CONVERGENCE OF CAPITAL AND INSURANCE MARKETS: 
CONSISTENT PRICING OF INDEX-LINKED CATASTROPHIC LOSS INSTRUMENTS

Nadine Gatzert, Sebastian Pokutta, Nikolai Vogl

This version: February 26, 2014

ABSTRACT

Index-linked catastrophic loss instruments have become increasingly attractive for investors and play an important role in risk management. Their payout is tied to the development of an underlying industry loss index (reflecting losses from natural catastrophes) and may additionally depend on the ceding company’s loss. Pricing is currently not transparent and does not assume a liquid market. We show how arbitrage-free and market-consistent prices for such instruments can be derived by overcoming the crucial point of tradability of the underlying processes. We develop suitable approximation and replication techniques and – based on these – provide explicit pricing formulas using stock and cat bond prices.

Keywords: Alternative risk transfer; index-linked catastrophic loss instruments; industry loss warranties; pricing approaches; risk-neutral valuation; cat bonds.

JEL Classification: G13, G22

1. INTRODUCTION

Alternative risk transfer (ART) has become increasingly relevant in recent years for insurers and investors, especially due to a considerably growing risk of extreme losses from natural catastrophes caused by value concentration and climate change, as well as the limited (and volatile) capacity of traditional reinsurance markets (Cummins, Doherty and Lo (2002); Ibragimov, Jaffee and Walden (2009)). In this context, ART intends to provide additional (re)insurance coverage by transferring insurance risks to the capital market. This offers

---

* Nadine Gatzert and Nikolai Vogl are at the Friedrich-Alexander-University (FAU) of Erlangen-Nuremberg, Department of Insurance Economics and Risk Management, Lange Gasse 20, 90403 Nuremberg, Germany, Tel.: +49 911 5302884, nadine.gatzert@fau.de, nikolai.vogl@fau.de. Sebastian Pokutta is at the Georgia Institute of Technology, Department of Industrial and Systems Engineering (ISyE), 765 Ferst Dr, Atlanta, GA 30332, USA, Tel.: +1 404 385 7308, sebastian.pokutta@isye.gatech.edu.

1 The volume of outstanding cat bonds, for instance, reached $17.5bn in 2013 (see AON (2013)). Investors include, e.g., specialized funds, institutional investors, mutual funds, and hedge funds (see AON (2013)).
considerably higher capacities and can thus help satisfy the high demand as well as reduce the market power of reinsurance companies (Froot, 2001). Moreover, ART could fix the problem of “nondiversification traps” in the catastrophe insurance markets by offering new risk transfer opportunities besides the pooling of risks in the reinsurance market as described in Ibragimov, Jaffee and Walden (2009). Among the most commonly used ART instruments are index-linked catastrophic loss instruments such as index-based cat bonds\(^2\) or industry loss warranties (ILWs), for instance, whose defining feature is their dependence on an industry loss index and which may also depend on the company-specific loss resulting from a natural catastrophe.\(^3\) However, the current degree of liquidity of the various index-linked instruments considerably differs. While the market for cat bonds is fairly well developed with an increasingly relevant secondary market (Albertini, 2009), for instance, the (secondary) market for ILWs is less liquid and limited (Elementum Advisors, 2010). In this paper, we focus on how these products are priced in a consistent way and discuss under which assumptions (e.g., regarding a liquid underlying market) risk-neutral valuation can be used. This procedure can considerably simplify pricing and enhance transparency, making the market as a whole more efficient. In addition, risk-neutral valuation is of great relevance for the inclusion of such instruments in enterprise risk management strategies as it provides a mark-to-market valuation approach, allowing for partial hedging, versus the traditional mark-to-model approaches with the associated model risk (which is very hard to quantify). We develop new pricing approaches by means of approximations and replication techniques and apply them to industry loss warranties (ILWs) as a representative of index-linked catastrophic loss instruments under the assumption of a liquid cat bond and stock market, while carefully addressing the necessary prerequisites and limitations. We study binary ILWs, whose payout depends on the index only, and indemnity-based ILWs, where the payout depends on both the industry index and the individual company losses, thus representing a double-trigger product. The approaches derived in this paper can also be transferred to the consistent pricing of other index-linked catastrophic loss instruments.

In the literature, several papers examine the impact of the actuarial and financial pricing of index-linked catastrophic loss instruments and especially ILWs (see, e.g., Gatzert and Schmeiser (2012), Gatzert, Schmeiser and Toplek (2011), or Ishaq (2005)), but the underlying assumptions of risk-neutral pricing have not been discussed in detail so far in this context. However, a few papers have already dealt with risk-neutral valuation in the context of cat bonds (see, e.g., Nowak and Romaniuk (2013) and Haslip and Kaishev (2010)) and compute explicit pricing formulas. The main assumption is the tradability of an underlying process.

\(^2\) There are various versions of cat bonds with different types of triggers, including indemnity-based and non-indemnity based triggers with parametric, modelled loss, and industry loss triggers, for instance (see, e.g., Hagedorn et al. (2009)).

\(^3\) See, e.g., Cummins and Weiss (2009) and Barrieu and Albertini (2009) for an overview of the ART market.
Since the underlying process can usually not be traded directly like a stock, one has to assume there is a liquid market for (certain) derivatives. We derive a general approach for dealing with this issue, describe the underlying assumptions and apply this approach to binary and indemnity-based ILWs. This is done by means of direct or approximate replication with traded derivatives using cat bonds, which leads to explicit and consistent prices. In the case of instruments that are only index-linked (i.e., binary / non-indemnity-based), one major advantage is that no assumptions concerning the distribution of the underlying industry loss index are necessary. In the case of indemnity-based index-linked catastrophic loss instruments, whose payout in addition to the industry loss index also depends on the company-specific loss caused by a catastrophe, one major problem is the behavior of the company loss, which should be tradable in some weak sense when using risk-neutral valuation. We thus propose three different approaches for solving this issue and finally discuss problems associated with them. We assume 1) a functional (monotonic increasing) relationship between the industry loss index and the company loss motivated by the typically high degree of dependence in order to reduce basis risk, thereby showing how pricing can again be conducted in the same way as in the case of non-indemnity-based index-linked catastrophic loss instruments, 2) independence between company loss and industry loss index (included for illustration), or 3) that the company loss is reflected in the stock price. The latter, for instance, results in structured risk management products, which are studied by Cox, Fairchild and Pedersen (2004). Thus, several of these three different approaches lead to situations that have been treated before (case 2), but the others (cases 1 and 3), to the best of the authors’ knowledge, are studied here the first time in this context. Furthermore, we examine the adequacy of our assumptions and indicate in which cases they can be used.

Using ILWs as an example, we assume the existence of a liquid cat bond market to handle the tradability of the industry loss index and to apply arbitrage-free valuation. Since there is a growing secondary market for cat bonds, this assumption appears to be at least appropriate in the foreseeable future (see, e.g., Albertini (2009) for a description of the secondary market). Moreover, we show that liquidity assumptions are not needed to the same extent as in classical option pricing theory because continuous trading is not necessary to replicate ILWs when using cat bonds, i.e., a static hedging approach is sufficient. Therefore, the liquidity requirement is reduced to the availability of suitable cat bonds at the time of replication. In a first step, we derive prices for binary / non-indemnity-based ILWs, where the payout only depends on the industry loss index exceeding a contractually defined trigger level during the contract term. If a suitable cat bond is not available for deriving ILW prices, we provide

---

4 The prices of available index-linked catastrophic loss instruments such as ILWs should be consistent with the prices of other derivatives traded on an already liquid market such as in the case of cat bonds. To ensure this consistency, the prices of ILWs should equal the prices of replicating portfolios consisting of tradable derivatives (cat bonds).
proper approximations under some additional assumptions. Second, when calculating prices for *indemnity-based* ILWs, we apply the three approaches presented above and provide explicit pricing formulas.

The presented approaches for pricing index-linked catastrophic loss instruments are of high relevance today and especially for the future (both for practical as well as academic endeavors), when index-linked catastrophic loss hedging instruments will become even more widespread than today and when some markets for derivatives like the cat bond market are truly liquid. One main contribution of our work is to overcome the crucial point of the tradability of the company loss through suitable approximations and to provide explicit pricing formulas using replication techniques. In a next step, this can be used to empirically study and compare ILW prices with the ones derived based on the replication and approximation techniques using available cat bond prices as presented in this paper.

