Precautionary Motives Under Multiple Instruments

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Abstract

We study precautionary behavior when more than one instrument is available to the decision-maker to optimize expected utility of intertemporal consumption. Specifically, we analyze saving, self-protection, and self-insurance and also study how pairs of these instruments behave when employed simultaneously. We examine a loss risk – which is a prerequisite of self-protection and self-insurance – and an income risk. When controlling for consumption smoothing we find that each risk implies a positive precautionary effect and a negative substitution effect for each instrument. Consequently, the total precautionary motive, which is the net effect of all the individual effects, can be decomposed into precautionary motives according to the different risks and the different instruments. We study their relative contributions to the total precautionary motive numerically. Our study documents that the availability of other instruments has important implications for measuring the intensity of precautionary motives both empirically and experimentally.

Keywords: prudence · saving · self-protection · self-insurance · precaution

JEL classification: D81, D90, D91

1 Introduction

Agents often dispose of a variety of channels to cope with risks. For example, they may reduce the probability or severity of potential damages, buffer risks using market instruments, or save or work more to prepare and forearm for future risks. These instruments will, in general, interact in their use and determine jointly an individual’s effective risk exposure. Accordingly, studies on risk management or underlying preferences should address the portfolio of instruments available to an agent and account for potential interactions between these.

We study the intertemporal risk responses of a decision maker who disposes of one, two, or three instruments out of saving, self-protection, and self-insurance. By investing some effort, self-protection allows the decision maker to decrease the probability of a potential loss, while self-insurance reduces the size of the potential loss keeping its probability (Ehrlich and Becker 1972). For observing the decision maker’s precautionary motives in this context, two aspects are crucial. First, as with all intertemporal

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decisions under risk, optimal behaviors involve both a consumption-smoothing and a precautionary component. Second, the decision maker has two risks to cope with, the endogenous loss risk and an exogenous background risk. The two aspects are important because a change in the loss risk involves, by definition, both a first-order effect and a change in pure riskiness. As a consequence, the loss risk will immediately activate an agent’s consumption smoothing and precautionary preference, even in the absence of the background risk.

In our two-period framework, we model the background risk as an independent zero-mean risk on period-two income. We define an agent’s precautionary motive as the extent to which the agent chooses an instrument for the response to a future risk. Even in the cases with a single risk-management instrument, the agent’s total precautionary motive is thus made up of two components, the precautionary reactions to the loss risk and to the income risk.

When two instruments are available, precautionary reactions arise, if the agent is prudent, in each instrument to each risk. In addition to the positive precautionary effects, in each instrument now also a negative substitution effect occurs. While the total precautionary reaction to a risk across the two instruments will always be positive, the interaction between the instruments may be so strong that the substitution effect prevails in one instrument. In that case, it is chosen less in reaction to the risk. These effects extend to the case where all three instruments are available.

We quantify the precautionary motives in the different settings of our theoretical analysis numerically for a median agent.\textsuperscript{1} The simulations show that all instruments react to each risk, however in different degrees. In the single-instrument cases, saving shows by far the largest precautionary reaction. Self-protection reacts still more than self-insurance, whence self-insurance is mostly used for consumption smoothing. In the cases with two instruments, the total precautionary motive is lower than in each corresponding single-instrument case. The precautionary motive in self-insurance dominates the precautionary motive in self-protection, which, in turn, dominates precautionary saving. When self-insurance is available, the total precautionary motives of saving or self-protection, respectively, turn even negative for a large enough loss size. In the three-instrument case, all effects are still more pronounced. The total precautionary motive across all instruments is once more smaller than in the corresponding two-instrument cases.

Knowing the intensity of precautionary motives is crucial in many applications, notably of public policy. Investments to fight climate change, optimal risk mitigation strategies by individuals and organizations, and recommendations for individual asset allocation decisions are but some areas of application where spurious estimates of individual risk and time preferences may bias predictions. Our

\textsuperscript{1}To the best of our knowledge, there are thus far no empirical studies on precautionary choices with respect to multiple risks or precautionary self-protection or self-insurance.
analysis provides a potential explanation for the difficulty to quantify the intensity of precautionary motives using field data. One possible reason is that the use of other instruments can pick up some of the precautionary response. Consequently, if additional instruments are not (fully) identified, results will be biased against higher intensities of precautionary motives.

Precautionary saving has been analyzed in a long-standing literature (e.g., Drèze and Modigliani 1966, 1972, Leland 1968, Sandmo 1970, Rothschild and Stiglitz 1971). By contrast, only some recent contributions have concerned precautionary choices in self-protection (e.g., Courbage and Rey 2012, Eeckhoudt et al. 2012). Precautionary self-insurance occurs in an intertemporal setup only as a special case in Hofmann and Peter (2012). Menegatti (2009) investigates optimal saving in the presence two risks, of which our case with saving only can be seen as an application.

Aspects of our two-instrument cases with intertemporal choices have been studied from different angles. Dionne and Eeckhoudt (1984), for example, analyze the Hicksian demand for saving and insurance, showing that under a decreasing shape of utility curvature (‘decreasing temporal risk aversion’) insurance and saving are pure substitutes. Gollier (1994) analyzes saving and insurance demand in continuous time in a lifecycle context and treats also precautionary saving. For saving and self-protection, Menegatti and Rebessi (2011) show that the two instruments are substitutive, so that the investment in one action is a decreasing function of the investment in the other. However, none of the above papers makes the distinction between consumption-smoothing and precautionary motivations for self-protection or self-insurance choices or studies the interaction between precautionary choices in different instruments.

The Section 2 explains the benchmark model and briefly recaps precautionary behaviors in single-decision cases. In Section 3, precautionary behaviors with two instruments are studied. Here we develop our main results that explain when the presence of unidentified endogenous instruments leads to the under- or overestimation of the intensity of precautionary motives. Section 4 generalizes our findings to the three-instrument cases. We provide comparative statics results in section 5, before Section 6 concludes.

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2 Various studies have sought to measure the precautionary saving motive from field observations, with diverse methodologies and datasets (e.g., Kuehlwein 1991, Dynan 1993, Merrigan and Normandin 1996, Eisenhauer 2000, Ventura and Eisenhauer 2006, Lee and Sawada 2007). The results have remained inconclusive regarding prevalence and strength.

3 For a monoperiodic set-up, Eeckhoudt and Kimball (1992) already established that a zero-mean income risk creates a precautionary demand for insurance if the agent has a positive third utility derivative.


2 Intertemporal Decisions with One Instrument and Two Risks

Consider an agent who lives for two periods, and receives the exogenous income $y$ in period one and $Y$ in period two. The agent’s intertemporal consumption stream $(c, \tilde{C})$ consists of certain consumption $c$ in period one and potentially risky consumption $\tilde{C}$ in period two. We mark random variables with a tilde. The following intertemporal utility objective characterizes the agent’s preferences:

$$U(c, \tilde{C}) = u(c) + \beta E v(\tilde{C}),$$

where $u$ is the agent’s first-period felicity, $v$ the second-period felicity, and $\beta$ the utility discount factor. We assume $u$ and $v$ are strictly increasing and concave.

The riskiness of second-period consumption may arise from two sources. First, we consider the risk of a property loss of size $L$ occurring with probability $p \in (0,1)$. The mitigation of this risk represents the focal aspect in the literature on self-protection or self-insurance. Second, there can be a background risk $\varepsilon$, $E \tilde{\varepsilon} = 0$, associated with the agent’s second-period income $Y$.\footnote{Gollier and Pratt (1996) study unfair background risks whose mean is allowed to be negative. To focus on the effects of riskiness we assume that expected consumption does not depend on the presence of the background risk, which is consistent with most of the literature.} A background risk of this kind is generally taken as the trigger for precautionary choices. Whereas the income risk, as a zero-mean risk, adds only riskiness to second-period consumption, the loss risk has also a first-order effect. In particular, there is no loss risk if the potential loss is zero. We stick in this paper to the simple case where loss risk and income risk are uncorrelated.\footnote{Several papers treat optimal choices in the face of correlated risks (e.g., Tsetlin and Winkler 2005, Courbage and Rey 2007, Courbage et al. 2013).}

Given a loss risk, the agent can influence the intertemporal consumption stream in three ways, via saving, self-protection, or self-insurance. In contrast to saving, self-protection and self-insurance require the presence of the loss risk to become operative.

