Benchmark-Based Stochastic Dominance

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Within the expected utility framework, utility function and stochastic dominance have long been applied separately in evaluating risky prospects. The former yields a complete ranking but is sensitive to functional specification, while the latter is robust yet often too partial to guide decisions. These limitations stem from their rigidity in adapting to varying granularity of preference information. To enhance flexibility, we propose an integrated approach that combines the respective advantages of both methods. It employs a most plausible utility as a benchmark and allows utility specifications to deviate within interpretable and elicitable bounds. These bounds characterize the extent of incompleteness in preferences and support a closed-form expression for the consensus across admissible utilities in comparing distributions. The resulting ranking criterion, termed benchmark-based stochastic dominance, centers dominance relations within a controllable neighborhood around the benchmark utility, providing a practical semi-parametric tool that yields more comprehensive insights than either a single utility function or standard stochastic dominance alone.

Key words: expected utility, stochastic dominance, benchmark, robustness.

1. Introduction

Expected utility, as a framework for evaluating risky prospects, has been central to decision analysis since the seminal work of von Neumann and Morgenstern (1944). Their theory showed that if four logical axioms—continuity, transitivity, completeness, and independence—hold, preferences over monetary outcomes can be represented by an expected utility measure, which assigns a utility value to each outcome and computes overall utility as the probability-weighted sum of these values. While expected utility offers a practical way to represent preferences, its real-world application faces challenges. It assumes that all prospects are fully comparable, overlooking perceptual and cognitive limitations that may hinder comparisons, and depends on accurate and sufficient information to construct a valid utility function. These challenges are particularly acute when reliable estimation is impeded by insufficient data or when agreement on a common utility function is hard to reach in complex settings such as collective decision-making (Baucells and Shapley 2008). Consequently, the sensitivity of evaluation results to utility function specification has been a persistent concern in the literature (Armbruster and Delage 2015).

Within the expected utility framework, integer-degree stochastic dominance provides an alternative to single utility function in modeling preferences. It captures consensus among utility functions that share certain qualitative characteristics, such as non-satiation and risk aversion. Despite its robustness and minimal information requirements, using a broad set of utility functions that completely ignore quantitative aspects often fails to yield clear preference relations over prospects of interest (Meyer 1977a).

Criticisms of both single utility function and integer-degree stochastic dominance arise from a common source: the rigidity in adapting to varying granularity of preference information. The former method identifies the most plausible representation of decision-makers' preferences but overlooks fluctuations and indecisiveness, while the latter captures broad consensus but disregards any available preference information that could inform decision-makers' typical behavior. The prior literature has separately addressed their limitations, for instance by incorporating utility deviations into optimization processes (Armbruster and Delage 2015, Hu et al. 2018) and imposing restrictions on utility functions in defining stochastic dominance (Meyer 1977b, Leshno and Levy 2002), yet no attempts have been made to integrate the two methods to enhance adaptability.

In this paper, we introduce an integrative method for modeling preferences within the expected utility framework to enhance flexibility and adaptability. The core idea is to retain the most plausible utility function as a benchmark for representing preferences. However, rather than relying solely on this utility, we account for fluctuations in preferences arising from perceptual imperfections and cognitive challenges. Specifically, we expand the set of permissible utility functions to include candidates that deviate from the benchmark to a controllable extent. By regulating the extent of deviation, decision-makers can tailor the utility set and thus the degree of indecisiveness to match the granularity of preference information. The consensus of this utility set, termed *benchmark-based stochastic dominance*, reconciles the two rigid extremes (i.e., single utility function and integer-degree stochastic dominance) and offers users the flexibility to balance decisiveness and robustness.

The success of the proposed method depends on constructing a proper bound that sets the limits of acceptable deviations from the benchmark while satisfying:

(i) *Connectivity.* The bound enables a continuous transition from the benchmark utility function to the utility set underlying stochastic dominance, facilitating the connection of both methods.

(ii) *Interpretability.* The bound has a clear and meaningful interpretation, allowing users to understand the deviation in terms of well-defined economic behavior.