The remainder of the paper is structured as follows. Section 2 gives an overview of related literature with a focus on the underlying theory and assumptions. Section 3 introduces index-linked catastrophic loss instruments and ILWs as representatives of index-linked catastrophic loss hedging instruments, their basic properties and the underlying industry loss indices. In Section 4 we present our arbitrage-free pricing framework and as an example apply it to ILWs, while Section 5 concludes.

### 2. Literature Review

There are several papers which deal with the pricing of index-linked catastrophic loss instruments such as ILWs or strongly related products. Actuarial pricing principles are applied in Gatzert, Schmeiser and Toplek (2011) or Gatzert and Schmeiser (2012). Gründl and Schmeiser (2002) compare actuarial pricing approaches with the capital asset pricing model to calculate prices for double-trigger reinsurance contracts, which are similar to indemnity-based ILWs. Furthermore, there are papers that combine financial and actuarial pricing approaches such as Møller (2002, 2003), who discusses the valuation and hedging of insurance products that depend on both the financial market and insurance claims. Regarding the arbitrage-free pricing of ART instruments related to index-linked catastrophic loss instruments, there is a wide literature, which is outlined in the following.

The pricing of options written on the average level of a Markov process and its application to the pricing of catastrophe options is studied by Bakshi and Madan (2002). They derive explicit pricing formulas for loss distributions with a low probability of high losses. Haslip and Kaishev (2010) also price reinsurance contracts, but with a specific focus on catastrophe
losses. They assume a liquid market of indemnity-based cat bonds and that the aggregated loss process of a company follows a compound Poisson process, and then calculate arbitrage-free prices for the excess of loss reinsurance contracts using Fourier transformation. The setting is similar to ours, as we compute prices for indemnity-based ILWs, which can be treated as a securitized excess of loss reinsurance contracts with an additional trigger. In contrast with their setting, we only assume a liquid cat bond market with comparable index-based cat bonds and weaken the distribution assumptions. Prices for catastrophe equity puts (CatEPuts), which are double-trigger contracts that can only be exercised if the insured loss rises above a certain level, are derived in Jaimungal and Wang (2006), who extend the work of Cox, Fairchild and Pedersen (2004) by allowing a non-constant interest rate and non-constant losses for every occurring catastrophe. The authors derive an explicit formula and the Greeks for the price of the option using standard arbitrage-free option pricing theory and based on the Merton (1976) approach. This assumes the insured loss follows a compound Poisson process, the stock price process follows a geometric Brownian motion between the jumps of the compound Poisson process, and that the influence of the insured loss on the stock price process is relative, i.e., the absolute loss of the stock price depends on its value before a jump happens. In this paper, we provide one approximation of the company loss through the stock price process leading to a situation similar to that in Jaimungal and Wang (2006) and Cox, Fairchild and Pedersen (2004) and discuss problems that might arise from using such structured risk management products.

Integral to one of our pricing approaches for ILWs is the valuation of cat bonds. Bantwal and Kunreuther (2000) was one of the first papers to observe that the spreads of cat bonds can hardly be reconciled with standard investor preferences. Furthermore, due to jumps induced by natural catastrophes, the market of cat bonds is generally incomplete (see, e.g., Merton (1976)). To deal with incompleteness, one approach is offered by Merton (1976), who argues that the risk of jumps can be completely neutralized by diversifying them away, as jumps are of an idiosyncratic nature. Under this “risk-neutralized” assumption, one can employ arbitrage-free pricing.\(^5\) In Cox and Pedersen (2000) and Cox, Fairchild and Pedersen (2004), the pricing of catastrophe bonds has also been discussed. The latter was the first work that dealt with the dynamics and interactions of losses and share value. In Poncet and Vaugirard (2002), a pricing approach is introduced that uses stochastic interest rates and a diffusion process for the industry loss index without jump risk from catastrophes, while in Vaugirard (2003a) jumps are accounted for by means of a jump diffusion process.\(^6\)

---

\(^5\) Alternatively, an equilibrium model can be employed. See also Cox and Pedersen (2000) for a brief discussion on the relation of the two frameworks.

\(^6\) Other pricing approaches are based on equilibrium models (see Dieckmann (2010)) or employ the pricing techniques of credit risk by means of probability-of-default and loss-given-default, which translate to probability-of-catastrophe and loss-given-catastrophe (see Jarrow (2010)). Moreover, Dieckmann (2010)
3. **Modeling Index-Linked Catastrophic Loss Instruments**

One main property of index-linked instruments such as ILWs is that they depend on an industry loss index, which generally does not instantaneously reflect the exact insured damage. Instead, if a catastrophe occurs, the first value of the index is a preliminary estimation (see, e.g., Kerney (2013)), which is adjusted if necessary or at predetermined points in time (see PCS (2013) and PERILS (2013)). To take into account this development, we follow the approach of Vaugirard (2003a, 2003b) and assume the actual (contractually defined) maturity $T'$ of the index-linked catastrophic loss instruments to exceed the *risk exposure period* $T$, during which the catastrophe must occur in order to trigger a payoff. Newly occurring catastrophes between the end of the risk exposure period $T$ and the maturity $T'$ are not taken into account, but adjustments of the loss estimates from the catastrophes that occurred during the risk exposure period (between time 0 and $T$) are taken into account and are reflected in the index. Hence, while the time interval from 0 to $T$ represents the *risk exposure period*, the time interval from $T$ to $T'$ is referred to as the *development period*. Thus, an industry loss index incorporates data from insured losses arising from an occurring catastrophe and each qualifying event is reflected in the index.

In the following, we consider an aggregated industry loss index $I^t$ for the risk exposure period until time $T$ by aggregating estimated losses from all catastrophes (e.g., of the same and contractually defined type) that occurred during the risk exposure period.\(^7\) Hence, following, e.g., Biagini, Bregman and Meyer-Brandis (2008), the aggregated industry loss index $I^T$ for the risk exposure period until $T$, taking into account the development of the estimations until time $t \leq T'$, is given by

\[ I^t_T = \sum_{i=1}^{N_{t,T}} X^i_t, \]

where $N_{t,T}$ (with $t \wedge T = \min(t, T)$) is the number of catastrophes that occurred up to time $t$ within the risk exposure period until $T$ and $X^i_t$ denotes the time-dependent estimation of the insured loss arising due to the $i$-th catastrophe at time $t$. In the following, we focus on index-linked catastrophic loss instruments, whose payoffs depend on the industry loss index at maturity $T'$ and which can be defined by

---

\(^7\) This loss information is provided by specialized industry loss index providers. Note that this also depends on the type of index-linked catastrophic loss instruments. E.g., if dealing with occurrence ILWs (see Ishaq (2005)), whose payoff does not depend on the sum of insured damage but only on the loss caused by the first event), one needs a different approach, but our techniques are applicable as well.
where \( f \) is a non-negative measurable function. There are also *indemnity-based* index-linked catastrophic loss instruments, which additionally depend on the company loss. The payoff at time \( T' \) of such indemnity-based index-linked catastrophic loss instrument is analogously given by

\[
\begin{align*}
P^T_{T'} &= f \left( I^T_{T'}, L^T_{T'} \right),
\end{align*}
\]

where \( f \) is a non-negative measurable function and \( L^T \) denotes (analogously to \( I^T \)) the estimated aggregated company loss for the risk exposure period until \( T \). In Figure 1, we depict the development of an industry loss index for a single catastrophe. After the catastrophe occurred, the loss estimations are adjusted several times, and additionally occurring catastrophes during the risk exposure period would cause additional jumps. The solid dot on the right side corresponds to the value of the industry loss index at maturity and hence the payoff is given by applying the function \( f \) to this value. Note that while downward adjustments of the estimation are possible in principle, they are unlikely and adjustments are typically upward.\(^8\)

*Figure 1:* Exemplary development of an industry loss index in the case of one catastrophe and adjustments of loss estimations over time

We assume that \( I^T \) is given by an industry loss index provider such as, e.g., the Property Claim Service (PCS) index for the U.S. or the PERILS index for Europe. It represents an estimation of the total insured loss caused by a catastrophe. The differences between the two indices are summarized in Table 1.

