ P_i^*}{\partial I_i} \geq \sum_{\mathbf{s} \in \Omega_Z \cap \Omega^i} \Pr(\mathbf{s}) + \phi_i A \left(\sum_{\mathbf{s} \in \Omega_D} \Pr(\mathbf{s}) + \tau \right). \quad (21)$$

A similar analysis using (16) and (19) yields:

$$\frac{\partial^- P_i^*}{\partial I_i} \leq \sum_{\mathbf{s} \in \Omega_Z \cap \Omega^i} \Pr(\mathbf{s}) + \phi_i^0 A \left(\sum_{\mathbf{s} \in \Omega_D \setminus \Omega_D^0} \Pr(\mathbf{s}) + \tau \right). \quad (22)$$

$$\phi_i^0 = \frac{\sum_{\mathbf{s} \in \Omega^i \cap \Omega_D \setminus \Omega_D^0} \Pr(\mathbf{s}) \sum_{k \in \Gamma(\mathbf{s})} \mu_k^{\mathbf{s}} \left(\frac{I_k}{\sum_{j \in \Gamma(\mathbf{s})} I_j} \right) \left(\frac{1}{\sum_{j \in \Gamma(\mathbf{s})} I_j} \right)}{\sum_{\mathbf{s} \in \Omega_D \setminus \Omega_D^0} \Pr(\mathbf{s}) \sum_{k \in \Gamma(\mathbf{s})} \mu_k^{\mathbf{s}} \left(\frac{I_k}{\sum_{j \in \Gamma(\mathbf{s})} I_j} \right)} \quad (23)$$

The right sides of (21) and 22) coincide when $\Omega_D^0 = \emptyset$, implying that a single marginal pricing condition (i.e., (10)) would satisfy both restrictions. When $\Omega_D^0 \neq \emptyset$, however, the conditions restrict the behavior on the pricing function in both directions away from the optimum, with the restrictions referencing the cost allocation rules that would apply on either side of the optimum.

The marginal restriction on the cost sharing function in the neighborhood of the optimum (14), can also be adapted to handle optima located in non-differentiable regions of the objective function or constraints. Specifically, we may write:

$$\frac{\partial^+ g_i(A^*, I_1^*, \dots, I_N^*)}{\partial A} \geq \frac{\frac{\partial^+ V_i}{\partial A}}{\frac{\partial V_i}{\partial W}}$$

and use (19) to obtain:

$$\frac{\partial^+ g_i(A^*, I_1^*, \dots, I_N^*)}{\partial A} \geq \frac{\frac{\partial^+ V_i}{\partial A}}{\sum_k \frac{\partial^+ V_k}{\partial A}} \left(\sum_k \frac{\partial^+ E_k}{\partial A} + \tau \right). \quad (24)$$

Similarly,

$$\frac{\partial^- g_i(A^*, I_1^*, \dots, I_N^*)}{\partial A} \leq \frac{\frac{\partial^- V_i}{\partial A}}{\sum_k \frac{\partial^- V_k}{\partial A}} \left(\sum_k \frac{\partial^- E_k}{\partial A} + \tau \right). \quad (25)$$

As before, the right sides of the inequalities (24) and (25) coincide when $\Omega_D^0 = \emptyset$, which means that a single marginal pricing condition could satisfy both inequalities (i.e., (14) holds). Otherwise, the foregoing inequalities restrict the behavior of the cost sharing function in both directions in the neighborhood of the optimal asset level.

6.2 Allocating Capital rather than Assets

Instead of assuming that frictional costs are proportional to assets, as in (1), we could also assume that frictional costs are proportional to capital—as in (2).