(iii) *Elicitability.* The bound is estimable in practice, so its numerical value can be elicited through simple lottery choices.

(iv) *Tractability.* The consensus of the utility set can be expressed analytically in terms of distribution functions.

A seemingly straightforward way to bound utility deviations is to constrain the pointwise differences between the well-known Arrow-Pratt risk aversion indices (Pratt 1964). This idea was explored by Meyer (1977b), who modified second-degree stochastic dominance by using an interval of the Arrow-Pratt risk aversion index to define the underlying set of utility functions. While this way satisfies criteria (i) to (iii), it falls short with respect to (iv), since the consensus ranking must rely on numerical computation (Meyer 1977a). To date, no research has provided a bound on utility deviations that satisfies all four criteria.

Building on Müller et al. (2017)'s method of assessing risk attitudes via the ratio of marginal utilities between two wealth levels, we propose to bound utility deviations within the utility set underlying first-degree stochastic dominance by limiting the variation of this ratio. When the variation vanishes, the deviating utility function coincides with the benchmark. As the variation increases without bound, the utility set expands continuously to encompass all monotone increasing functions, ensuring a smooth transition to first-degree stochastic dominance and thereby satisfying (i). By ascertaining the role of marginal utility ratio in influencing choice behavior, we show that our bound exactly constrains the difference between the probability premia in simple lottery choices implied by two utility functions, thus meeting criterion (ii). Since the probability premium can be elicited using the widely applied multiple price list approach (Andersen et al. 2006), the bound is naturally elicitable, thus satisfying (iii). Finally, criterion (iv) is also met, as the bound is structurally parallel to that of Müller et al. (2017), ensuring comparable analytical tractability. The corresponding consensus ranking, defined as benchmark-based first-degree stochastic dominance, is both economically meaningful and computationally explicit, offering a tractable interpolation between the benchmark utility function and first-degree stochastic dominance.

Extending the above approach to utility sets underlying higher-degree stochastic dominance is challenging, as imposing first-degree-like conditions on all successive derivatives overly constrains the utility function due to the interdependence between higher- and lower-order derivatives. To address this, we apply the first-degree-like condition only to the highest-degree marginal utility to bound its deviation, and supplement this with an array of endpoint conditions to bound the deviations in all lower-degree marginal utilities. Conceptually, this resembles the use of a higher-order differential equation to characterize a deviating utility function, specifying the behavior of the highest-order derivative at interior points while employing boundary conditions on lower-order derivatives to uniquely determine the function. In this way, the constraints on derivatives of different orders are mutually independent, ensuring analytical tractability comparable to that of the first-degree case. These conditions, whether interior or boundary, carry clear economic meaning, as they all can be interpreted as placing direct bounds on differences in probability premia in simple lottery choices. The premia can be readily elicited using the price list method, with the procedure no more difficult than that for the first-degree case. Overall, our approach to bounding deviations through a combination of interior and boundary constraints satisfies all four criteria, resulting in a tractable benchmark-based higher-degree stochastic dominance rule that interpolates between the benchmark utility and higher-degree stochastic dominance.

Dubra et al. (2004) show that removing the completeness axiom in expected utility leads to preferences representable as the consensus of a set of utility functions. Our study offers a concrete formulation of incomplete preferences through an explicitly constructed utility set, where the extent of incompleteness is governed by bounds on deviations from the benchmark. By allowing users to determine the benchmark at their discretion and calibrate deviation bounds based on personalized information, benchmark-based stochastic dominance centers dominance relations within a controllable neighborhood around a selectable utility function, yielding more comprehensive insights than either a single utility function or standard stochastic dominance alone. This practical advantage is illustrated through an application that ranks stocks and bonds for different investment horizons.