---

\(^8\) See, e.g., McDonnell (2002) for the development of the Property Claim Service estimations of the ten U.S. natural catastrophes with the highest insured damage.
Table 1: Comparison of PCS and PERILS industry loss index

<table>
<thead>
<tr>
<th>Event considered “catastrophe” if insured</th>
<th>PCS</th>
<th>PERILS</th>
</tr>
</thead>
<tbody>
<tr>
<td>loss exceeds</td>
<td>$25 million</td>
<td>€200 million</td>
</tr>
<tr>
<td>Data update boundary</td>
<td>No</td>
<td>36 months after event</td>
</tr>
<tr>
<td>Perils</td>
<td>All essential</td>
<td>Windstorm Europe; Flood</td>
</tr>
<tr>
<td>U.S. perils</td>
<td>U.K.</td>
<td></td>
</tr>
</tbody>
</table>

PCS provides insured property loss and defines a catastrophe as an event that causes at least $25 million in direct insured property losses and affects a significant number of policyholders and insurers. The PCS index covers perils including earthquake, fire, hail, hurricane, terrorism, utility service disruption and winter storm. The loss data is updated approximately every 60 days, if the event caused more than $250 million in insured property losses, until PCS believes the estimated loss reflects the insured loss for the industry (see PCS (2013) and Kerney (2013)). PERILS defines a catastrophe as an event exceeding a total loss of €200 million and estimates only property windstorm losses in Europe and property flood losses in the U.K. It publishes the first index value report six weeks after the event at the latest and updates after three, six and twelve months. More updates are provided if necessary but the reporting is final in any case after 36 months (see PERILS (2013)). It is important to keep in mind that these indices do not incorporate information instantaneously but with a certain delay; however under the assumption of sufficient liquidity in the derivatives market, we can assume that information is incorporated almost instantaneously (similar to the CDS market for credit defaults) into the derivatives prices (e.g., cat bonds that play a crucial role here).

One main problem related to the use of an index-linked catastrophic loss instrument in the context of hedging is basis risk (see, e.g., Gatzerl and Schmeiser (2012)), which arises if the dependence between the index and the company’s losses (that are to be hedged) is not sufficiently high. In particular, the company loss could be high, but the industry loss index could be too low to trigger a payoff. There are several definitions of basis risk and we refer to Cummins, Lalonde and Phillips (2004) for a discussion of basis risk associated with index-linked catastrophic loss instruments.
4. Pricing Index-Linked Catastrophic Loss Instruments

4.1 Pricing under no-arbitrage

In general, actuarial valuation methods for insurance contracts are based on the expected loss plus a specific loading, which depends, e.g., on the insurers’ risk aversion and its already existing portfolio (see Gatzert and Schmeiser (2012)). Financial pricing approaches generally allow the derivation of market-consistent prices, i.e., the price generally does not depend on individual risk preferences. For arbitrage-free valuation, one has to calculate the expected value under a certain risk-neutral measure $Q$ given the existence of a liquid market. If there is a unique risk-neutral measure (i.e., the market is complete) the arbitrage-free price is unique and the price corresponds to the initial value of a self-financing portfolio replicating the cash flow.

*Pricing by replication for non-indemnity-based instruments*

In the present setting, we focus on the two stochastic processes of importance: the industry loss index $I^T$ and the company loss $L^T$. These processes need to be tradable in a more general sense, as there is no possibility to trade $I^T$ or $L^T$ directly like a stock and it is unrealistic to assume that there is a possibility to buy or sell this index, but we assume there is a liquid market for certain derivatives on the industry loss index $I^T$. To avoid arbitrage opportunities, the prices of these derivatives have to coincide with the expectation of the discounted cash flow under a risk-neutral measure, and the prices of other index-linked catastrophic loss instruments should be consistent with the prices of the derivatives already traded on a liquid market. To ensure this consistency, the prices of these other instruments should equal the expectations under the same risk-neutral measure, or, equivalently, should equal the prices of replicating portfolios consisting of tradable derivatives, for instance.

We thus assume that the liquid market for certain index-linked catastrophic loss instruments (here: cat bonds) is at least large enough to allow the derivation of an exact replicating portfolio or at least a close approximation. For instance, in case suitable cat bonds are

---

9 These derivatives also represent index-linked catastrophic loss instruments. Note, however, that the assumption of liquidity only holds for certain types of derivatives, as standard call and put options on an industry loss index are not traded on a liquid market. Call and Put options were introduced 1995 by CBOT, but due to limited trading these options were delisted in 2000 (see, e.g., Cummins and Weiss (2009)). Hence, the derivatives we focus on in the following are cat bonds, which exhibit a considerable market volume with a relevant secondary market. In case that there is a liquid derivatives market in the sense that calls and puts on the underlying industry loss index are traded at all strikes, the Litzenberg formula could be applied to for static replication.
available for perfectly replicating the respective instrument’s cash flows (here: ILW), a unique arbitrage-free price for the considered index-linked catastrophic loss instrument (here: ILW) can be derived.\textsuperscript{10} Alternatively, in case the required cat bonds are not available (e.g., mismatching trigger level or time to maturity), we derive suitable approximations for the replicating portfolio.

In case of \textit{indemnity-based} index-linked catastrophic loss instruments, one crucial point and major problem in addition to the treatment of the industry loss index is the tradability of the company loss $L^T$. As a solution, we suggest three approaches.

\textit{Pricing indemnity-based instruments: Independence between company loss and industry loss index}

First, in case $I^T$ and $L^T$ are independent, one can use the approach in Møller (2002, 2003) and calculate prices, using risk-neutral valuation for the part concerning the industry loss index and an actuarial valuation approach for the part concerning the company loss. However, as independence between the index and the company loss would imply a substantial degree of basis risk, this assumption is not realistic and, moreover, would defy the risk management purpose of an indemnity-based index-linked catastrophic loss instrument.

\textit{Pricing indemnity-based instruments: Approximating the company loss based on the stock price loss}

Second, one could approximate the company loss $L^T$ with the loss of the stock price, which in turn is assumed to be traded, thus allowing arbitrage-free valuation. As a result, one obtains a structured risk management product as considered in Cox, Fairchild and Pedersen (2004). We assume a company with traded stock and approximate the individual company loss $L^T$ through the loss of market value by means of its stock price, e.g.,

$$L^T_t = f\left(\left(S_s\right)_{t \geq s \geq 0}\right)$$

for some function $f$ satisfying mild regularity conditions (e.g., such that expected values can be derived), where the entire path of the stock price $0 \leq s \leq t$ is considered (e.g., in order to sum up all jumps between 0 and $t$). We thereby assume that the market estimates the true and entire industry loss caused by a catastrophe instantly, as reflected by the function $f$, such that

\textsuperscript{10} The instruments we focus on are not path-dependent and there could be various measures resulting in the same prices for instruments with a payoff depending only on $I^T_t$. Since there is no difference for the prices of the instruments we consider, it is not relevant which of these measures we take.
there is no time lag for receiving loss data, which can be considered somewhat similar to the CDS market, where beliefs are also reflected in the price. The function $f$ provides the company loss $L_t^T$ by just observing the stock price and, therefore, since the stock is tradable on a liquid market, one can revert to arbitrage-free pricing methods for the company loss. For instance, one can assume that the only events causing a jump of the stock price are catastrophic events. In this case, the function $f$ should sum up all jumps until time $t$.

Using this approximation, then, essentially leads to a situation as considered in Cox, Fairchild and Pedersen (2004). However, there are several problems related to the use of such structured risk management products. A joint-stock company is needed and there is a risk of moral hazard, as companies may even try to publish negative information or overestimated company loss data (which could be adjusted after the maturity of the instrument) if the issued product is triggered to make it more valuable. Furthermore, one fundamental problem is that the stock price of the company will decline in a less pronounced way than in cases without hedging. Thus, the stock price loss would not reflect the actual loss resulting from the catastrophe. Hanke and Pötzelberger (2003) formally illustrate this issue for the case of arbitrage-free prices of options on a company’s own stock. They show that the resulting prices differ from prices obtained through classical option pricing theory and that ignoring this effect implies arbitrage opportunities.