**Saving.** From the literature on intertemporal consumption, it is well known that agents commit to saving for two main reasons, to smooth consumption and for precautionary purposes. The origin of precautionary saving is risk on future resources for consumption. Given that the agent is prudent, that is, $v'' > 0$ (Kimball 1990), precautionary saving will in the present context already arise when only the loss risk is present. Let $r$ be the interest rate in the market, which we take to be non-random. Then, the agent chooses saving $s$ such as to maximize

$$U^l(s) = u(y - s) + \beta [pv(Y + s(1 + r) - L) + (1 - p)v(Y + s(1 + r))]$$

where superscript $l$ indicates that the utility objective includes the loss risk, but no other risk. We
denote the optimal saving choice resulting from this problem by $s^{Is}$. As usual in treatments of consumption/saving problems, the consumption-smoothing component of optimal saving can be determined by maximizing

$$U^0(s) = u(y - s) + \beta v(Y + s(1 + r) - pL)$$

Superscript 0 indicates that the risk on future consumption is set to zero because the loss risk is replaced by its mean. The optimal saving choice is denoted by $s^{0s}$. Precautionary saving is, in turn, the extra amount of saving that arises due to the riskiness of second-period consumption, $s^{Is} - s^{0s}$.

The income risk constitutes an alternative, or an additional cause of precautionary saving, depending on whether it is considered as the only source of risk or the two risks are jointly admitted. In the first case, we denote the optimal saving choice by $s^{Is}$, so that precautionary saving due to the income risk arises as $s^{Is} - s^{0s}$. Here $s^{Is}$ is obtained via maximization of

$$U^i(s) = u(y - s) + \beta Ev(Y + \tilde{\varepsilon} + s(1 + r) - pL).$$

In the second case, optimal saving is $s^{li}$, and precautionary saving due to the income risk is $s^{li} - s^{Is}$. The latter case with loss risk and income risk is an example of optimal saving with two risks as in Menegatti (2009), and the optimal saving decision is given as the maximand of

$$U^{li}(s) = u(y - s) + \beta \left[ pEv(Y + \tilde{\varepsilon} + s(1 + r) - L) + (1 - p)Ev(Y + \tilde{\varepsilon} + s(1 + r)) \right].$$

In our notation superscript $i$ denotes the case where only the income risk is present (and the loss risk is replaced by its mean), and superscript $li$ is for the case where both the loss risk and the income risk are present.

**Self-protection.** The other two risk-management instruments endogenize the loss risk. Self-protection implies an investment to decrease the probability of the loss. Formally, spending $x \geq 0$ dollars on self-protection reduces the loss probability to $p(x) < p$, with $p'(x) < 0$ and $p''(x) \geq 0$. The agent will choose self-protection expenditures $x$ such as to maximize

$$U^I(x) = u(y - x) + \beta \left[ p(x)v(Y - L) + (1 - p(x))v(Y) \right]$$

yielding optimal expenses on self-protection of $x^{Is}$. Similar to the saving decision, also for self-protection consumption smoothing and precaution can be distinguished as basic motivations in the intertemporal case. The consumption-smoothing component of self-protection expenditures $x^{0s}$ arises, in analogy to the saving case, by maximizing the utility objective with future utility depending on
expected future consumption. The precautionary component is then \( x^{ls} - x^{0s} \).

The introduction of a zero-mean income risk in the future raises the optimal investment in self-protection, if the agent is prudent (Courbage and Rey 2012, Eeckhoudt et al. 2012). In analogy to before, we denote its level by \( x^{lis} \). Because the presence of the loss risk is a prerequisite for self-protection, the precautionary demand for self-protection due to the income risk, \( x^{lis} - x^{ls} \), is always additional to the precautionary demand already induced by the loss risk.

(Self-)insurance. Self-insurance involves an investment to mitigate the loss size. Formally, spending \( z \) dollars on self-insurance reduces the loss to \( L(z) \), with \( L'(z) < 0 \) and \( L''(z) \geq 0 \). The agent will choose self-insurance expenditures \( z \) such as to maximize

\[
U^l(z) = u(y - z) + \beta [pv(Y - L(z)) + (1 - p)v(Y)]
\]

giving rise to the optimal level \( z^{ls} \). The reasoning to obtain the consumption-smoothing component, \( z^{0s} \), and precautionary component, \( z^{ls} - z^{0s} \), of self-insurance, as well as the additional precautionary expenses due to the income risk, \( z^{lis} - z^{ls} \), is analogous to the self-protection case. The aspect that adding a zero-mean income risk in the future raises the optimal self-insurance, if the agent is prudent, is a special case of Hofmann and Peter (2012). Moreover, already Eeckhoudt and Kimball (1992) establish for a monoperiodic set-up that a zero-mean income risk creates a precautionary demand for insurance if the agent has a positive third utility derivative.

To clarify the different precautionary motives and their relations operative in the different cases, and to compare them later on across the different instruments, we introduce finally the following definition. This definition applies interchangeably to each instrument, because our modeling supposes that the agent’s resources and choices are measured in the same dimension, namely money.

**Definition 1** Be instrument \( a \) either saving \( s \), self-protection \( x \), or self-insurance \( z \). The

- Loss-Risk Precautionary Motive in Instrument \( a \) (LPM\(_a\)) is the additional use of this instrument beyond consumption smoothing due to the riskiness of the loss risk in the future, \( a^{ls} - a^{0s} \);

- Income-Risk Precautionary Motive in Instrument \( a \) (IPM\(_a\)) is the additional use of this instrument due to the income risk in the future, \( a^{lis} - a^{ls} \);

- Total Precautionary Motive in Instrument \( a \) (TPM\(_a\)) is the additional use of this instrument due to the riskiness of future consumption, \( TPM_a = LPM_a + IPM_a = a^{lis} - a^{0s} \);

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\(^{8}\)The coinsurance problem is a special case of optimal self-insurance. It is obtained by setting \( z = \alpha(1 + \lambda')pL \) and \( L(z) = (1 - \alpha)L \) with \( \alpha \in [0,1] \) being the coinsurance rate and \( \lambda' \geq 0 \) being a loading factor.
• shares induced by loss risk, \( l \), or income risk, \( i \), respectively, in the total precautionary reaction of instrument \( a \) are, for \( TPM_a > 0 \),

\[
sh^l_a = \frac{LPM_a}{TPM_a}, \quad sh^i_a = \frac{IPM_a}{TPM_a}
\]

Our definition of the \( IPM_a \) corresponds to the notion of precautionary effort found in the literature on self-protection and self-insurance. Alternatively, it could be defined as \( a^{i*} - a^{0*} \), which would retain the first-order effect of the loss risk but not account for its riskiness. In the present paper we stick to the former definition. Our distinct definition of the \( LPM_a \), however, points to the fact that in intertemporal decisions with risk, also in the cases with self-protection and self-insurance, a precautionary motive is operative already in response to the loss risk. Precautionary choices arise thus even when the second-period income risk is absent. As a consequence, the total precautionary response in instrument \( a \) to future risk is composed of the two components, \( LPM_a \) and \( IPM_a \). Of course, the shares induced by loss and income risk in the total precautionary reaction of the instrument add up to one, \( sh^l_a + sh^i_a = 1 \). For the given risks, the \( TPM_a \) is positive as soon as the agent is prudent.

3 Precautionary Behaviors with Two Decisions

We turn to the settings with two instruments out of saving, self-protection, and self-insurance. We put a particular focus on the interaction of the choices within each instrument pair. In order to emphasize the structural similarity of the results, we start with an overarching analysis for two instruments \( a_1 \) and \( a_2 \), with \( a_1, a_2 \in \{ s, x, z \} \) and \( a_1 \neq a_2 \).

Starting from the intertemporal utility objective (1) for the case of certainty, we first consider the introduction of the loss risk. As noted above, the loss risk will generate two sets of effects. On the one hand, the first-order effect in form of the lower expected future consumption will activate consumption-smoothing preference. On the other hand, the associated riskiness may induce precautionary choices. In the two-instrument cases, the agent will, in general, use both instruments to deal with these two aspects, and the instruments may interact in a non-trivial way. As the focus in this paper is on precautionary motives, we compare the agent’s choices in two situations: the face of the loss risk, and under the first-order effect only.