More broadly, our approach unifies utility-function-based and stochastic-dominance-based methods in decision analysis. Although both methods are widely used in applied decision-making (Levy 2016), they have traditionally been treated as separate strands, often distinguished as parametric and nonparametric. The key difference is that the former relies on a parameterized utility function for analysis, while the latter uses only qualitative constraints to characterize preferences. Benchmark-based stochastic dominance offers a third, semi-parametric approach. It takes a parametric utility function as a benchmark but characterizes deviations nonparametrically, imposing no assumptions on the functional form of the deviating utility functions. While previous studies have explored various types of utility perturbations (Hu et al. 2018, Luo et al. 2025), our method uniquely connects the benchmark utility function to integer-degree stochastic dominance, thereby offering the potential to bridge parametric and nonparametric methods in various applications.

Last but not least, benchmark-based stochastic dominance integrates several existing but fragmented notions of stochastic dominance, particularly almost stochastic dominance and fractional-degree stochastic dominance. The first-, second-, and higher-degree almost stochastic dominance rules, introduced by Leshno and Levy (2002), Tzeng et al. (2013) and Tsetlin et al. (2015) respectively, correspond to benchmark-based stochastic dominance with linear, quadratic and higher-degree polynomial utility functions as the benchmark, while excluding extreme deviations. Some further generalizations of almost stochastic dominance explored by Tsetlin et al. (2015) fall outside our framework due to different utility constraints. As for fractional degree stochastic dominance, the three rules proposed by Müller et al. (2017), Bi and Zhu (2019) and Huang et al. (2020) are all special cases of benchmark-based stochastic dominance. The former two rules, respectively using linear and polynomial utility functions as the benchmark, accommodate some degree of risk-seeking deviations, while the third uses a risk-seeking utility function as the benchmark and allows only risk-averse deviations. This unified perspective clarifies the relationships between existing stochastic dominance rules and may guide future developments of stochastic dominance theory.

2. Bounding Deviations via First Derivative

Throughout the paper, the wealth domain is a fixed bounded interval $[a,b] \subset \mathbb{R}$. Random variables, denoted by $\tilde{x}$, $\tilde{y}$, etc., are supported on this domain, while x, y, etc., represent their corresponding realizations. von Neumann-Morgenstern utility functions are denoted by u and v, assumed to be continuous, strictly increasing, and differentiable to the desired degree. Derivatives at the endpoints of $[a,b]$ are understood as the corresponding right- and left-sided derivatives, respectively. The benchmark utility function is always denoted by v. As usual, $\mathbb{E}(\cdot)$ is the expectation operator.

In this section, we establish a meaningful bound for deviations from the benchmark utility that extend to the utility set underlying first-degree stochastic dominance (FSD). FSD characterizes the consensus of all increasing utility functions in comparing random variables (Levy 1992). Formally, $\tilde{x}$ dominates $\tilde{y}$ by FSD, written as $\tilde{x}$ FSD $\tilde{y}$, if $\mathbb{E}u(\tilde{x}) \geq \mathbb{E}u(\tilde{y})$ for all $u \in U_1$, where

$$U_1 = \{u : \; u'(x) > 0 \text{ for all } x \in [a,b]\}.$$

2.1. Definition

Given a benchmark utility $v \in U_1$, for any $u \in U_1$, $\kappa(x) = u'(x)/v'(x)$ captures how the marginal utility under u differs from that under v. It is a standard result that u represents the same preference as v if and only if u is a positive affine transformation of v (Pratt 1964), a property equivalently reflected by $\kappa(x)$ being constant on $[a,b]$. Thus, deviations of $\kappa(x)$ from constancy capture departures from the benchmark preference. To control such deviations, we impose uniform multiplicative bounds on the relative variation of $\kappa(x)$ across wealth levels

$$\frac{1}{\lambda_1} \leq \frac{\kappa(x)}{\kappa(y)} \leq \lambda_2 \text{ for all } x,y \in [a,b] \text{ with } x < y,$$

where $\lambda_1, \lambda_2 \geq 1$ are constants. When $\lambda_1 = \lambda_2 = 1$, $\kappa(x)$ is constant and u and v describe the same preference. As λ_1 or λ_2 exceeds 1, u is permitted to deviate from v in ways that go beyond affine transformations, with the allowable range expanding to U_1 as $\lambda_1 \to \infty$ and $\lambda_2 \to \infty$. Substituting $\kappa(x) = u'(x)/v'(x)$ gives the following characterization of controllable deviations within U_1.