**Pricing indemnity-based instruments: Exploiting the dependence: functional relationship between $I$ and $L$**

Third, one can assume a functional relationship between the industry loss index and the company loss motivated by the typically high degree of dependence between these processes.\(^{11}\) This approach reduces indemnity-based index-linked catastrophic loss instruments to non-indemnity-based instruments and allows applying the replication techniques mentioned at the beginning of this section. A high degree of dependence is realistic, since basis risk arises if the company loss and the industry loss are not fully dependent (see, e.g., Harrington and Niehaus (1999), Doherty and Richter (2002), Zeng (2000) and Gatzert and Kellner (2011)), which is why index-linked catastrophic loss instruments are typically purchased in case of a sufficient degree of dependence. For instance, Cummins, Lalonde and Phillips (2004) show that 36% of Florida hurricane insurers could effectively use instruments based on a statewide index without being exposed to a high degree of basis risk. Following these arguments, we assume a functional dependence between $L_t^T$ and $I_t^T$, i.e.,

\(^{11}\) Alternatively, one can assume stochastic dependence to reflect the case where the company is not affected by a catastrophe.
\[ L^T_T = g \left( I^T_T \right), \]  

where we assume \( g \) to be a function satisfying the following rational properties:

- If \( I^T_T \) increases, the increase in \( L^T_T \) at most amounts to the increment of \( I^T_T \), i.e., \( g \) is Lipschitz continuous with Lipschitz constant less or equal 1 (one can think of a differentiable function with \( |g'| \leq 1 \)).
- If \( I^T_T \) increases, \( L^T_T \) also increases, i.e., \( g \) is strictly monotone increasing and hence invertible.
- If there is no industry loss, the insurer’s aggregated loss is also zero, i.e., \( g(0) = 0 \).

Using Equations (1) and (2) we observe that the payoff can be described as

\[ P^T_{I^T_T} = f \left( I^T_T, L^T_T \right) = f \left( g \left( I^T_T \right) \right), \]

which leads to an analogous situation as in the case of non-indemnity-based instruments, since the payoff only depends on the industry loss index at maturity. The price can again be calculated as the expectation of the discounted cash flow under the risk-neutral measure \( Q \) or is given by the initial price of a replicating portfolio.

### 4.2 Industry Loss Warranties

We apply the proposed approaches to price binary and indemnity-based ILWs, whose payoff depends on whether the industry loss index exceeds the trigger level \( Y \) at the end of the development period \( T' \) (see McDonnell (2002)). The payoff at maturity \( T' \) of a binary ILW (denoted “b”) with trigger level \( Y \) and risk exposure period until \( T \) is given by

\[ P^b_{I^T_T,Y,T'} = D \cdot 1_{\{I^T_T > Y\}}, \]  

where \( D \) represents the possible payout and \( 1_{\{I^T_T > Y\}} \) denotes the indicator function, which is equal to 1 if \( I^T_T > Y \), i.e., if the industry loss index at maturity \( I^T_T \) exceeds the trigger level \( Y \) (due to catastrophes that occurred during the risk exposure time \( T \)), and 0 otherwise.\(^{12}\)

The payoff at time \( T' \) of an indemnity-based ILW with attachment point \( A \), maximum payoff \( M \), trigger level \( Y \), risk exposure period until \( T \) and maturing at time \( T' \) additionally depends

\(^{12}\) Note that an alternative but less common representation of an ILW would be that the payoff is triggered if the index exceeds the trigger limit during the contract term (knock-in barrier options). The following analysis can be extended to this case as well.
on the (estimated) aggregated company loss for the risk exposure period until $T$ denoted by $L^T$

$$P_{TILW,T}^{A,M,Y} = \min\left(M, (L^T - A)\right) \mathbf{1}_{[L^T > A]} = \left((L^T - A) - (L^T - (A + M))\right) \mathbf{1}_{[L^T > A]},$$

(4)

where $(\cdot)_+ = \max(\cdot, 0)$. From Equations (3) and (4), one can see the option-like structure of ILWs. A binary ILW is a binary call on the industry loss index $I_T$, and an indemnity-based ILW corresponds to a call spread option on the company loss $L^T$ with an additional trigger for the industry loss.

Under the assumption of a liquid derivatives market, in the case of binary ILWs, the option character of ILWs can be used to directly compute prices, i.e., in case the distribution (under the risk-neutral measure $Q$) of the industry loss index is known, the price of a binary ILW is given through the expected value of the discounted cash flow under $Q$, i.e., the discounted (risk-neutral) probability of the index exceeding the trigger level. Alternatively, one can directly replicate the cash flow of the ILW with available derivatives. For example, in case there is a liquid market for binary options on the industry loss index $I_T$, one can directly observe prices at the market as binary ILWs are itself binary options. Alternatively, if there is a liquid cat bond market, in the following we illustrate how to derive a replicating portfolio that can be used to derive market-consistent prices. In case fitting cat bonds are not available, we propose approximation techniques to derive a replicating portfolio as described in detail in the following sections. These techniques can also be applied for pricing indemnity-based ILWs or other index-linked catastrophic loss instruments.

To illustrate our method more specifically, we assume that there is a liquid cat bond market consisting of cat bonds (with comparable maturities and strikes etc.) that can be used to derive consistent ILW prices by means of replicating the ILWs’ cash flows. This extends the approach of Haslip and Kaishev (2010), who also assume a liquid cat bond market and then evaluate excess of loss reinsurance contracts under additional assumptions concerning the risk-neutral measure $Q$. The assumption of a liquid cat bond market can be considered reasonably realistic given the high volume of cat bonds (more than $17$ billion of capital outstanding in $2013$ (see AON (2013)) and a growing secondary market with a considerable

\[13\]

In general, in case the market is incomplete and ILWs cannot be fully replicated, under suitable assumptions it can be still possible to at least approximately (dynamically) replicate the ILWs (see, e.g., Bertsimas, Kogan and Lo (2001), Brunel (2002), Xu (2006), where arbitrage-free price windows are derived by means of sub- and super-hedges). With the derivation of sub- and super-hedges, one obtains a price interval in which any arbitrage-free price of the ILW has to be contained. Within the interval, the chosen price will depend on the risk preference of the investor and might still be subject to inefficiencies.
volume of cat bond transactions (see Moody (2013)), even though it still does not provide the
same degree of liquidity as the stock market. However, we show that liquidity is not needed to
the same extent as in classical option pricing as continuous trading is not necessary to
replicate (European-style) ILWs when using cat bonds, i.e., a static hedging approach is
sufficient. Moreover, the Bermuda Stock Exchange, among others (see Artemis (2013)), has
started to list cat bonds, which is a first step to trading them on the exchange. Thus, the
secondary market is growing and even if the market is not yet enough liquid, it appears
reasonable to assume that it will become liquid in the foreseeable future against the
background of the current developments.

The approach using available cat bonds in order to (at least approximately) replicate cash
flows can also be used for pricing other index-linked catastrophic loss instruments covering a
wide variety of instruments. Hence, once the replicating portfolio is known, prices can easily
be calculated as they can directly be observed on the market. In addition, this approach can be
used to test the degree of liquidity in the market. Consistent prices would encourage the
assumption of a liquid market, while inconsistent prices would contradict this assumption and
lead to an arbitrage opportunity or indicate the existence of an external risk that might not
have been incorporated into market prices.

4.3 Pricing binary ILWs by replication using cat bonds

In the following, we present different ways of replicating binary ILWs’ cash flows. In case of
a liquid cat bond market, the whole market is represented by the filtered probability space
\( \left( \Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, P \right) \), where \( (\mathcal{F}_t)_{t \geq 0} \) is a filtration satisfying the usual assumptions and \( \mathcal{F}_t \)
represents the whole information up to time \( t \). Following the fundamental theorem of asset
pricing (see, e.g., Delbaen and Schachermayer (1994)) the price of every contingent claim is
given by the expectation of the discounted payoff under an equivalent martingale measure \( Q \).

The discounted payoff (at time 0) of a typical index-based cat bond with trigger level \( Y \), risk
exposure period until \( T \), coupon payment \( c \) and maturing at time \( T' \) and without loss of
generality an assumed nominal of 1 is given by

\[
P_{cat,Y}^{c,T} = \sum_{j=1}^{n} c \cdot e^{-rT'} 1_{[\frac{j-1}{n},\frac{j}{n})} + e^{-rT} 1_{[\frac{n}{n},\infty)}.
\]

where \( r \) is the constant risk-free interest rate (this assumption can be weakened) and \( n \) is the
number of coupon payments (coupon time intervals). In the following, we first assume that
there is a perfectly fitting cat bond, i.e., a zero coupon \( (c=0) \) cat bond with the same trigger
level \( Y \), the same risk exposure period until \( T \) and maturing at the same time \( T' \), and then extend our formula to imperfectly fitting cat bonds under certain additional assumptions.