We can then rewrite the social planning problem as:

$$\max_{A, \{I_i\}, \{P_i\}} \sum V_i,$$

subject to:

$$\sum P_i = \sum E_i + \tau(A - \sum E_i). \quad (26)$$

We now revert to the custom of focusing on solutions in smooth regions (as in the main body of the paper):

$$[I_i]: \frac{\partial V_i}{\partial I_i} - \frac{\partial V_i}{\partial W}(1 - \tau) \left(\frac{\partial E_i}{\partial I_i} + \sum_{j \neq i} \frac{\partial E_j}{\partial I_i} \right) + \sum_{j \neq i} \frac{\partial V_j}{\partial I_i} = 0 \quad (27)$$

$$[A]: \sum_k \frac{\partial V_k}{\partial A} - \frac{\partial V_i}{\partial W} \left((1 - \tau) \sum_k \frac{\partial E_k}{\partial A} + \tau \right) = 0 \quad (28)$$

$$[P_i]: \frac{\partial V_i}{\partial W} - \frac{\partial V_j}{\partial W} = 0, \quad (29)$$

Condition (9) becomes:

$$\frac{\partial V_i}{\partial I_i} - \frac{\partial V_i}{\partial W}(1 - \tau) \frac{\partial E_i^Z}{\partial I_i} + \sum_{j \neq i} \frac{\partial V_j}{\partial I_i} = 0, \quad (30)$$

with the marginal pricing condition associated with decentralized implementation being

$$\frac{\partial P_i^*}{\partial I_i} = (1 - \tau) \frac{\partial E_i^Z}{\partial I_i} - \frac{\sum_{j \neq i} \frac{\partial V_j}{\partial I_i} - \sum_{\mathbf{s} \in \Omega_D \cap \Omega^i} \Pr(\mathbf{s}) \left(\mu_i^{\mathbf{s}} \frac{A}{(\sum_{j \in \Gamma(\mathbf{s})} I_j)^2} I_i \right)}{\frac{\partial V_i}{\partial W}},$$

and reducing, after substitution, to:

$$\frac{\partial P_i^*}{\partial I_i} = (1 - \tau) \sum_{\mathbf{s} \in \Omega_Z \cap \Omega^i} \Pr(\mathbf{s}) + \phi_i A \left((1 - \tau) \sum_{\mathbf{s} \in \Omega_D} \Pr(\mathbf{s}) + \tau \right). \quad (31)$$

with ϕ_i defined as before in (11). Adding up follows from:

$$\sum_i \phi_i I_i = 1.$$

Specifically,

$$\begin{aligned} \sum_i \frac{\partial P_i^*}{\partial I_i} I_i &= \sum_i (1 - \tau) \left(\sum_{\mathbf{s} \in \Omega_Z \cap \Omega^i} \Pr(\mathbf{s}) \right) I_i + \sum_i (\phi_i I_i) \left[\left((1 - \tau) \sum_{\mathbf{s} \in \Omega_D} \Pr(\mathbf{s}) + \tau \right) A \right] \\ &= \sum_i E_i + \tau \left(A - \sum_i E_i \right). \end{aligned}$$

Cost sharing functions properties can also be derived on the basis of (2), with the analog of (14) being:

$$\frac{\partial g_i(A^*, I_1^*, \dots, I_N^*)}{\partial A} = \frac{\frac{\partial V_i}{\partial A}}{\sum_k \frac{\partial V_k}{\partial A}} \left((1 - \tau) \sum_{\mathbf{s} \in \Omega_D} \Pr(\mathbf{s}) + \tau \right). \quad (32)$$

6.3 Allocation and Security Market Equilibrium

We use the N person model and start by defining the relevant probability spaces. Recall the set of vectors that define consumer loss experience, denoted by Ω , whose members are row vectors \mathbf{s} of length N , with the vector elements all taking a value of zero or one: $\mathbf{s}(i) = 1$ means that consumer i experienced a loss, while $\mathbf{s}(i) = 0$ means that she did not.