Definition 1. Let $v \in U_1$ be a benchmark utility function and $\lambda_1, \lambda_2 \geq 1$ be finite constants. We say that $u \in U_1$ has its deviation bounded within $[1/\lambda_1, \lambda_2]$, denoted by $u \in U_1^v(\lambda_1, \lambda_2)$, if it satisfies

$$\frac{1}{\lambda_1}\frac{v'(x)}{v'(y)} \leq \frac{u'(x)}{u'(y)} \leq \lambda_2 \frac{v'(x)}{v'(y)} \text{ for all } x,y \in [a,b] \text{ with } x < y. \tag{1}$$

In expected utility theory, the Arrow-Pratt index of absolute risk aversion (Pratt 1964) is a widely recognized measure of risk attitudes, as it intuitively links the curvature of a utility function to an individual's willingness to accept risky outcomes. To enhance the understanding of condition (1), it is helpful to relate it to this index. Let $A_u(x) = -u''(x)/u'(x)$ and $A_v(x) = -v''(x)/v'(x)$ represent the Arrow-Pratt risk aversion indices for u and v, respectively.

Proposition 1. *Let $v \in U_1$ be a benchmark utility function and $\lambda_1, \lambda_2 \geq 1$ be finite constants. Under twice differentiability, $u \in U_1$ satisfies condition (1) if and only if*

$$-\log\lambda_1 \leq \int_x^y [A_u(t) - A_v(t)]dt \leq \log\lambda_2 \text{ for all } x, y \text{ such that } a \leq x < y \leq b.$$

Previous efforts based on bounding *pointwise* differences between the Arrow-Pratt indices, $A_u(x) - A_v(x)$, in constructing stochastic dominance failed to yield an analytical expression for the corresponding ranking rule (Meyer 1977a,b). Instead, Definition 1 imposes uniform bounds on *cumulative* differences between these indices, $\int_x^y [A_u(t) - A_v(t)]dt$, across all subintervals. The bounds reduce to pointwise comparison only when $\lambda_1 = 1$ or $\lambda_2 = 1$. When $\lambda_1 = 1$, the left inequality holds if and only if $A_u(x) \geq A_v(x)$ for all x, implying that u is more risk averse than v everywhere. When $\lambda_2 = 1$, the right inequality requires that u be less risk averse than v everywhere.

Example 1. Let $[a, b] \subset (0, \infty)$ and consider the power utility $v(x) = x^{1-\gamma_v}/(1-\gamma_v)$ (with $v(x) = \log x$ when $\gamma_v = 1$), whose Arrow-Pratt index is $A_v(x) = \gamma_v/x$. For $u(x) = x^{1-\gamma_u}/(1-\gamma_u)$ (with $u(x) = \log x$ when $\gamma_u = 1$), we have $A_u(x) = \gamma_u/x$. Then, $u \in U_1^v(\lambda_1, \lambda_2)$ if and only if

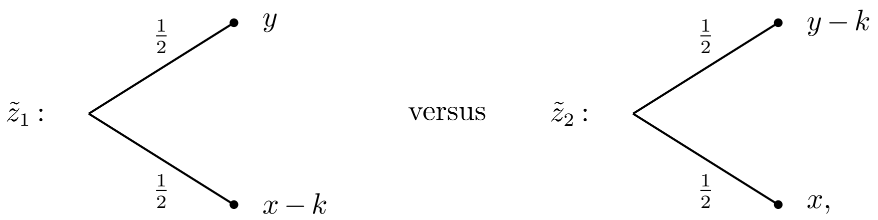
$$-\frac{\log\lambda_1}{\log(b/a)} \leq \gamma_u - \gamma_v \leq \frac{\log\lambda_2}{\log(b/a)}.$$

This example shows that condition (1) is global, depending on the interval $[a, b]$ where deviations are evaluated. It also requires finite marginal utilities on $[a, b]$, so the lower bound a (interpretable as a subsistence wealth level) must be strictly positive to include power utilities.