**Matching maturity and trigger level**

First, if there are binary zero coupon cat bonds with the same maturity and the same trigger level, we can proceed as follows. According to Equation (5), the risk-neutral price at time \( t \) of a binary (denoted “\( b \)”) zero coupon (denoted “\( 0 \)”) cat bond with trigger level \( Y \), risk exposure period until \( T \), maturing at time \( T' \) is (see Haslip and Kaishev (2010))

\[
V_{cat,T}^{b,0,Y,T}(t) = E^0\left( e^{-r(T'-t)} \mathbb{1}_{[I_t\leq Y]} \big| F_t \right) = e^{-r(T'-t)} E^0\left( e^{-r(T'-t)} \mathbb{1}_{[I_t>Y]} \big| F_t \right).
\]

Hence, together with (3), Equation (6) turns into

\[
V_{cat,T}^{b,0,Y,T}(t) = e^{-r(T'-t)} - V_{ILW,T}^{b,Y,T}(t),
\]

where \( V_{ILW,T}^{b,Y,T} \) is the price of a binary ILW with risk exposure period until \( T \), trigger level \( Y \) and maturing at time \( T' \). Thus, in the presence of a zero coupon binary cat bond with the same trigger level \( Y \) and same maturity, the price is given by

\[
V_{ILW,T}^{b,Y,T}(t) = E^0\left( e^{-r(T'-t)} \mathbb{1}_{[I_t>Y]} \big| F_t \right) = e^{-r(T'-t)} - V_{cat,T}^{b,0,Y,T}(t),
\]

where \( e^{-r(T'-t)} \) is the price of the related zero coupon bond without catastrophe (or any other default) risk under the assumption of a constant interest rate. In case of a non-constant interest rate this could be replaced with the price of a zero coupon bond maturing at time \( T' \) without default risk. Note that since direct replication is used, independence between the risk-free rate and the industry loss index is not necessary in this case. Equation (7) thus also yields a replicating portfolio for a binary ILW. In particular, the cash flow of a binary ILW with risk exposure period until \( T \), trigger level \( Y \) maturing at time \( T' \) can be perfectly replicated by buying a zero coupon bond without default risk and selling a binary zero coupon cat bond with risk exposure period until \( T \), trigger level \( Y \) and maturing at time \( T' \).

However, one has to take into account that the chance of finding a perfectly fitting cat bond in the market might be low. Therefore, we further provide approximations if there are only cat bonds with different maturity, different trigger level, non-zero or non-binary coupons available.
Matching trigger level but mismatching maturity

We next assume that there is a liquid market for binary zero coupon cat bonds with the same trigger level \( Y \) but different maturity \( T' \) and different risk exposure period until \( T' \). The aim is to approximate the binary ILW price \( V^{b,Y,T}_{ILW,T} \) with the price of this cat bond. We obtain this through Equation (7)

\[
V^{b,Y,T}_{ILW,T}(t) = e^{-r(T-t)}E^{Q}\left(1_{[I^{T}_t > Y]} \mid F_t \right) = e^{-r(T-t)} - V^{b,0,Y,T}_{cat,T} (t)
\]

for the price of a binary ILW with risk exposure period until \( T', \) trigger level \( Y \) and maturing at time \( T' \). To proceed, an assumption regarding the underlying distribution is needed. Hence, we assume (see motivation below)

\[
Q(I^{T}_T > Y) = F^{T,T}_{T',T'}\left(Q(I^{T}_T > Y)\right) \tag{8}
\]

for a given \( F^{T,T}_{T',T'} \) mapping from \([0,1]\) to \([0,1]\), and without loss of generality the price is calculated at time \( t=0 \). This assumption leads to

\[
V^{b,Y,T}_{ILW,T}(0) = e^{-rT}Q(I^{T}_T > Y) = e^{-rT}F^{T,T}_{T',T'}\left(Q(I^{T}_T > Y)\right) \\
= e^{-rT}F^{T,T}_{T',T'}\left(e^{rT}V^{b,Y,T}_{ILW,T}(0)\right) = e^{-rT}F^{T,T}_{T',T'}\left(1 - e^{rT}V^{b,0,Y,T}_{cat,T} (0)\right). \tag{9}
\]

for the price of a binary ILW with trigger level \( Y \), risk exposure period until \( T \) and maturing at time \( T' \). Assumption (8) is motivated by the following consideration. Let the risk exposure period be equal to the maturity \((T=T', \bar{T} = \bar{T}')\) and assume the industry loss index shows the real catastrophe losses instantly when they occur. Furthermore, let \( I^{T}_T \) follow a compound Poisson process under \( Q \), which is not too restrictive if \( I^{T}_T \) follows a compound Poisson process under the real world measure \( P \). Delbaen and Haezendonck (1989) showed under some reasonable assumptions, which should be fulfilled in most non-life insurance cases, that \( I^{T}_T \) remains a compound Poisson process under any equivalent risk-neutral measure. For a further treatment, we refer to Mürmann (2008) and Aase (1992). These assumptions lead to

\[
I^{T}_T = \sum_{i=1}^{N} X^i, \tag{10}
\]

where \( N_t \) and \( X^i \) are independent, \( N_t \) denotes the claims arrival process (i.e., a Poisson process with intensity \( \lambda \)) and \( X^i \) the i.i.d. claims (see, e.g., Levi and Partrat (1991)).
Furthermore, we assume that the industry loss index incorporates the actual catastrophe losses instantaneously and if two or more catastrophes occur, the industry loss index almost surely exceeds the trigger level. Using these assumptions and a resulting mapping $F_{T,T}'$ as shown in Appendix A, we can show that the ILW price can be approximated by (see Appendix A)

$$V_{ILW,T}^{b,Y,T}(0) = e^{-rT} \left( 1 - e^{-\lambda (T-T')} \frac{Q(I_T > Y)}{T} + (1 - e^{-\lambda (T-T')}) \right).$$

**Matching maturity but mismatching trigger level**

If the maturity of the cat bond and the ILW is the same, but the trigger level differs, assumptions regarding the underlying distribution (under $Q$) of the industry loss index $I_T$ are needed. We therefore assume

$$Q(I_T > Y) = F_{Y}' \left( Q(I_T > \tilde{Y}) \right)$$

for a given $F_Y'$ mapping from $[0,1]$ to $[0,1]$ and a different trigger level $\tilde{Y}$. Equation (11) leads to

$$V_{ILW,T}^{b,Y,T}(0) = e^{-rT} Q(I_T > Y) = e^{-rT} F_{Y}' \left( Q(I_T > \tilde{Y}) \right)$$

$$= e^{-rT} F_{Y}' \left( e^{rT} V_{ILW,T}^{b,Y,T}(0) \right) = e^{-rT} F_{Y}' \left( 1 - e^{rT} V_{cat,T}^{b,\tilde{Y},T}(0) \right).$$

for the price of a binary ILW with trigger level $Y$, risk exposure period until $T$ and maturing at time $T'$. Using

$$Q(I_T > Y) = Q(I_T > \tilde{Y}) + Q(I_T > Y) - Q(I_T > \tilde{Y})$$

leads to $F_Y' (\bullet) = \bullet + Q(I_T > Y) - Q(I_T > \tilde{Y})$ (see Equation (11)). One problem is that the distribution of $I_T$ under $Q$ is usually not known, but we give a motivation following the approach presented in the last subsection. Using the same assumptions as in the previous subsection and (10) for $I_T$ and following Burnecki, Kukla and Weron (2000) and Katz (2002), who showed that the log-normal distribution provides the best fit to the PCS index among typical claim size distributions, we assume that the i.i.d. claims $X_i$ follow a log-normal distribution with parameter $\mu$ and $\sigma^2$. In this case we can derive an approximation for $Q(I_T > Y)$ using the approximation in the last subsection (see Appendix A), i.e.,

---

14 Note that the distribution under the risk-neutral measure $Q$ can be different, but when using the Wang transform (see Wang (2000)), for instance, one obtains a log-normal distribution under $Q$, too, but with adjusted parameters.
\[
Q(I_T^T > \bar{Y}) = 1 - Q(I_T^T \leq \bar{Y}) = 1 - e^{-\lambda_T} - e^{-\lambda_T} Q^1 \{X^1 \leq \bar{Y}\} \\
= 1 - e^{-\lambda_T} - e^{-\lambda_T} \lambda_T \left( \frac{\ln(\bar{Y}) - \mu}{\sigma} \right),
\]

(13)

where \( \Phi \) is the standard normal distribution function. Equation (13) leads to

\[
F_t^Y(\bullet) = 1 + e^{-\lambda_T} \lambda_T \left( \Phi \left( \frac{\ln(Y) - \mu}{\sigma} \right) - \Phi \left( \frac{\ln(\bar{Y}) - \mu}{\sigma} \right) \right),
\]

which can be used to solve Equation (12) under our simplifying assumptions.