We now introduce a set of M securities, each security with distinct payoffs in X states of the world. Let Ψ be the set of those states (with the associated σ -algebra \mathcal{F}_Ψ), and define state prices, consistent with the absence

of arbitrage, denoted by π_x for each $x \in \Psi$. Let D be an $M \times X$ matrix, with D_{ij} describing the payoff of the i -th security in the j -th state. We assume that

$$\text{span}(D) \equiv \mathbb{R}^X.$$

This condition is typically known as a “complete markets” condition—that any arbitrary menu of state-contingent consumption can be purchased at time zero. In our case, however, it would be misleading to characterize markets as complete, since Ψ does not provide a complete description of the states of the world.

Instead, we characterize the full probability space as $(\Theta, \mathcal{F}_\Theta, \omega_\Theta)$, with

$$\Theta \equiv \{\boldsymbol{\theta} = [\mathbf{s}(1) \ \mathbf{s}(2) \dots \mathbf{s}(N) \ x] \mid \mathbf{s} \in \Omega, x \in \Psi\}$$

The state variable $\boldsymbol{\theta} \in \Theta$ is a row vector of length $N + 1$ that provides a complete description of one possible state of the world. The first N elements of $\boldsymbol{\theta}$ describe which consumers experienced losses (and which ones did not), while the last element describes the state of the securities markets. The entire set Θ contains all possible states of the world.

The following set definitions are useful:

$$\Theta^i = \{\boldsymbol{\theta} : \boldsymbol{\theta}(i) = 1\},$$

the set of all states in which agent i suffers a loss, and

$$\Gamma(\boldsymbol{\theta}) = \{i : \boldsymbol{\theta}(i) = 1\},$$

the set of all agents that lose in state $\boldsymbol{\theta}$. In addition, for every x and every agent i , define:

$$\Upsilon_i^{x0} \equiv \{\boldsymbol{\theta} : \boldsymbol{\theta} \notin \Theta^i, \boldsymbol{\theta}(N + 1) = x\},$$

the set of all states $\boldsymbol{\theta}$ where agent i does not suffer a loss and the security market “sub-state” is x , and

$$\Upsilon_i^{x1} \equiv \{\boldsymbol{\theta} : \boldsymbol{\theta} \in \Theta^i, \boldsymbol{\theta}(N + 1) = x\},$$

the set of all states $\boldsymbol{\theta}$ where agent i *does* suffer a loss and the security market “sub-state” is x . Finally, note that for any i the entire sub-state space defined

by x can be written as:

$$\Upsilon^x \equiv \Upsilon_i^{x1} \cup \Upsilon_i^{x0}$$

We now extend the state prices to define prices for events that are not measurable with respect to \mathcal{F}_Ψ . Define extended state prices as follows. For each $x \in \Psi$

$$\pi^\theta = \pi_x, \forall \theta \in \Theta \mid \theta(N+1) = x$$

Recall that we are extending the state-prices to include the price of claims associated with the hazards being insured. This approach implicitly assumes that there is no variation in “sub-state” prices within the states priced by the security market equilibrium. As noted earlier, the absence of arbitrage does not pin down the state prices for events that are not measurable with respect to \mathcal{F}_Ψ , but this assumption provides a basis for insurance pricing that is logically consistent with the security market equilibrium, with the implicit assumption that the insurance market is “small” in the context of the security market equilibrium.

Consumer utility now depends on wealth purchased in the respective security market states (W_{ix}), consumer utility may now be expressed as:

$$\sum_{x \in \Psi} \left(\sum_{\theta \in \Upsilon_i^{x0}} \Pr(\theta) U_i(W_{ix} - P_i) + \sum_{\theta \in \Upsilon_i^{x1}} \Pr(\theta) U_i(W_{ix} - P_i - L_i + R_i^\theta) \right),$$