2.2. Interpretation and Elicitation

Definition 1 is not only analytically amenable, but also economically meaningful, as it directly delineates a comparison of the choice behavior under two utility functions over simple lotteries.

In the expected utility framework, risk preferences are shaped by how marginal utility varies with wealth. One can infer the direction of risk attitudes (risk aversion or risk seeking) from choices between the following 50-50 binary lotteries



where $y > x$ are two wealth levels and $k > 0$ denotes a wealth reduction such that $x, y, x-k, y-k \in [a, b]$. Choosing $\tilde{z}_2$ over $\tilde{z}_1$ indicates that a wealth reduction at the lower level causes a greater utility loss than the same reduction at the higher level, revealing risk aversion. Conversely, choosing $\tilde{z}_1$ suggests greater utility loss at the higher wealth level, implying risk seeking.

Beyond direction, the intensity of risk preferences can be measured by probability premium, a concept introduced by Pratt (1964) that has since become widely adopted (Abdellaoui et al. 2011, Liu and Neilson 2019). It refers to the probability shift toward the higher wealth level required to compensate for the greater utility loss from subtracting k at x rather than at y, revealing a decision-maker's evaluation of the relative appeal of bearing a loss at a higher wealth level rather than at a lower level. Formally, it is the value of $q \in [-1,1]$ that achieves indifference between

$$\tilde{z}_1 : \begin{cases} \frac{1}{2}(1+q) & y \\ \frac{1}{2}(1-q) & x-k \end{cases} \qquad \text{and} \qquad \tilde{z}_2 : \begin{cases} \frac{1}{2}(1+q) & y-k \\ \frac{1}{2}(1-q) & x. \end{cases}$$

A simple calculation shows that the probability premium associated with a utility function u, denoted by $q_u(x,y;k)$, is given by

$$q_u(x,y;k) = \frac{2}{\left[\frac{u(y)-u(y-k)}{u(x)-u(x-k)}\right]+1} - 1,$$

where a larger $q_u(x,y;k)$ implies a faster decline in marginal utility with wealth, indicating greater risk aversion. As q increases from -1 to 1, the probability that $\tilde{z}_1$ yields a higher (lower) payoff than $\tilde{z}_2$ increases (decreases), thus causing the expected utility difference between $\tilde{z}_1$ and $\tilde{z}_2$ to rise monotonically from negative to positive. Therefore, $q_u(x,y;k)$ is identified as the unique switch point in a list of successive choices (Holt and Laury 2002). It is determined by $[u(y) - u(y-k)]/[u(x) - u(x-k)]$, which depends on the ratio of marginal utilities. Thus, ensuring similar marginal utility ratios between two utility functions as in (1) is equivalent to requiring their corresponding probability premia to be similar as well.

Proposition 2. *Let $v \in U_1$ be a benchmark utility function and $\lambda_1, \lambda_2 \geq 1$ be finite constants. Under continuously differentiability, $u \in U_1$ satisfies condition (1) if and only if*

$$\xi(q_v(x,y;k), 1/\lambda_1) \leq q_u(x,y;k) \leq \xi(q_v(x,y;k), \lambda_2)$$

for all $a < x < y < b$ and $0 < k < \min\{x-a, y-x\}$, where

$$\xi(q_v, \eta) = \frac{\eta - 1 + (\eta+1)q_v}{\eta + 1 + (\eta - 1)q_v}$$

measures the maximum allowable deviation in probability premium and satisfies: (i) $\xi(-1,\eta) = -1$ and $\xi(1,\eta) = 1$ for all $\eta > 0$; (ii) $\xi(q_v, 1) = q_v$ for all $q_v \in [-1,1]$; and (iii) $\xi(q_v,\eta)$ is strictly increasing in η for all $q_v \in (-1,1)$.