**Non-zero cat bonds**

If there is no market for zero coupon cat bonds (as is currently the case), binary non-zero cat bonds need to be used for approximating ILW prices. The decomposition follows the standard bond stripping approach, where each coupon itself is understood as a zero coupon bond with adjusted nominal, interest rate and maturity. To express the price involving only \( V_{\text{cat},T}^{b,0,Y,T} \), we use assumption (8). Recall Equation (5) for the payoff of such a cat bond and Equation (9) for adjusting a mismatching maturity to obtain

\[
V_{\text{cat},T}^{b,0,Y,T}(0) = \sum_{j=1}^{n} c \cdot E^Q \left( e^{-r_{j,T}} 1_{[T_j^T,T]} \right) + E^Q \left( e^{-r_{j,T}} 1_{[T_j^T,T]} \right) \\
= \sum_{j=1}^{n} c \cdot e^{-r_{j,T}} E^Q \left( 1 - 1_{[T_j^T,T]} \right) + e^{-r_{j,T}} E^Q \left( 1 - 1_{[T_j^T,T]} \right) \\
= \sum_{j=1}^{n} c \cdot e^{-r_{j,T}} - \sum_{j=1}^{n} V_{\text{ILW},T}^{b,Y,T}(0) + e^{-r_{j,T}} - V_{\text{ILW},T}^{b,Y,T}(0) \\
= \sum_{j=1}^{n} c \cdot e^{-r_{j,T}} - \sum_{j=1}^{n} F_{T,T,j}^{T,T} \left( e^{r_{j,T}} V_{\text{ILW},T}^{b,Y,T}(0) \right) + e^{-r_{j,T}} - V_{\text{ILW},T}^{b,Y,T}(0)
\]

(14)

Solving this equation for \( V_{\text{ILW},T}^{b,Y,T}(0) \) leads to the price of a binary ILW with trigger level \( Y \), risk exposure period until \( T \), and development period until \( T' \).
**Non-binary cat bonds**

We now assume a non-binary zero coupon cat bond with risk exposure period until $T$, trigger level $Y$, limit $M$, and maturing at time $T'$ to have nominal 1 and that the payment at time $T'$ depends on the extent (captured by $I_T^T$) to which the industry loss index exceeds the trigger $Y$ at time $T'$. The price of this cat bond is given by

$$V_{\text{cat}, T'}^{Y, M, T} (0) = E^Q \left( e^{-rT'} \left( 1 - \frac{\min \left( (I_T^T - Y)_+, M \right)}{M} \right) \right)$$

$$= e^{-rT'} - \frac{1}{M} \left( C_T^{I_T^T, Y} (0) - C_T^{I_T^T, Y+M} (0) \right)$$

$$= e^{-rT'} - \frac{1}{M} C_T^{I_T^T, Y+M} (0), \quad (\text{15})$$

where $C_T^{I_T^T, Y} (0)$ is the price of a call option on the industry loss index $I_T^T$ with strike price $Y$ maturing at time $T'$ and $C_T^{I_T^T, Y+M} (0)$ is the price of the associated call spread. We use the midpoint rectangle rule to approximate $C_T^{I_T^T, Y+M} (0)$ by

$$C_T^{I_T^T, Y+M} (0) = E^Q \left( e^{-rT'} \min \left( (I_T^T - Y)_+, M \right) \right)$$

$$= \int_{Y}^{Y+M} e^{-rT'} Q \left( I_T^T > x \right) dx = \int_{Y}^{Y+M} V_{\text{ILW}, T'}^{b,x,T} (0) dx$$

$$= M \cdot e^{-rT'} Q \left( I_T^T > Y + \frac{M}{2} \right) = M \cdot V_{\text{ILW}, T'}^{b,y+M/2,T} (0) \quad (\text{16})$$

and use Equations (15) and (16) to obtain

$$V_{\text{ILW}, T'}^{b,y+M/2,T} (0) = e^{-rT'} - V_{\text{cat}, T'}^{Y, M, T} (0) \quad (\text{17})$$

for the price of a binary ILW with trigger level $Y + \frac{M}{2}$, risk exposure period until $T$ and maturing at time $T'$. The approximation generally works well for small layers. Figure 3 illustrates the relative pricing error assuming compound Poisson losses with parameters from Katz (2002).
Figure 3: Exemplary relative pricing error in percent through the approximation in Equation (17) (layer limit $M$ and trigger level $Y$ in bn. U.S. $)

Notes: Monte-Carlo simulation based on 500,000 sample paths; layer and trigger level in bn. U.S. $. Relative pricing error in Equation (17) = \left( \frac{1}{\mu} \right) \left( \frac{1}{\sigma} \right) \left( \frac{1}{\lambda} \right) \left( \frac{1}{T} \right) \left( \frac{1}{V_{TILW, T}^b} \right) \left( \frac{1}{V_{TILW, T}^b} \right). Parameters of the log-normal distribution according to Katz (2002): $\mu = -1.090$, $\sigma = 2.307$, the intensity of the compound Poisson process is $\lambda = 1.8$, maturity is set to 1.

Summary: Pricing binary ILWs

We presented different approaches for the valuation of binary ILWs under the assumption of no-arbitrage. In the presence of a liquid cat bond market, we obtain Equation (7) under no further assumptions. Since there may not be enough variety of cat bonds to match the parameters of every ILW, we further provided approximations for different trigger levels and maturities. To compute prices for ILWs (and similarly for other index-linked catastrophic loss instruments), the following three steps should be taken to obtain $V_{TILW, T}^b$:

1. From non-zero cat bonds to zero coupon cat bonds.
2. Adjust the trigger level.
3. Adjust the maturity.

The possible cases, steps, and underlying assumptions are summarized in Table 2.
Table 2: Overview of steps and assumptions in the occurring cases for approximately calculating the prices of binary ILWs by means of replications by cat bonds

<table>
<thead>
<tr>
<th>Cases</th>
<th>Steps and solutions</th>
<th>Assumptions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Zero coupon cat bond matching maturity, matching trigger level</td>
<td>$V_{ILW,T}^{0,Y,T}(0) = e^{-rT} - V_{cat,T}^{0,Y}(0)$</td>
<td>-</td>
</tr>
<tr>
<td>Zero coupon cat bond mismatching maturity, matching trigger level</td>
<td>$V_{ILW,T}^{0,Y,T}(0) = v'/T - e^{-r(1-T_T)}V_{ILW,T}^{0,Y}(0)$ [ \text{with } V_{ILW,T}^{0,Y,T}(0) = e^{-rT} - V_{cat,T}^{0,Y,T}(0) ]</td>
<td>(8)</td>
</tr>
<tr>
<td>Zero coupon cat bond, matching maturity, mismatching trigger level</td>
<td>Use Equation (12) with $V_{ILW,T}^{0,Y,T}(0) = e^{-rT} - V_{cat,T}^{0,Y}(0)$</td>
<td>(11)</td>
</tr>
<tr>
<td>Zero coupon cat bond, mismatching maturity, mismatching trigger level</td>
<td>Use Equation (12) to obtain $V_{ILW,T}^{0,Y,T}(0)$ with $V_{ILW,T}^{0,Y,T}(0) = e^{-rT} - V_{cat,T}^{0,Y}(0)$</td>
<td>(8), (11)</td>
</tr>
<tr>
<td>Non-zero coupon cat bond</td>
<td>Use Equation (14) to obtain $V_{ILW,T}^{0,Y,T}(0)$, then adjust trigger level and maturity if necessary</td>
<td>(8)</td>
</tr>
<tr>
<td>Non-binary cat bonds</td>
<td>Use Equation (17) to obtain $V_{ILW,T}^{b,Y,0.5M,T}$, then adjust trigger level and maturity if necessary</td>
<td>Small layer ( \text{see Figure 3} )</td>
</tr>
</tbody>
</table>

4.4 Pricing indemnity-based (double-trigger) ILWs

For pricing indemnity-based ILWs by means of replication, the tradability of the company loss $L_T$ must be examined. Toward this end, we apply the three approaches presented in Section 4.1.