When the loss risk, but no further risk is present, the agent chooses the levels \((a_1^{l*}, a_2^{l*})\) of the two instruments maximizing the objective

\[
U^l(a_1, a_2) = u(y - a_1 - a_2) + \beta V^l(a_1, a_2)
\]

(2)
$V^l(.)$ represents the second-period expected utility with endogenous loss risk according to the instrument pair under consideration. $V^l$ inherits its properties from its constituent functions. We show in Remark 1 that it is in each case concave in its two arguments and has a negative cross-derivative. The associated first-order conditions are, for $j = 1, 2$,\footnote{These conditions can also be stated in the Euler form, $\beta \frac{\partial V_j(a_1, a_2)}{\partial (y - a_1 - a_2)} = 1$.}

\[-u'(y - a_1 - a_2) + \beta V^l_j(a_1, a_2) = 0 \tag{3}\]

We focus on interior solutions in this paper, and assume throughout that the second-order conditions hold. Intuitively, conditions (3) claim that, at the optimum, the effect on expected utility of a marginal dollar is the same whichever instrument is used.

The consumption-smoothing and precautionary components of the choices of the two instruments can be distinguished, in analogy to above, by determining the optimal choices $(a^{0*}_1, a^{0*}_2)$ maximizing utility under the first-order effect only:

\[U^0(a_1, a_2) = u(y - a_1 - a_2) + \beta v\left(E\tilde{C}^l(a_1, a_2)\right) \tag{4}\]

where $\tilde{C}^l(.)$ is the second-period consumption with endogenous loss risk. In the following, we will write for the second-period utility in this case also $V^0(a_1, a_2) = v\left(E\tilde{C}^l(a_1, a_2)\right)$. We denote the solution to the analogous problem without risk $(a^{0*}_1, a^{0*}_2)$.

Their potential interaction makes the optimal decisions non-separable. Indirect, or substitution, effects between the instruments are to be considered. We summarize the arising effects in

**Proposition 1** Let $V^l_j$, for $j = 1, 2$, be convex in consumption, and $V^l_{12} < 0$. The riskiness of the loss risk generates two effects. On each instrument, there is a positive precautionary effect and a negative substitution effect. At least one instrument will be used more compared to the situation where only the first-order effect is considered.

The intuition is quite clear. $V^l_j$ represents the marginal expected utility benefit of instrument $j$. Convexity in wealth implies that the riskiness of the loss risk increases the marginal benefit whereas the marginal cost is unaffected. The direct effect is that both instruments should be used more. However, the two instruments also compete for resources in the first period, and the use of one instrument diminishes the marginal benefit of using the other. Consequently, if one instrument is used more, this is bad news for the attractiveness of the other, which explains the negative indirect effects.

To derive formally the two effects, evaluate $U^l$ in equation (2) at $a^{0*}_j$ and solve for the other instrument, $\hat{a}_k = \arg\max_{a_k} U^l_j(a^{0*}_j, a_k)$, with $j, k = 1, 2, k \neq j$. Consider $j = 1$. To compare $a^{0*}_1$ to
evaluate

\[
U_1^l(a_1^{0*}, \hat{a}_2) = -u_1(y - a_1^{0*} - \hat{a}_2) + \beta V_1^l(a_1^{0*}, \hat{a}_2)
\]

\[
= \beta \left[ V_1^l(a_1^{0*}, a_2^{0*}) - V_1^0(a_1^{0*}, a_2^{0*}) \right] - u_1(\hat{c}) + u_1(c_0) + \beta \left[ V_1^l(a_1^{0*}, \hat{a}_2) - V_1^0(a_1^{0*}, a_2^{0*}) \right].
\]

The first summand is the positive precautionary effect. The marginal benefit of the first instrument is convex in consumption. Therefore, when considering the riskiness of the loss risk on top of its first-order effect, the marginal expected utility benefit of using the first instrument becomes larger.

The second term refers to the marginal cost of using the first instrument. The pure change in the second instrument is positive (\(\hat{a}_2 > a_2^{0*}\)) so that \(\hat{c} < c_0\), because more money is spent on the second instrument. As a result, the marginal cost of using the first instrument is larger. Finally, the third term is negative due to the substitution between the two instruments. The negative cross-derivative \((V_{12}^0 < 0)\) indicates that the marginal benefit from using the first instrument is lower when the second instrument is used more. We see that the second together with the third term represent the negative substitution effect.

The last statement in Proposition 1 follows by analyzing \(U_1^l\) in the \((a_1, a_2)\)-plane. We take the directional derivative into the direction \(\nu = (\nu_1, \nu_2) \in \mathbb{R}^2\) and evaluate it at \((a_1^{0*}, a_2^{0*})\):

\[
\partial_{\nu}U_1^l(a_1^{0*}, a_2^{0*}) = \nu_1 U_1^l(a_1^{0*}, a_2^{0*}) + \nu_2 U_1^l(a_1^{0*}, a_2^{0*})
\]

\[
= \nu_1 \beta \left[ V_1^l(a_1^{0*}, a_2^{0*}) - V_1^0(a_1^{0*}, a_2^{0*}) \right] + \nu_2 \beta \left[ V_1^l(a_1^{0*}, \hat{a}_2) - V_1^0(a_1^{0*}, a_2^{0*}) \right].
\]

The last equation follows by using the first-order condition from the problem where the loss risk is replaced by its mean. Due to the assumption that the marginal expected utility benefit of each instrument is convex in wealth, both bracketed expressions are positive. As a result, not both \(\nu_1\) and \(\nu_2\) can be negative when moving towards the new equilibrium \((a_1^{l*}, a_2^{l*})\). Consequently, at most one of the two substitution effects can prevail at the new optimum. Figure 1 illustrates this aspect.

The next step is to introduce the income risk. The agent chooses the levels \((a_1^{li*}, a_2^{li*})\) maximizing

\[
U^{li}(a_1, a_2) = u(y - a_1 - a_2) + \beta V^{li}(a_1, a_2)
\]

where superscript \(li\) indicates the presence of both loss and income risk. \(V^{li}\) is expected second-period consumption utility where the expectation operator refers to the joint distribution. Due to the assumption of an independent income risk, it is equal to the product of the two individual distributions.
The first-order conditions arise, for \( j = 1, 2 \), as

\[-u'(y - a_1 - a_2) + \beta V_{lj}(a_1, a_2) = 0 \quad (5)\]

The intuition is as before: at the optimum a marginal dollar has the same effect on intertemporal expected consumption utility whichever instrument is used. As before, instruments will, in general, interact in their use implying the presence of indirect effects. The proof of the arising effects is analogous to the one of Proposition 1. We summarize the result in

**Proposition 2** Let \( V_{lj} \), for \( j = 1, 2 \), be convex in consumption, and \( V_{12}^{li} < 0 \). Then, the income risk has on each instrument a positive precautionary effect and a negative substitution effect. At least one instrument will be used more.

Propositions 1 or 2 claim that future utility in objective (4) with the first-order effect of the loss risk only is concave, the future marginal utilities associated with the problems with loss risk only (3) and loss and income risks (5) are convex in consumption, and in all three situations the cross-partial derivatives of future utility with respect to the two arguments are negative. Remark 1 states that these assumptions hold in all three two-instrument cases, if second-period utility exhibits prudence.

**Remark 1** Given the above assumptions on second-period utility \( v \) and the self-protection and self-insurance technologies as well as prudence in \( v \) (\( v''' > 0 \)), Propositions 1 or 2 apply in all three two-instrument cases.

See Appendix A.1 for the proof.
We renounce in this paper from stating a large set of theoretical comparative statics. While all single effects we obtain are intuitive, the overall effects are, in general, ambiguous. In Section 5, we study rather the risk reactions in the different settings numerically for a median agent. Preparing these quantitative comparisons, we extend now Definition 1 to the case with two instruments. The definitions of the loss-risk, the income-risk, and the total precautionary motive for one specific instrument naturally apply also in the two-instrument cases. We add the following concepts.