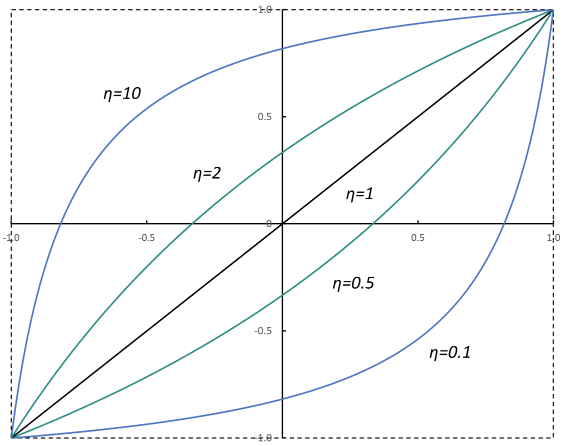


Figure 1 **The graph of $\xi(q_v, \eta)$ as a function of q_v with various values of η.**

A graphical illustration of $\xi(q_v, \eta)$ is provided in Figure 1. In the extreme cases where $q_v = -1$ or $q_v = 1$, $\xi(q_v, \eta) = q_v$ for all values of η. This reflects the limited room for deviation in the probability premium when the benchmark utility function exhibits extreme risk attitudes. For $q_v \in (-1, 1)$, $\xi(q_v, \eta)$ equals q_v when $\eta = 1$, but deviates from q_v upwards as η increases and downwards as η decreases. As η goes to infinity or 0, $\xi(q_v, \eta)$ approaches 1 or -1, permitting the decision-maker's risk attitude to vary arbitrarily.

The choice-based interpretation links the preference setup to observable behavior, enabling practical elicitation of preferences from choice data. The benchmark v can be constructed using standard methods, such as via certainty-equivalent (Becker et al. 1964), probability-equivalent (Wakker and Deneffe 1996) or maximum-likelihood estimation (Harrison 2008). Assuming v is given, we focus on eliciting λ_1 and λ_2 based on Proposition 2. Present the decision-maker with $k = 1, \ldots, K$ distinct pairs of lotteries in the form of $(\tilde{z}_1, \tilde{z}_2)$ as above. For each pair, elicit the probability premium q_k either by directly asking for the indifference probability à la Becker et al. (1964), or by identifying the switching point using a price list procedure similar to Deck et al. (2024). Let $\bar{q}_k$ denote the corresponding probability premium implied by the benchmark utility v. Define

$$\lambda_1^* = \max\left\{1, \max_{k=1,\ldots,K} \frac{(1+\bar{q}_k)/(1-\bar{q}_k)}{(1+q_k)/(1-q_k)}\right\} \text{ and } \lambda_2^* = \max\left\{1, \max_{k=1,\ldots,K} \frac{(1+q_k)/(1-q_k)}{(1+\bar{q}_k)/(1-\bar{q}_k)}\right\}.$$

Then, setting $\lambda_1 = \lambda_1^*$ and $\lambda_2 = \lambda_2^*$ ensures that condition (1) holds across all K lottery pairs and binds for some of them. Assuming robustness across similar lotteries, these values characterize the maximal extent to which the decision-maker may deviate from the benchmark utility. To mitigate the influence of outliers that produce excessively large deviation bounds, one may follow Levy et al. (2010) by choosing λ_1 and λ_2 such that condition (1) is satisfied for a specified proportion (e.g., 95%) of the observed choices.

3. Benchmark-Based First-Degree Stochastic Dominance

Benchmark-based first-degree stochastic dominance captures the consensus ranking of risky prospects among utility functions within $U_1^v(\lambda_1,\lambda_2)$.

Definition 2. Given a benchmark utility function $v \in U_1$ and two finite constants $\lambda_1,\lambda_2 \geq 1$, $\tilde{x}$ dominates $\tilde{y}$ by *benchmark-based first-degree stochastic dominance parameterized by* (v,λ_1,λ_2), denoted by $\tilde{x}$ (v,λ_1,λ_2)-BFSD $\tilde{y}$, if $\mathbb{E}u(\tilde{x}) \geq \mathbb{E}u(\tilde{y})$ for all $u \in U_1^v(\lambda_1,\lambda_2)$.