Independence between company loss and industry loss index

First, we assume that the industry loss index $I_T$ and the company loss $L_T$ are independent and, using Equation (4), obtain
for the price of an indemnity-based ILW with attachment point $A$, maximum payoff $M$, risk exposure period until $T$ and maturing at time $T'$, where $C_{T'}^{L,A}(0)$ is the price of a call option on the company loss $L^r$ with strike price $A$ maturing at time $T'$. This setting can be treated as in Møller (2003), where prices for $C_{T'}^{L,A}(0)$ are derived using, e.g., the traditional variance principle for the company loss part and assuming that the company loss is log-normally distributed. However, in Møller (2003), a trigger for a financial asset is used instead of an additional industry loss trigger. In this case, independence is a more realistic assumption than in the present setting, but nevertheless this calculation could provide a lower bond for an efficient price in our case.

**Approximating the company loss based on the stock price loss**

Thus, second, we approximate $L^r$ through the stock price loss and obtain a structured risk management product, which can be treated as in Cox, Fairchild and Pedersen (2004). As described in Section 4.1, we assume a company with traded stock and approximate the individual company loss $L^s$ through the loss of market value by means of its stock price, e.g.,

$$L^s_t = f\left((S_s)_{t \geq 20}\right)$$

for a function $f$ satisfying the assumptions presented in Section 4.1. The price of an indemnity-based ILW is given through

$$V_{ILW,T}^{A,M,Y,T}(0) = E^Q\left(e^{-rT}\left((L^s_{T'} - A)_{s} - (L^s_{T'} - (A + M))_{s}\right)\right)1_{[t^r,T]}$$

Now arbitrage-free valuation can be applied, and in case the distribution of $S$ and $I^T$ is known, one can calculate the expectation either directly or by means of simulation methods. Regarding the problems associated with this approach, we refer to the discussion in Section 4.1.
Exploiting the dependence: approximating \( L \) by \( g(I) \)

Third, we assume a functional relationship between the company loss and the industry loss index and calculate prices exploiting this relation, as ILWs will not be used if the degree of dependence is not sufficiently high. We can hence assume functional dependence between \( L^T \) and \( I^T \), i.e.,

\[
L_t^T = g\left(I_t^T\right), \quad \text{or} \quad L_t^T = g\left(I_t^T\right) + Z_t,
\]

where \( Z_t \) is supposed to have increments, which have the expectation 0 under \( Q \) (or under \( P \), see Merton (1976) if we meaningfully assume that \( Z_t \) is diversifiable). The existence of such a representation of \( L^T \) is motivated by

\[
L_t^T = E\left[L_t^T \mid I_t^T\right] + \left(L_t^T - E\left[L_t^T \mid I_t^T\right]\right).
\]

With a standard property of the conditional expectation (regarding taking the expected value), one can observe that \( Z_t := L_t^T - E\left[L_t^T \mid I_t^T\right] \) fulfills the last assumption. We assume \( g \) to be a function satisfying the properties presented in Section 4.1. Moreover, we assume without loss of generality that if \( I_t^T > Y \) then \( L_t^T > A \) almost surely (in practice this is often the case in order to increase the probability of the payout, see Gatzert and Schmeiser (2012)). One example is \( g(x) = a \cdot x \) with \( 0 \leq a \leq 1 \) representing an indication of the market share. We start with Equation (4) for the payoff of an indemnity-based ILW with attachment point \( A \), maximum payoff \( M \), trigger level \( Y \), risk exposure period until \( T \) and maturing at time \( T' \),

\[
P_{\text{ILW},T}^{A,M,Y,T} = \left( (L_{T'}^T - A)_+ - (I_{T'}^T - (A + M))_+ \right) 1_{\{I_{T'}^T > Y\}},
\]

and assume that \( L_t^T = g\left(I_t^T\right) \), thus obtaining

\[
V_{\text{ILW},T}^{A,M,Y,T}(0) = \left( E^Q\left(e^{-rT}\left(g\left(I_{T'}^T\right) - A\right)_+ 1_{\{I_{T'}^T > Y\}}\right) - E^Q\left(e^{-rT}\left(g\left(I_{T'}^T\right) - (A + M)\right)_+ 1_{\{I_{T'}^T > Y\}}\right) \right)
\]

for the arbitrage-free price of an indemnity-based ILW with attachment point \( A \) of the company loss and an equivalent martingale measure \( Q \). We show in Appendix B that

\[
V_{\text{ILW},T}^{A,M,Y,T}(0) = \int_{x > Y} (e^{-rT} - V_{\text{cat},T}^{b,0,x,T}(0)) g'(x) dx + (g(Y) - A)(e^{-rT} - V_{\text{cat},T}^{b,0,Y,T}(0))
\]

\[
- \int_{x > Y} (e^{-rT} - V_{\text{cat},T}^{b,0,x,T}(0)) g'(x) dx
\]
holds for the price of an ILW. Hence, the price can be calculated by only using the observable cat bond prices. If \( g(x) = a \cdot x \) (at least approximately for \( T \) in the area from \( g(Y) \) to \( g(A + M) \)) then (recall the non-binary cat bond from Section 4.2)

\[
V_{ILW,T}^{A,M,Y,T}(0) = a \cdot (a^{-1} (A + M) - Y) V_{cat,T}^{Y,A^{-1}(A+M)-Y,T}(0) + (a \cdot Y - A) (e^{-rt} - V_{cat,T}^{b,0,Y,T}(0)).
\]

If there is no suitable cat bond, one could again use approximations, which, however, would be considerably more complicated than in the binary case, thus suggesting that it is advisable to use numerical methods.

5. SUMMARY

This paper presents new approaches and techniques of how prices for index-linked catastrophic loss instrument can be derived using arbitrage-free pricing by proposing replication techniques and approximations that aim to overcome the requirement of direct tradability of the underlying company loss and industry loss index. In particular, contrary to traditional option pricing theory, one cannot necessarily assume that the underlying industry loss index is tradable itself; however, there may be a liquid market for certain derivatives, including cat bonds. This is of great relevance today in the academic literature, but it will be even more relevant in the future for the insurance industry and financial investors, when index-linked catastrophic loss instruments become even more widespread than today and when there are truly liquid markets for derivatives (e.g., cat bonds) on the industry loss index.

We apply the proposed approaches and considerations to the arbitrage-free pricing of ILWs, where we assume a liquid cat bond market to ensure the tradability of the underlying industry loss index. We hence consider a liquid index-linked cat bond market (equivalent to a liquid option market) for various trigger levels and maturities, and first derive prices for binary ILWs. We thereby show that we do not need any assumption concerning the distribution of the underlying industry loss index, which represents a major advantage as compared to other pricing approaches. If a suitable cat bond is not available for approximating ILW prices, we provide proper approximations under some reasonable assumptions.

When calculating prices for indemnity-based index-linked catastrophic loss instruments, one major problem is the behavior of the company loss, which should be tradable in some sense when using risk-neutral valuation. We thus propose three different approaches for solving this issue and discuss problems associated with them. First we assume independence between the industry index and the company loss, which leads to a situation treated by Møller (2003). We next replace the company loss by the stock price loss, which leads to a structured risk
management product as studied by Cox, Fairchild and Pedersen (2004). As a third approach, we propose to approximate the company loss by means of the industry loss index using a monotonic increasing function and then show how pricing can again be conducted similarly to the case of non-indemnity-based index-linked catastrophic loss instruments. We further apply the three proposed approaches to deriving arbitrage-free prices for indemnity-based ILWs under the assumption of a liquid cat bond market.

Thus, one main contribution of the paper is to propose approaches to overcome the crucial point of tradability of the underlying company loss and industry loss processes in case of index-linked catastrophic loss instruments through suitable approximations and by deriving explicit replicating portfolios or close approximations using traded derivatives. This allows us to provide explicit pricing formulas, which in a next step can be used to empirically verify how liquid these markets already are by comparing ILW prices with the ones derived based on the replication techniques presented in this paper. In addition, one further aspect to be taken into account in future work is the counterparty risk that is potentially associated with static replication by means of, e.g., cat bonds.