**Definition 2** For instruments $a_j$, $j = 1, 2$, with $a_j \in \{s, x, z\}$ and $a_1 \neq a_2$, the

- Loss-Risk Precautionary Motive ($LPM$) is the sum of the loss-risk precautionary motives in the two instruments, $LPM = LPM_1 + LPM_2$;

- Income-Risk Precautionary Motive ($IPM$) is the sum of the income-risk precautionary motives in the two instruments, $IPM = IPM_1 + IPM_2$;

- Total Precautionary Motive ($TPM$) arises, equivalently, as the sum of the loss-risk and income-risk precautionary motives across the two instruments or the sum of the total precautionary motives of the two instruments, $TPM = LPM + IPM = TPM_1 + TPM_2$;

- share that instrument $a_j$ contributes to the $TPM$ is, for $TPM > 0$, $sh_j = \frac{TPM_j}{TPM}$.

We visualize the definitions for the precautionary motives the two-instrument cases in Table 1. Obviously, across the instruments the shares add up to one, $\sum_{j=1}^{2} sh_j^{TPM} = 1$.

<table>
<thead>
<tr>
<th>Instrument 1</th>
<th>Instrument 2</th>
<th>Both Instruments</th>
</tr>
</thead>
<tbody>
<tr>
<td>Loss Risk</td>
<td>$LPM_1$</td>
<td>$LPM_2$</td>
</tr>
<tr>
<td>Income Risk</td>
<td>$IPM_1$</td>
<td>$IPM_2$</td>
</tr>
<tr>
<td>Total Risk</td>
<td>$TPM_1$</td>
<td>$TPM_2$</td>
</tr>
</tbody>
</table>

Using these concepts, Propositions 1 and 2 have the following

**Corollary 1** Under prudence, the shares $sh_j^{TPM}$, $j = 1, 2$, in the total precautionary motive cannot be zero simultaneously for all instruments. Given a positive (negative) $TPM$, the share $sh_j^{TPM}$ is negative (positive) for instrument $j$ if and only if the substitution effect prevails in that instrument.

We finally turn to the question whether and to what extent the possibility of precautionary choices in a second instrument leads to systematic increases or decreases in precautionary choices in the first instrument. This has important implications for empirical identification strategies and the measurement of the strength of precautionary motives. If risks are endogenous, e.g., via self-protection or self-insurance, but are treated as exogenous due to the lack of good proxies for individual measures
of risk mitigation, this may lead to the systematic over- or underprediction of other precautionary choices. This in turn would imply that predictions about individual preference parameters deviate systematically from their true values. To achieve comparability, we compare the precautionary choice in one instrument in a setting with second instrument to a setting in which this second instrument is exogenous but the latter is fixed at the level it had in the endogenous case when the zero-mean income risk is present. This allows us to isolate the effect of neglecting the endogeneity of the second instrument on the choice of the first. The following proposition summarizes our results.

**Proposition 3** Treating the second instrument as exogenous leads to overestimating the first if and only if the expenditures on the second increase due to the introduction of a zero-mean income risk.

The intuition is that when the second instrument is productive, in the sense that it is desirable for the decision-maker to invest more in this instrument as a reaction to the zero-mean income risk, then the agent will depend less on precautionary choices in the first instrument and its level will be lower, compared to a situation in which the first instrument is indeed the only decision. Consequently, neglecting other endogenous determinants of risk exposure leads us to theoretically overestimate the precautionary response in the first instrument. For example, if self-protection or self-insurance is available as a second instrument in addition to saving, mere inference from consumption data can provide unrealistically low intensities of prudence.

### 4 Saving, Self-Protection, and Self-Insurance

The extension of the analysis for two instruments in Section 3 to three and more is formally straightforward. This section sets out the case with all three instruments we consider in this paper. We discuss with respect to it the particularities that arise in the cases with more than two instruments.\(^\text{10}\)

If saving, self-protection, and self-insurance are available to optimize intertemporal consumption, the agent’s problem reads

\[
\max_{x,z,s} \left\{ u(y - x - z - s) + \beta [p(x)v(Y + s(1 + r) - L(z)) + (1 - p(x))v(Y + s(1 + r))] \right\},
\]

with associated first-order conditions

\[
U_x^l = -u'(c) + \beta p'(x) (v(Y + s(1 + r) - L(z)) - v(Y + s(1 + r))) = 0,
\]

\[
U_z^l = -u'(c) - \beta p(x)L'(z)v'(Y + s(1 + r) - L(z)) = 0,
\]

\(^{10}\)Note that Menegatti and Rebessi (2011) study the three instruments saving, self-protection and (market) insurance but do not consider future income risk.
\[ U_s^t = -u'(c) + \beta(1 + r) \left( p(x)v'(Y + s(1 + r) - L(z)) + (1 - p(x))v'(Y + s(1 + r)) \right) = 0. \]

In terms of notation, we use the same conventions as above.

Because the analog of Proposition 1 is analogous to the one corresponding to Proposition 2, we only state the extension of the latter to the case with three instruments here.

**Proposition 4** For a prudent individual, the income risk has on each instrument among saving, self-protection, and self-insurance a positive precautionary effect and a negative substitution effect. At least one of the three instruments will be used more in the new equilibrium.

See Appendix A.3 for the proof.

When all three instruments can be used, also the effect of a precautionary response in one instrument to some additional risk to second-period consumption on the cross-effect between the two other instruments can be considered. The cross-effects are described by the cross-derivatives of the utility objective with respect to the two other variables as in system (8). Clearly, all instruments are mutual substitutes. In each case, the effect from a positive precautionary response in one variable on the cross-effect can be seen from the derivative of these cross derivatives with respect to the third decision variable:

\[ U_{xzs}^{hi} = -u'''(c) - \beta(1 + r) p'(x) L'(z) \frac{E v''(\bar{Y} + s(1 + r) - L(z))}{E v'(\bar{Y} + s(1 + r) - L(z))} \cdot \frac{u'(c)}{u'(c)}. \] (6)

The effect from a positive precautionary response on the cross-effects on the marginal cost of the other instruments is negative, while its effect on the cross-effect between the marginal benefit of the other instruments is positive. The prevalence of one of these two effects determines the sign of derivative (6). Remark 2 states the arising endogenous condition.

**Remark 2** A positive precautionary response in the third instrument will strengthen (weaken) the cross-effect between the first and second variables if and only if Ross prudence of present consumption is sufficiently low (high) in the following sense:

\[ -\frac{u'''(c)}{u'(c)} < (>) \beta(1 + r) p'(x) L'(z) \frac{E v''(\bar{Y} + s(1 + r) - L(z))}{E v'(\bar{Y} + s(1 + r) - L(z))} \cdot \frac{u'(c)}{u'(c)}. \] (7)

The intuition for this finding is straightforward. The substitution effect between two instruments leads to lower precautionary responses than when each instrument is considered in isolation. If (Ross) prudence is very high,\(^{11}\) the presence of the third instrument weakens the cross-effect between the two

\(^{11}\) - as a coefficient of prudence has been treated by Modica and Scarsini (2005). The present terminology has been introduced by Denuit and Eeckhoudt (2010), who generalize coefficients of higher-order risk aversion that carry \( u'(c) \) in the denominator to the \( n^{th} \) order.
so that it is less likely that the substitution effect dominates. In this sense, high intensities of prudence give less room for weak or even negative precautionary responses in multiple-instrument situations.

Finally, we turn again to the question whether and to what extent the possibility to use other risk-management instruments leads to systematic increases or decreases in precautionary saving. The following proposition summarizes our results.

**Proposition 5** Treating the risk of loss as exogenous implies an overestimation of precautionary saving if and only if the sum of the expenditures on self-protection and self-insurance increases due to the introduction of a zero-mean income risk.

See Appendix A.4 for the proof.

Similar to the case with two decisions, it is apparent that the precautionary saving response will depend on the use of the other two instruments at hand. Consequently, these choices need to be fully identified to correctly quantify the overall precautionary motive.