This recovery R_i^θ depends now both on insurance loss activity and on the portfolio decision made within the insurance company. To elaborate, the budget constraint of the insurance company may be expressed as:

$$A = \sum_{x \in \Psi} \sum_{\theta \in \Upsilon^x} \pi^\theta k_x A, \quad (33)$$

where we have expressed the contingent consumption purchased in security market state x as the product of the present value of available assets and a state specific factor k_x . Consumer i 's recovery in state θ can thus be expressed as:

$$\begin{aligned}
R_i^\theta &\equiv \min \left\{ I_i, \frac{k_x A}{\sum_{j \in \Gamma(\mathbf{s})} I_j} I_i \right\} \text{ for } \theta \in \Upsilon_i^{x1} \\
R_i^\theta &= 0 \text{ for } \theta \in \Upsilon_i^{x0}
\end{aligned}$$

So our problem becomes

$$\max_{A, \{W_{ix}\}, \{P_i\}, \{k_x\}, \{I_i\}} \left\{ \sum_{x \in \Psi} \left(\sum_{\theta \in \Upsilon_i^{x0}} \Pr(\theta) U_i(W_{ix} - P_i) + \sum_{\theta \in \Upsilon_i^{x1}} \Pr(\theta) U_i(W_{ix} - P_i - L_i + R_i^\theta) \right) \right\},$$

subject to (33) and:

$$\sum_i P_i = \sum_i \sum_{x \in \Psi} \sum_{\theta \in \Upsilon_i^{x1}} \pi^\theta R_i^\theta + \tau A. \quad (34)$$

Before proceeding, we adapt some set notation used earlier to decompose the states of the world into those in which the company defaults (Θ_D) and those in which it is solvent (Θ_Z). Accordingly, we define:

$$\begin{aligned}
\Theta_D &= \left\{ \theta \in \Theta : \sum_{j \in \Gamma(\mathbf{s})} I_j \geq \mathbf{A} \right\} \\
\Theta_Z &= \left\{ \theta \in \Theta : \sum_{j \in \Gamma(\mathbf{s})} I_j < \mathbf{A} \right\}
\end{aligned}$$

and extend the notation to decompose the financial value for each consumer's loss into that associated with states of default and that associated with solvent states:

$$\begin{aligned}
E_i^D &\equiv \sum_{\theta \in \Theta_D} \pi^\theta R_i^\theta, \\
E_i^Z &\equiv \sum_{\theta \in \Theta_Z} \pi^\theta R_i^\theta,
\end{aligned}$$

with:

$$E_i = \sum_{\theta \in \Theta} \pi^\theta R_i^\theta = E_i^D + E_i^Z$$

As before, we sacrifice technical rigor by assuming a solution in a smooth part of the function and obtain the following optimality conditions:

$$[I_i] : \frac{\partial V_i}{\partial I_i} - \frac{\partial V_i}{\partial W} \left(\frac{\partial E_i^Z}{\partial I_i} \right) + \sum_{j \neq i} \frac{\partial V_j}{\partial I_i} = 0 \quad (35)$$

$$[A] : \sum_k \frac{\partial V_k}{\partial A} - \frac{\partial V_i}{\partial W} \left(\sum_k \frac{\partial E_k}{\partial A} + \tau \right) = 0 \quad (36)$$

$$[P_i] : \frac{\partial V_i}{\partial W} - \frac{\partial V_j}{\partial W} = 0, \quad (37)$$

Proceeding as before, we arrive at the marginal pricing condition associated with a decentralized implementation:

$$\begin{aligned} \frac{\partial P_i^*}{\partial I_i} &= \frac{\partial E_i^Z}{\partial I_i} - \frac{\sum_{\theta \in \Theta_D \cap \Gamma_i^x} \Pr(\theta) \sum_{j \in \Gamma(\theta)} \mu_j^\theta \frac{I_j^{kx}}{(\sum_{k \in \Gamma(\theta)} I_k)^2}}{\sum_{\theta \in \Theta_D} \Pr(\theta) \sum_{j \in \Gamma(\theta)} \mu_j^\theta \frac{I_j^{kx}}{(\sum_{k \in \Gamma(\theta)} I_k)}} \left(\sum_{\theta \in \Theta_D} \pi^\theta A + \tau A \right), \\ \frac{\partial P_i^*}{\partial I_i} &= \sum_{\theta \in \Theta_Z \cap \theta^i} \pi^\theta + \phi_i \left(\sum_{\theta \in \Theta_D} \pi^\theta A + \tau A \right) \end{aligned} \quad (38)$$