REFERENCES


Appendix A

$I^{\tilde{t}}_t$ is a compound Poisson process defined through

$$I^{\tilde{t}}_t = \sum_{i=1}^{N} X^i,$$

where $N_t$ and $X^i$ are independent, $N_t$ denotes the claims arrival process (i.e., a Poisson process with intensity $\lambda$) and $X^i$ the i.i.d. claims. The probability for exceeding the trigger level is given through

$$Q(I^{\tilde{t}}_t > Y) = 1 - Q(I^{\tilde{t}}_t \leq Y) = 1 - e^{-2\tilde{t}} \sum_{k=0}^{\infty} \left( \lambda \tilde{t} \right)^k \frac{1}{k!} \cdot Q \left( \sum_{i=1}^{k} X^i \leq Y \right). \quad (18)$$

Furthermore, we assume that the industry loss index incorporates the actual catastrophe losses instantaneously and if $k = 2$ or more catastrophes occur, the industry loss index almost surely exceeds the trigger level. Hence, we can approximate Equation (18) through (sum until $k = 1$)

$$Q(I^{\tilde{t}}_t > Y) \approx 1 - e^{-\tilde{t}} - e^{-\tilde{t}} \lambda \tilde{t} Q(X^1 \leq Y).$$

An interpretation is that there are just two possibilities for the index not to be triggered: no claim or one claim with size $X^1$ lower than the trigger level $Y$. This should approximately be true if we model only the occurrence of very large events. Solving this equation for $Q(X^1 \leq Y)$ leads to

$$Q(X^1 \leq Y) = \frac{-Q(I^{\tilde{t}}_t > Y) + 1 - e^{-\tilde{t}}}{e^{-\tilde{t}} \lambda \tilde{t}}.$$

We use $T = T^\prime$, $\tilde{T} = \tilde{T}^\prime$ and Equation (18) again to calculate $Q(I^{t\prime}_T > Y)$

$$Q(I^{t\prime}_T > Y) = 1 - e^{-\lambda T} - e^{-\lambda \tilde{T}} \lambda T Q(X^1 \leq Y)$$

$$= 1 - e^{-\lambda \tilde{T}} - e^{-\lambda \tilde{T}} \lambda T \left[ -Q(I^{\tilde{t}}_t > Y) + 1 - e^{-\tilde{t}} \right] \frac{1}{e^{-\lambda \tilde{T}} \lambda \tilde{T}}$$

$$= 1 - e^{-\lambda \tilde{T}} - e^{-\lambda (T - \tilde{t})} \frac{T}{\tilde{T}} \left[ -Q(I^{\tilde{t}}_t > Y) + 1 - e^{-\tilde{t}} \right] = \tilde{F}^{TT}_{t, \tilde{t}} (Q(I^{\tilde{t}}_t > Y)).$$
Appendix B

We obtained

\[ V_{ILW,T}^{AM,Y} (0) = \left( E^Q \left( e^{-rT} \left( g(I_T) - A \right)_+, 1_{[\tilde{T} > \gamma]} \right) - E^Q \left( e^{-rT} \left( g(I_T) - (A + M) \right)_+, 1_{[\tilde{T} > \gamma]} \right) \right) \]

for the arbitrage-free price of an indemnity-based ILW with attachment point \( A \) of the company loss and an equivalent martingale measure \( Q \). We assume without loss of generality that if \( T > Y \) then \( T = g(I_T) > A \), and \( g(Y) \leq A + M \). These assumptions also imply that the ILW payoff does not always amount to the maximum \( M \) every time \( T > Y \), i.e., \( T > Y \) does not necessarily imply that \( I_T = g(I_T) > A + M \). This results in

\[ V_{ILW,T}^{AM,Y} (0) = E^Q \left( e^{-rT} \left( g(I_T) - A \right)_+, 1_{[\tilde{T} > \gamma]} \right) - E^Q \left( e^{-rT} \left( g(I_T) - (A + M) \right)_+, 1_{[\tilde{T} > \gamma]} \right) \]

\[ = E^Q \left( e^{-rT} \left( g(I_T) - A \right)_+, 1_{[\tilde{T} > \gamma]} \right) - E^Q \left( e^{-rT} \left( g(I_T) - (A + M) \right)_+, 1_{[\tilde{T} > \gamma]} \right) \]

\[ = E^Q \left( e^{-rT} g(I_T)_+, 1_{[\tilde{T} > \gamma]} \right) - e^{-rT} A 1_{[\tilde{T} > \gamma]} - C^{T,A+M}_T (0) \]

\[ = e^{-rT} E^Q \left( g(I_T)_+, 1_{[\tilde{T} > \gamma]} \right) + \left( g(Y) - g(Y) \right)_+, 1_{[\tilde{T} > \gamma]} \right) - C^{T,A+M}_T (0) \]

\[ = e^{-rT} E^Q \left( g(I_T)_+, 1_{[\tilde{T} > \gamma]} \right) + \left( g(Y) - g(Y) \right)_+, 1_{[\tilde{T} > \gamma]} \right) - C^{T,A+M}_T (0) \]

\[ = C^{T,g(Y)}_T (0) + \left( g(Y) - A \right) C^{T,b,g(Y)}_T (0) - C^{T,A+M}_T (0) \]

where \( C^{T,g(Y)}_T (0) \) is the price of a call option on \( L_T \) with strike price \( g(Y) \) and \( C^{T,b,g(Y)}_T (0) \) is the price of a binary call option with strike price \( g(Y) \). We calculate \( C^{T,g(Y)}_T (0) \) in terms of binary ILWs and \( g \), i.e.,

---

15 Since \( A \) is typically low and the correlation between industry index and company loss is high; otherwise, a similar result would be obtained, but this represents the more realistic alternative; see Gatzert and Schmeiser (2012).

16 For the first transformation, we use \( g(I_T) \geq (A + M) \geq g(Y) \). Thus, if \( g(I_T) - (A + M) \) is positive, \( 1_{[g(I_T) - (A + M)]} = 1_{[g(I_T) > (A + M)]} \) is equal to 1.
\[ C^T_{\mathcal{T}, \sigma(Y)}(0) = E^Q \left( e^{-rT} \left( I^T_{\mathcal{T}} - g(Y) \right) \right) = e^{-rT} \int_{\mathbb{R}_+} Q \left( g \left( I^T_{\mathcal{T}} \right) > g(Y) + x \right) dx \]

\[ = e^{-rT} \int_{x \geq g(Y)} Q \left( g \left( I^T_{\mathcal{T}} \right) > x \right) dx = e^{-rT} \int_{x \geq g(Y)} Q \left( I^T_{\mathcal{T}} > g^{-1}(x) \right) dx \]

\[ = e^{-rT} \int_{x \geq g^{-1}(g(Y))} Q \left( I^T_{\mathcal{T}} > x \right) g'(x) dx = \int_{x \geq Y} \left( e^{-rT} - V^b_{\text{cat}, T}^{b, 0, Y, T}(0) \right) g'(x) dx \]

(20)

and

\[ C^T_{\mathcal{T}, \phi(Y)}(0) = e^{-rT} - V^b_{\text{cat}, T}^{b, 0, Y, T}(0). \]

\[ V^{A_M Y, T}_{\text{ILW}, T}(0) \text{ can be calculated using Equations (19) and (20) by only using the observable cat bond prices.} \]

\[ V^{A_M Y, T}_{\text{ILW}, T}(0) = C^T_{\mathcal{T}, \sigma(Y)}(0) + \left( g(Y) - A \right) C^T_{\mathcal{T}, \phi(Y)}(0) - C^T_{\mathcal{T}, A+M}(0) \]

\[ = \int_{x \geq Y} \left( e^{-rT} - V^b_{\text{cat}, T}^{b, 0, X, T}(0) \right) g'(x) dx + \left( g(Y) - A \right) \left( e^{-rT} - V^b_{\text{cat}, T}^{b, 0, Y, T}(0) \right) \]

\[ - \int_{x \geq Y^{-1}(A+M)} \left( e^{-rT} - V^b_{\text{cat}, T}^{b, 0, X, T}(0) \right) g'(x) dx. \]

If \( g(x) = a \cdot x \) (at least approximately for \( I^T \) in the area from \( g(Y) \) to \( g(A+M) \)) then (recall the non-binary cat bond from Section 4.2)

\[ V^{A_M Y, T}_{\text{ILW}, T}(0) = a \cdot C^T_{\mathcal{T}, Y}(0) + \left( a \cdot Y - A \right) C^T_{\mathcal{T}, \phi(Y)}(0) - a \cdot C^T_{\mathcal{T}, a^{-1}(A+M)}(0) \]

\[ = a \left( a^{-1} \left( A+M \right) - Y \right) V^Y_{\text{cat}, T,a^{-1}(A+M)-Y,T}(0) + \left( a \cdot Y - A \right) \left( e^{-rT} - V^b_{\text{cat}, T}^{b, 0, Y, T}(0) \right). \]