## 5 Numerical Analysis

We study now the precautionary motives and their interaction among the different risk-mitigation instruments from above numerically. The goal is to quantify and compare the proportions the precautionary choices in the different instruments take in settings with one or more instruments. To analyze the determinants, we consider the comparative statics of the technology and risk parameters on the optimal choices of a median subject which we define below.\(^\text{12}\)

For the analysis, we need to operationalize the decision-maker’s preferences, the technologies to influence the loss risk, and the income risk, and adopt appropriate parameter values. Regarding preferences, we stick to Expected Utility in this paper. In that case, utility curvature represents mostly intertemporal, not risk, preferences, whence risk responses are somewhat stronger than under a utility specification that disentangles preferences in the risk and time domains (Bostian and Heinzl 2014). We use expo-power utility to specify preferences (Saha 1993, Holt and Laury 2002),

\[
f(c) = \frac{1}{\alpha} \left[ 1 - \exp \left( -\alpha \cdot \frac{c^{1-\rho}}{1-\rho} \right) \right]
\]

The expo-power form has the ability to capture increasing, decreasing, or constant shapes of the relative resistance to intertemporal substitution, and exhibits a decreasing absolute resistance to in-

\(^{12}\)Bostian and Heinzl (2014) study influences of preference and financial parameters on precautionary saving choices. Similar results would obtain here.
temporal substitution (except in the edge case of $\rho = 0$, where it is constant).\(^{13}\)

For the self-protection and self-insurance technologies there is no standard modeling in the literature. In coherence with the assumptions above, we model the self-protection technology as:

$$p(x) = p_0 e^{-\mu x}$$

where $p_0 \in (0, 1)$ is the baseline probability of loss $L$ and $\mu > 0$ is an efficiency parameter. For the self-insurance technology, we assume

$$L(z) = L_0 e^{-\nu z}$$

where $L_0 < Y + s(1 + r)$ is the baseline level of the potential loss and $\nu > 0$ is an efficiency parameter.

We implement the zero-mean risk on future income as mean-preserving spreads around the certain baseline level of $Y$ using a factor $\delta \in (0, 1)$, which shrinks or increases $Y$ by multiplication with $(1 - \delta)$ or $(1 + \delta)$, respectively, with probability 1/2.

We consider a one-month time horizon between the decision and the realization of the risk, for ease of presentation. The effects are qualitatively the same for longer time horizons. We assume that the per-period felicity function does not change during this period, so that $u \equiv v$. Our baseline values for the utility and economic parameters are

$$\alpha = 3 \quad \rho = 1.5 \quad \beta = 0.99 \quad y = Y = $1,000 \quad r = 0.01$$

The $\rho$ parameter corresponds to typical empirical macroeconomic estimates, and the positive $\alpha$ adds a decreasing shape to the relative resistance to intertemporal substitution (increasing elasticity of intertemporal substitution) (e.g., Meyer and Meyer 2005). We set the discount factor $\beta$ to 0.99 (corresponding to a monthly discount rate of 1.01 percent and an annual rate of 12.82 percent) (e.g., Andersen et al. 2008). The baseline income of $1,000 is on the order of monthly income for the first and second quintiles of the US population. The baseline return $r$ to saving of 1 percent gives a fairly strong incentive to save.

For the baseline risk and technology parameters, we assume

$$L_0 = $250, \quad p_0 = 0.5, \quad \mu = \nu = 0.2, \quad \delta = 0.3$$

\(^{13}\)Its coefficients of absolute and relative resistance to intertemporal substitution are, respectively,

$$ARIS(c) = \frac{\alpha}{e^c + \frac{\rho}{c}}, \quad RRIS(c) = \alpha c^{1-\rho} + \rho$$

Excluding pathological parametrizations, the expo-power function has derivatives with alternating sign.
The size of $L_0$ can be interpreted in relation to the assumed monthly income of $1,000. Thus, the baseline loss risk we consider posits that the agent will suffer with a chance of one in two a loss on the order of a quarter of monthly income. For the efficiency parameters, we are not aware of studies that could give empirical support for one or another value. We choose the values of $\mu$ and $\nu$ so that coherent results arise across all scenarios we consider based on the modeling framework in this paper. We set them equal so as to allows to detect eventual differences in their comparative effectiveness. $\delta$ measures directly fractions of the monthly income. Income fluctuations on the order of 30 percent certainly correspond to the normal experience, for example, of self-employed persons, including small farmers, or freelance workers in the service industry.

5.1 One Instrument and Two Risks

We study first the comparative statics of the precautionary motives in each of the three instruments separately. The theoretical literature has thus far focused on these single-instrument cases. To the best of our knowledge, past empirical or numerical studies have only concerned precautionary saving. Our saving case is still different from the typical approach in the literature, because we consider a setting where, with the potential loss, an additional risk is present. We start with the saving case as a benchmark and compare subsequently the cases with self-protection or self-insurance only.

Saving

Figure 2 shows the comparative statics of the precautionary saving motives in response to the loss risk and the additional income risk, as well as the total precautionary motive for $L \in (0, 800]$, $p \in (0, 0.5]$, $\delta \in (0, 0.4]$, and $r \in (0, 1]$. The precautionary motives are represented as the precautionary fraction of the total choice in this instrument. The two upper graphs show that, for a given income risk, increases in $L$ or $p_0$ expand the share of the $LPM_s$ in the $TPM_s$ at the expense of the share of the $IPM_s$. In addition, due to the inverse wealth effect, the $TPM_s$ tends to shrink in favor of consumption smoothing. For the considered parameter ranges, the latter first-order effect is stronger in the loss size than in its probability. Moreover, increases in loss size stimulate the $LPM_s$ more strongly than increases in the loss probability. For the baseline parameter values, the $IPM_s$ is clearly larger than the $LPM_s$. For example, if $L$ or $p_0$ tend towards zero, the $IPM_s$ is at about 0.9, whereas, if $\delta$ is zero, the $LPM_s$ is at about 0.16. However, the $LPM_s$ may exceed the $IPM_s$ if the loss size is large enough. $\delta$ adds a pure risk to the decision maker’s choice environment. Its expansion increases the $IPM_s$ at the expense of both the $LPM_s$ and consumption smoothing. The graph at the bottom-right shows the comparative statics is for a monthly interest rate of up to 100 percent. An expansion of $r$
leaves the shares of $LPM_s$ and $IPM_s$ in the $TPM_s$ unchanged, but the $TPM_s$ decreases in favor of consumption smoothing due to the related first-order effect.

![Graphs showing comparative statics of precautionary saving motives for different parameters.](image)

**Figure 2**: Comparative statics of precautionary saving motive for $L \in (0, 800]$, $p \in (0, 0.5]$, $\delta \in (0, 0.4]$, and $r \in (0, 1]$.

**Self-Protection**

If self-protection is the agent’s only instrument, the precautionary motives in response to the two risks are similarly operative (Figure 3). But the fractions of the loss-risk and income-risk as well as the total precautionary motives in the total effort choice are, in general, much lower than in the case with saving only. Thus, for $L$ towards zero the $IPM_x$ is at about 0.22, and for $\delta$ towards zero the $LPM_x$ is at about 0.08. For the most part of the considered parameter ranges, the $TPM_x$ stays below 0.25. Self-protection seems thus, in general, much more effective to cope with the loss and income risks than saving. An exception is the case of small baseline loss probabilities ($p_0 < 0.05$), where with decreasing $p_0$ also the $LPM_x$ rises and the $TPM_x$ can reach 100 percent. In general, the $LPM_x$ is smaller than the $IPM_x$, but may exceed the $IPM_x$ for $L$ large enough or $\delta$ small enough. The graphs show only very few interaction between the risk-wise precautionary motives. Notably, $\delta$ has not a
Figure 3: Comparative statics of precautionary motive in $x$ for $L \in (0,500]$, $p_0 \in (0,0.5]$, $\delta \in (0,0.4]$, and $\mu \in (0,1]$.

large influence on the $LPM_x$. An increase in the efficiency of self-protection $\mu$ reduces the $TPM_x$, decreasing similarly $LPM_x$ and $IPM_x$.