with

$$\phi_i = \frac{\sum_{\theta \in \Theta_D \cap \Gamma_i^x} \left[\left(\Pr(\theta) \sum_{j \in \Gamma(\theta)} \mu_j^\theta \frac{kx I_j}{(\sum_{k \in \Gamma(\theta)} I_k)} \right) \left(\frac{1}{(\sum_{k \in \Gamma(\theta)} I_k)} \right) \right]}{\sum_{\theta \in \Theta_D} \left(\Pr(\theta) \sum_{j \in \Gamma(\theta)} \mu_j^\theta \frac{kx I_j}{(\sum_{k \in \Gamma(\theta)} I_k)} \right)} \quad (39)$$

The “adding up” property applies. Interestingly, the allocation factor ϕ_i does not directly depend on state prices, but again is driven by the average marginal consumer valuation of asset dollars in the various default states.

6.4 Proof of Theorem 1

Suppose not. Then there is a cost sharing equilibrium

$$\{A^*, I_1^*, \dots, I_N^*; g_1, \dots, g_N; W_1, \dots, W_N\}$$

such that an alternative cost share system and allocation

$$\{\bar{A}, \bar{I}_1, \dots, \bar{I}_N; \bar{g}_1, \dots, \bar{g}_N; W_1, \dots, W_N\}$$

yields:

$$V_i(\bar{A}, W_i - \bar{g}_i(\bar{A}, \bar{I}_1, \dots, \bar{I}_N), \bar{I}_1, \dots, \bar{I}_N) \geq V_i(A^*, W_i - g_i(A^*, I_1^*, \dots, I_N^*), I_1^*, \dots, I_N^*)$$

for all i , with strict inequality for some $j \in 1, \dots, N$. However, by definition of equilibrium,

$$V_j(\bar{A}, W_j - \bar{g}_j(\bar{A}, \bar{I}_1, \dots, \bar{I}_N), \bar{I}_1, \dots, \bar{I}_N) \leq V_j(A^*, W_j - g_j(A^*, I_1^*, \dots, I_N^*), I_1^*, \dots, I_N^*)$$

It follows that:

$$\bar{g}_j(\bar{A}, \bar{I}_1, \dots, \bar{I}_N), \bar{I}_1, \dots, \bar{I}_N < g_j(\bar{A}, \bar{I}_1, \dots, \bar{I}_N), \bar{I}_1, \dots, \bar{I}_N$$

and the definition of a cost share system implies that:

$$\bar{g}_k(\bar{A}, \bar{I}_1, \dots, \bar{I}_N), \bar{I}_1, \dots, \bar{I}_N > g_k(\bar{A}, \bar{I}_1, \dots, \bar{I}_N), \bar{I}_1, \dots, \bar{I}_N$$

for some $k \neq j$. But this implies further that:

$$V_k(\bar{A}, W_k - \bar{g}_k(\bar{A}, \bar{I}_1, \dots, \bar{I}_N), \bar{I}_1, \dots, \bar{I}_N) < V_k(\bar{A}, W_k - g_k(\bar{A}, \bar{I}_1, \dots, \bar{I}_N), \bar{I}_1, \dots, \bar{I}_N)$$

and this leads to:

$$V_k(\bar{A}, W_k - \bar{g}_k(\bar{A}, \bar{I}_1, \dots, \bar{I}_N), \bar{I}_1, \dots, \bar{I}_N) < V_k(A^*, W_k - g_k(A^*, I_1^*, \dots, I_N^*), I_1^*, \dots, I_N^*),$$

a contradiction. Q.E.D.

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