Self-Insurance

Also if the agent disposes only of self-insurance, both the loss- and the income-risk precautionary motives are active (Figure 4). However, the total precautionary motive is still smaller than in the case with self-protection only. While the $IPM_x$ is at about 0.22 when the baseline loss tends towards zero, the $LPM_x$ remains tiny unless for loss probabilities ($p < 0.2$). Self-insurance occurs thus as a most effective instrument to cope with the loss risk and, to a lesser degree, also with the income risk. By implication, self-insurance is mostly made to smooth consumption. Similar to $\mu$ in the self-protection case, $\nu$ has a monotonically decreasing effect on the precautionary motives. The share of the $LPM_x$ in the $TPM_x$ tends towards zero fairly quickly in favor of the $IPM_x$. 
5.2 Instrument Interaction

From a real-world perspective, assuming that an agent disposes only of one instrument to cope with future risks occurs certainly as stylized. How does admitting two or three instruments influence the precautionary motives an agent expresses? The graphs in this section present the $IPM_a$ and the $TPM_a$ for each instrument $a \in \{s, x, z\}$ as well as the $TPM$ across all instruments. (In each case, the $LPM_a$ are implicit as $TPM_a - IPM_a$.)

Two Instruments

Because the precautionary motives are generally lower than in the single-instrument cases, we reduce the scale of the precautionary motives in the graphs for two instruments to the range of $[-0.05, 0.2]$.

Saving and Self-Protection. Figure 5 shows the comparative statics of the precautionary motives for $L \in (0, 600]$, $p_0 \in (0, 0.5]$, $\delta \in (0, 0.4]$, and $\mu \in (0, 1]$. Generally, the precautionary motives are much smaller than in the cases with saving or self-protection only. Consistent with the single-instrument cases, also here the total as well as the risk- and instrument-wise precautionary motives decrease with
the baseline loss probability and the efficiency parameter $\mu$ and increase with $\delta$. However, in contrast to before, the $IPM_s$ and $IMP_x$ as well as, for $L$ above about $100$, the $TPM$, $TPM_s$, and $TPM_x$ rise with the loss size. At the same time, for a high enough loss size the $LPM_x$ crowds out the $LPM_s$ and, thus, precautionary saving. A similar instrument interaction can be observed in the comparative statics for $\delta$ and $\mu$. An increase in $\delta$ leads, interestingly, to a slight reduction in the $IPM_s$ and $IPM_x$. By contrast, the $LPM_s$ and $LPM_x$ and the $TPM_s$ and $TPM_x$ consistently rise, slightly more for self-protection than for saving. Increase in $\mu$ consistently decrease the precautionary motive across all instruments and risk sources. More efficient self-protection reduces precautionary saving more strongly than precautionary self-protection efforts, especially in response to the loss risk.

![Figure 5](image)

Figure 5: Comparative statics of precautionary motives for $L \in (0, 600]$, $p_0 \in (0, 0.5]$, $\delta \in (0, 0.4]$, and $\mu \in (0, 1]$.  

**Saving and Self-Insurance.** Under saving and self-insurance (Figure 6), most effects resemble the case with saving and self-protection, but they are typically more pronounced and involve, in general, lower precautionary choices. Contrary to the previous case, with an increase in loss size now all precautionary motives consistently decrease. For $L$ high enough, precautionary effort in self-insurance crowds out precautionary saving. At a loss size of around $340$, the $TPM_s$ becomes even negative.
Similarly, if the efficiency parameter $\nu$ is above 0.4, the $TPM_s$ turns negative. (The $IPM_s$ is negative already from $\nu$ of about 0.2 on.) In the $\delta$ comparative statics, the $IPM_z$ is consistently close to zero, and the $IPM_s$ is also slightly negative. The $TPM$ response to increases in $\delta$ is for the most part carried by the $LPM_z$ and only to some extent by $LPM_s$.

Figure 6: Comparative statics of precautionary motives for $L_0 \in (0, 350]$, $p \in (0, 0.5]$, $\delta \in (0, 0.4]$, and $\nu \in (0, 1]$.

**Self-Protection and Self-Insurance.** The graphs for the case with self-protection and self-insurance (Figure 7) resemble the case with saving and self-insurance, with some exceptions. The comparative statics in the loss size exhibits a strong crowding out of the precautionary efforts in self-protection by those in self-insurance. For $L_0$ above $200$, the $LPM_x$ and the $IPM_x$ start to shrink, whereas the $LPM_z$ and the $IPM_z$ rise. In both instruments, the former effect is stronger than the latter. At a loss size of about $480$, the $TPM_x$ turns negative. Self-insurance seems, thus, clearly more effective than self-protection for precautionary purposes. Increases in the baseline loss probability make all precautionary motives monotonically decrease. When $\delta$ increases, the rise of $TPM$ is again mostly carried by the $LPM_x$ and the $LPM_z$. The $IPM_x$ and the $IPM_z$ are very small but stay positive. The efficiency parameter of self-protection $\mu$ has only a very mild influence on the precautionary
motives. For about $\mu > 1.2$, the $LPM_z$ decreases and with it the $TPM$. All precautionary motives stay positive. An increase of $\nu$ decreases clearly the $TPM$ by reducing the $LPM_z$ from about 0.04 to almost zero, the $IPM_x$ is slightly negative from $\nu$ of about 0.4 on.

![Graphs showing comparative statics]

Figure 7: Comparative statics of precautionary motives for $L_0 \in (0, 500]$, $p_0 \in (0, 0.5]$, $\delta \in (0, 0.4]$, and $\nu \in (0, 1]$.  

Saving, Self-Protection, and Self-Insurance

Figures 8-10 show the comparative statics of the precautionary motives in the three instruments for $L_0 \in (0, 500]$, $p_0 \in (0, 0.5]$, $\mu \in (0, 1]$, and $\nu \in (0, 1]$. For a better summary impression, we add graphs with the share that each instrument contributes to the $TPM$. Because the precautionary motives are generally lower than in the two-instrument cases, we reduce the scale of the precautionary motives in the graphs, in general, to the range of $[-0.05, 0.15]$. Similar to the cases with self-insurance before, the comparative statics in the loss size exhibit a monotonically decreasing shape of the $TPM$. Now, investments in self-insurance crowd out both saving and self-protection, the former more strongly than the latter. At a loss size of $500$ even both the $TPM_z$ and the $TPM_x$ are negative, with both of their components. The shares in the $TPM$ show how strongly self-insurance appropriates the precautionary
reaction at the expense first of saving, but also of self-insurance. The comparative statics in the loss probability show the same consistently declining shape of all precautionary motives as in all cases with endogenous risk before. The crowding among the three instruments has the same tendency as under loss-size increases but is much less pronounced.

Figure 8: Comparative statics of precautionary motives and shares in TPM for $L_0 \in (0, 500]$ and $p_0 \in (0, 0.5]$.

The comparative statics in $\delta$ is very similar to the case with self-protection and self-insurance. However, the $TPM$ are slightly, and the $TPM_x$ and the $TPM_z$ clearly lower. Again, the $IPM_a$ for $a \in \{s, x, z\}$ stay virtually unchanged in the considered range of $\delta$. Most precautionary reaction is taken up by the $LPM_a$. At the same time, the shares in the $TPM$ stay mostly the same for all three instruments.

Also the comparative statics in the efficiency parameters $\mu$ and $\nu$ resemble previous cases. Like under self-protection and self-insurance, the precautionary motives first increase up to a level of $\mu$ of about 0.1 and then consistently decrease. The initial increase contrasts with the case with saving and self-protection. Generally, the levels of the precautionary motives are lower. Self-protection crowds out saving and self-insurance, the former more than the latter. For $\mu$ high enough, self-
Figure 9: Comparative statics of precautionary motives and shares in TPM for $\delta \in (0, 0.4]$.

Protection will be used the most for precautionary purposes. The $IPM$ stay tiny for all instruments. With $\nu$ all precautionary motives consistently decrease. The higher $\nu$ the more the $TPM$ is mostly determined by the $TPM_z$. For $\nu$ high enough, the $IPM_s$ and the $IPM_x$ as well as the $TPM_s$ and the $TPM_x$ go negative, but less strongly than, where this occurs, in the two-instrument cases. The comparison of the comparative statics of the shares in the $TPM$ in the two cases shows how much more strongly self-insurance may appropriate the precautionary reaction than self-protection for high levels of the efficiency parameters. In the presence of self-protection and self-insurance, saving has the least important role to play in the precautionary reaction in both cases, which still decreases with the efficiency levels.

5.3 Summary

We summarize our above simulation results and discuss some implications. In the cases with one instrument, each instrument reacts to each risk, whence in different degrees. The largest precautionary-saving motives by far are associated with saving. Self-protection and self-insurance react much less, self-protection still more than self-insurance. In the most part of the considered parameter ranges the $IPM_s$ clearly dominates the $LPM_s$. While the $IPM_x$ and the $LPM_x$ are roughly equal, the $LPM_z$ under self-insurance is only tiny. In all cases, the first-order effect associated with the loss risk is clearly visible. With rising loss size or probability, consumption smoothing prevails at the expense of precaution. Self-insurance is mostly used for consumption smoothing.

When a second risk-management instrument is admitted, the total precautionary motive is, in each case, lower than in the corresponding cases with only one instrument. The precautionary reactions in self-protection and in self-insurance dominate precautionary saving. Similarly, precautionary self-insurance dominates precaution in self-protection. If in the scenarios with self-insurance the loss size
is large enough, the total precautionary motive in saving or self-protection, respectively, may even be negative. Thanks to instrument interaction, the $IPM_z$ may be very small and the $IPM_s$ even negative for rising income risk $\delta$ in the setting with saving and self-insurance.

If all three instruments are available, all effects tend to be still more pronounced. The total precautionary motive across all instruments is once more smaller than in the corresponding two-instrument cases. Self-insurance choices clearly dominate the precautionary reactions in saving and self-protection, unless self-protection is particularly productive. While, in general, much smaller than self-insurance, self-protection is still more used than saving.

These results hinge, obviously, on the chosen parameter values. Especially, the implementation and parameterization of the self-protection and self-insurance technologies are purely illustrative in this paper. At the same time, the strong predicted interactions between the instruments in precautionary choices suggest the importance to control for a maximum of instruments if precautionary behaviors are studied from field data.
6 Conclusion

We study situations where a decision-maker has one, two, or three instruments available to maximize expected intertemporal consumption utility in the face of a future loss and a future income risk. As is well known, saving, self-protection and self-insurance each exhibit a precautionary increase after the introduction of a zero-mean income risk when the agent is prudent. We argue that the loss risk already stimulates consumption smoothing, due to its first-order effect, and precaution, due to its riskiness. We distinguish a precautionary motive in response to the loss risk, and one in response to the income risk. Both add up to the total precautionary motive in an instrument. We show that when the agent uses more than one instrument, controlling for consumption smoothing each risk implies a positive precautionary effect and a negative substitution effect in each instrument. Consequently, in the multi-instrument cases the total precautionary motive, which is the net of all individual effects, will typically differ from a mere summation of the effects in the single-instrument cases.

We study the different precautionary motives for the single- and multi-instrument cases numerically. In the single-instrument cases, saving shows clearly the highest precautionary choices before self-protection. Self-insurance is mostly used for consumption smoothing. If two instruments compete for resources, the total precautionary motive is consistently lower than in each single-instrument case, and saving is used the least among the three instruments. Self-insurance, in turn, tends to be more used than self-protection. In the three-instrument case, the total motive is still lower, and the other effects from the cases with two instruments are still more pronounced.

While our numerical results are bound to a particular parameterization, they still illustrate that instrument interaction may have important repercussions for the strength of the total precautionary motive and the expression of individual risk and time preferences. When other instruments are not well identified, the precautionary response of the observed variables will systematically deviate from its true value. Hence, when self-protection or self-insurance are productive in the sense that they react positively to the introduction of a risk, neglecting this endogeneity systematically overpredicts the precautionary saving response. This is a potential explanation for the difficulty that arises when measuring the intensity of prudence from field data.
References


A Appendix

A.1 Proof of Remark 1

We only consider the case with saving and self-protection explicitly. In that case,

\[ V_{xx}(x, s) = -p''(x)L + (p'(x)L)^2 v''(Y + s(1 + r) - p(x)L) < 0 \]
\[ V_{xy}(x, s) = (1 + r)^2 v''(Y + s(1 + r) - p(x)L) < 0 \]
\[ V_{xs}(x, s) = -p'(x)L(1 + r)v''(Y + s(1 + r) - p(x)L) < 0 \]
\[ V_{yy}(x, s) = p'(x) [v''(Y + s(1 + r) - L) - v''(Y + s(1 + r))] > 0 \]
\[ V_{yy}(x, s) = (1 + r) [p(x)v''(Y + s(1 + r) - L) + (1 - p(x))v''(Y + s(1 + r))] > 0 \]
\[ V_{xs}(x, s) = (1 + r)p'(x) [v'(Y + s(1 + r) - L) + (1 - p(x))v'(Y + s(1 + r))] < 0 \]

The signs of \( V_{xy}, V_{sy}, \) and \( V_{xs} \) in the situation with loss and income risk can be immediately traced back to the corresponding derivatives in the situation with loss risk only. The same set of signs is easily verified also for the cases with saving and self-insurance, and self-protection and self-insurance.

A.2 Proof of Proposition 3

Because we fix the second instrument in the case of its exogeneity at its optimal level in the case of endogeneity, we obtain that the levels of the first instrument in the presence of the zero-mean income risk are identical in both cases. Consistent with the notation above they are denoted by \( a_1^r \).

Consequently, we need to compare choices of \( a_1 \) in the absence of the zero-mean income risk with and without the opportunity to vary \( a_2 \). Consider the problem

\[ \max_{a_1, a_2} U(a_1, a_2) \quad \text{s.t.} \quad a_2 = a_2^r. \]

The Lagrangian is given by

\[ \mathcal{L}(a_1, a_2; \lambda) = U(a_1, a_2) + \lambda(a_2^r - a_2) \]

with associated first-order conditions

\[ \frac{\partial \mathcal{L}}{\partial a_1} = U_1 = 0, \]
\[ \frac{\partial \mathcal{L}}{\partial a_2} = U_2 - \lambda = 0, \]
\[ \frac{\partial \mathcal{L}}{\partial \lambda} = a_2^r - a_2 = 0. \]

We denote the associated Lagrange multiplier \( \lambda \). Consequently, we obtain that \( a_1^r(\lambda) \) is the level of \( a_1 \) when \( a_2 \) is fixed, whereas \( a_1^r(0) \) is the level of the first instrument when the second is allowed to vary.

Table 2 summarizes this approach.

To compare the two cases, we treat \( \lambda \) as exogenous and use the Implicit Function Theorem:

\[
\begin{pmatrix}
\frac{\partial a_1}{\partial \lambda} \\
\frac{\partial a_2}{\partial \lambda}
\end{pmatrix} = -\frac{1}{U_{11}U_{22} - U_{12}^2} \begin{pmatrix}
U_{22} & -U_{12} \\
-U_{12} & U_{11}
\end{pmatrix} \cdot \begin{pmatrix}
\frac{\partial \mathcal{L}}{\partial a_1} \\
\frac{\partial \mathcal{L}}{\partial a_2}
\end{pmatrix} = \begin{pmatrix}
-U_{22} & U_{12} \\
U_{12} & U_{11}
\end{pmatrix}.
\]
Table 2: Approach in Proof of Proposition 3

<table>
<thead>
<tr>
<th>Instrument(s)</th>
<th>(a_1)</th>
<th>(a_1, a_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No income risk</td>
<td>(a_1^\ast(\hat\lambda), \pi_2 = a_2^\ast)</td>
<td>((a_1^\ast, a_2^\ast))</td>
</tr>
<tr>
<td>Income risk</td>
<td>(a_1^\ast r, \pi_2 = a_2^\ast r)</td>
<td>((a_1^\ast r, a_2^\ast r))</td>
</tr>
</tbody>
</table>

Consequently, we obtain that

\[
a_1(0) - a_1(\hat\lambda) = \int_0^{\hat\lambda} \frac{\partial a_1}{\partial \lambda} d\lambda.
\]

Due to the substitution between saving and self-protection, it is

\[
U_{12} = u''(w) + \beta E_L V_{12}(a_1, a_2) < 0,
\]

so that the integrand is unambiguously positive. Hence, the sign entirely depends on \(\hat\lambda\). Now \(a_1^\ast(0) > a_1^\ast(\hat\lambda)\) which is equivalent to the fact that the precautionary choice of \(a_1\) is larger if \(a_2\) is exogenous, if and only if \(\hat\lambda\) is negative, which is the case if and only if \(a_2^\ast < a_2^\ast r\).

A.3 Proof of Proposition 4

The proof for the effects on the single decision variables is analogous to the ones for Propositions 1 and 2. In particular, the substitutive relationship between each two instruments can be seen from the respective cross-derivatives of the utility objective:

\[
U_{1x_L} = u''(c) - \beta p'(x)L'(z)E v'(\tilde{Y} + s(1 + r) - L(z)) < 0 \quad (8a)
\]

\[
U_{1z} = u''(c) + \beta(1 + r)p'(x)E(\tilde{Y} + s(1 + r) - L(z)) - v'(\tilde{Y} + s(1 + r))) < 0 \quad (8b)
\]

\[
U_{1s} = u''(c) - \beta(1 + r)p(x)L'(z)Ev''(\tilde{Y} + s(1 + r) - L(z)) < 0 \quad (8c)
\]

For the statement on the prevalence of positive precautionary responses in equilibrium, consider \(U_{1i}^\ast\) in the \((x, z, s)\)-plane and evaluate the directional derivative into the direction \(\nu = (\nu_1, \nu_2, \nu_3) \in \mathbb{R}^3\) at \((x^\ast, z^\ast, s^\ast)\):

\[
\partial_{\nu} U_{1i}^\ast(x^\ast, z^\ast, s^\ast) = \nu_1 \frac{\partial U_{1i}^\ast}{\partial x}(x^\ast, z^\ast, s^\ast) + \nu_2 \frac{\partial U_{1i}^\ast}{\partial s}(x^\ast, z^\ast, s^\ast) + \nu_3 \frac{\partial U_{1i}^\ast}{\partial s}(x^\ast, z^\ast, s^\ast)
\]

Under prudence, \(\frac{\partial U_{1i}^\ast}{\partial x}(x^\ast, z^\ast, s^\ast), \frac{\partial U_{1i}^\ast}{\partial z}(x^\ast, z^\ast, s^\ast), \text{ and } \frac{\partial U_{1i}^\ast}{\partial s}(x^\ast, z^\ast, s^\ast)\) are each positive so that at least one of \(\nu_1, \nu_2, \text{ or } \nu_3\) must be positive. Consequently, at most two of the three variables can be lower in the new optimum.

A.4 Proof of Proposition 5

Assuming that the probability and size of the loss are identical in the cases with and without self-protection and self-insurance, savings in the presence of the zero-mean income risk, denoted \(s^{li*}\), are identical in both cases. Compare the saving choices in the absence of the zero-mean income risk with
and without the opportunity to self-protect and self-insure. Consider the problem

\[
\max_{x,z,s} U^{li}(x, z, s) \quad \text{s.t.} \quad x = x^{*s}, \quad z = z^{*s}
\]

The Lagrangian is given by

\[
\mathcal{L}(x, z, s; \lambda_1, \lambda_2) = U^{li}(x, z, s) + \lambda_1 (x^{*s} - x) + \lambda_2 (z^{*s} - z)
\]

and the first-order conditions are

\[
\begin{align*}
\frac{\partial \mathcal{L}}{\partial x} &= U_x^{li} - \lambda_1 = 0 \\
\frac{\partial \mathcal{L}}{\partial z} &= U_z^{li} - \lambda_2 = 0 \\
\frac{\partial \mathcal{L}}{\partial s} &= U_s^{li} = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_1} &= x^{*s} - x = 0 \\
\frac{\partial \mathcal{L}}{\partial \lambda_2} &= z^{*s} - z = 0
\end{align*}
\]

The constrained optimum is given by \((x^{*s}, z^{*s}, \hat{s})\) where \(\hat{s}\) maximizes \(U^{li}(x^{*s}, z^{*s}, s)\). The associated Lagrange multipliers \(\lambda_i, i = 1, 2\), equal \(U^{li}_x(x^{*s}, z^{*s}, \hat{s})\) and \(U^{li}_z(x^{*s}, z^{*s}, \hat{s})\), respectively.

Now assume \(\lambda_1\) and \(\lambda_2\) are exogenous to (9) and define the optimal \(x\) and \(z\) as functions of \((\lambda_1, \lambda_2)\). Then, obviously, \((x(\hat{\lambda}_1, \hat{\lambda}_2), z(\hat{\lambda}_1, \hat{\lambda}_2)) = (x^{*s}, z^{*s})\) and \((x(0, 0), z(0, 0)) = (x^{*s}, z^{*s})\). To see how variations in \((\lambda_1, \lambda_2)\) affect the optimal values of \(x, z\), and \(s\), consider the following transformation of system (10):

\[
\begin{pmatrix}
\frac{dx}{ds} \\
\frac{dz}{ds}
\end{pmatrix} = -H^{-1} \left( U^{li}_x(x, z, s) \right) \cdot
\begin{pmatrix}
\frac{\partial \mathcal{L}}{\partial x} & \frac{\partial \mathcal{L}}{\partial s} \\
\frac{\partial \mathcal{L}}{\partial z} & \frac{\partial \mathcal{L}}{\partial s}
\end{pmatrix}
\begin{pmatrix}
\frac{d\lambda_1}{d\lambda_2} \\
\frac{d\lambda_2}{d\lambda_2}
\end{pmatrix} = \frac{\text{adj}(H)}{|H|} \cdot
\begin{pmatrix}
\frac{d\lambda_1}{d\lambda_2} \\
\frac{d\lambda_2}{d\lambda_2}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
\frac{(U_{xx} U_{ss} - U_{sx}^2) d\lambda_1 + (U_{xs} U_{zs} - U_{sx} U_{zs}) d\lambda_2}{U_{xx} U_{zz} (U_{zs} - U_{gs}) + U_{xs}^2 (U_{zs} - U_{gs}) + U_{xs} (U_{xx} U_{zs} - U_{xs} U_{zs})} \\
\frac{(U_{xx} U_{zs} - U_{sx} U_{ss}) d\lambda_1 + (U_{xs} U_{zs} - U_{sx}^2) d\lambda_2}{U_{xx} U_{zz} (U_{zs} - U_{gs}) + U_{xs}^2 (U_{zs} - U_{gs}) + U_{xs} (U_{xx} U_{zs} - U_{xs} U_{zs})}
\end{pmatrix}
\]

where \(H(\cdot)\) is the Hessian matrix of the objective \(U^{li}(x, z, s)\) and \(\text{adj}(H)\) its adjugate.

Consequently,

\[
ds = \frac{(U_{xx} U_{zs} - U_{sx} U_{zs}) d\lambda_1 + (U_{xs} U_{zs} - U_{sx} U_{zs}) d\lambda_2}{U_{xx} U_{zz} (U_{zs} - U_{gs}) + U_{xs}^2 (U_{zs} - U_{gs}) + U_{xs} (U_{xx} U_{zs} - U_{xs} U_{zs})},
\]

(11)
so that the sign of $ds$ is equal to the sign of the numerator in equation (11),

$$
\text{sgn}[ds] = \text{sgn}[(U_{xz}U_{zs} - U_{xs}U_{zz}) \, d\lambda_1 + (U_{xz}U_{xs} - U_{xx}U_{zs}) \, d\lambda_2].
$$

(12)

Given that $(U_{xz}U_{zs} - U_{xs}U_{zz})$ and $(U_{xz}U_{xs} - U_{xx}U_{zs})$ are positive, $ds > 0$, i.e., precautionary saving is larger in the absence of self-protection or self-insurance, if and only if the sum of $d\lambda_1$ and $d\lambda_2$ weighted, respectively, by $(U_{xz}U_{zs} - U_{xs}U_{zz})$ and $(U_{xz}U_{xs} - U_{xx}U_{zs})$ is positive. More generally, it holds thus that

$$
ds > 0 \iff d\lambda_1 > -\frac{U_{xz}U_{xs} - U_{xx}U_{zs}}{U_{xz}U_{zs} - U_{xs}U_{zz}} \, d\lambda_2.
$$