3.1. Integral Condition

The BFSD can be characterized by an explicit integral condition in terms of the cumulative distribution functions (cdfs) being compared. Let $z_+ = \max\{z,0\}$ for any real number z.

Theorem 1. *Given a benchmark utility function $v \in U_1$ and two finite constants $\lambda_1,\lambda_2 \geq 1$, $\tilde{x}$ (v,λ_1,λ_2)-BFSD $\tilde{y}$ if and only if*

$$\begin{aligned}&\min\left\{1,\frac{\lambda_2}{\lambda_1}\right\}\int_a^t v'(x)[F(x)-G(x)]_+dx+\min\left\{1,\frac{\lambda_1}{\lambda_2}\right\}\int_t^b v'(x)[F(x)-G(x)]_+dx\\ \leq&\frac{1}{\lambda_1}\int_a^t v'(x)[G(x)-F(x)]_+dx+\frac{1}{\lambda_2}\int_t^b v'(x)[G(x)-F(x)]_+dx \text{ for all } t\in[a,b],\end{aligned} \tag{2}$$

where F and G are the cdfs of $\tilde{x}$ and $\tilde{y}$, respectively.

To build intuition for condition (2), let $1_\Omega(x)$ be an indicator function that equals 1 if $x \in \Omega$ and 0 otherwise, and

$$w_{\lambda_1,\lambda_2}(x;t)=\min\left\{1,\frac{\lambda_2}{\lambda_1}\right\}1_{[a,t)}(x)+\min\left\{1,\frac{\lambda_1}{\lambda_2}\right\}1_{[t,b]}(x) \tag{3}$$

be a weight function depending on both x and t. Using the identities $1/\lambda_1 = \min\{1,\lambda_2/\lambda_1\}\cdot\max\{1/\lambda_1,1/\lambda_2\}$ and $1/\lambda_2 = \min\{1,\lambda_1/\lambda_2\}\cdot\max\{1/\lambda_1,1/\lambda_2\}$, condition (2) can be rewritten as

$$\int_a^b v'(x)[F(x)-G(x)]_+w_{\lambda_1,\lambda_2}(x;t)dx \leq \max\left\{\frac{1}{\lambda_1},\frac{1}{\lambda_2}\right\}\int_a^b v'(x)[G(x)-F(x)]_+w_{\lambda_1,\lambda_2}(x;t)dx \tag{4}$$

for all $t \in [a,b]$. Recall that the integral condition for $\tilde{x}$ FSD $\tilde{y}$ is $F(x) \leq G(x)$ for all $x \in [a,b]$ (Levy 1992). The integral on the left-hand side of (4) represents the "weighted violation area" between F and G where FSD is violated, and on the right-hand side represents the "weighted compliance area" where FSD holds. Hence, condition (4) states exactly that the weighted violation area is no greater than the fraction $\max\{1/\lambda_1,1/\lambda_2\}$ of the weighted compliance area. It reduces to $\mathbb{E}v(\tilde{x}) \geq \mathbb{E}v(\tilde{y})$ when $\lambda_1=\lambda_2=1$, and approaches the condition for $\tilde{x}$ FSD $\tilde{y}$ as λ_1 and λ_2 tend to infinity.

One can focus on three intuitive cases to obtain a mathematically simpler dominance rule. First, setting $\lambda_1=\lambda_2=\lambda$ imposes symmetric deviations in Arrow-Pratt risk aversion relative to the benchmark. In this case, $\tilde{x}$ (v,λ,λ)-BFSD $\tilde{y}$ if and only if

$$\int_a^b v'(x)[F(x)-G(x)]_+dx \leq \frac{1}{\lambda}\int_a^b v'(x)[G(x)-F(x)]_+dx.$$

Second, allowing only upward deviations by setting $\lambda_1 = 1$ means risk aversion can exceed but not fall below the benchmark. Then, $\tilde{x}$ $(v, 1, \lambda_2)$-BFSD $\tilde{y}$ if and only if

$$\int_a^t v'(x)[F(x) - G(x)]dx \leq \frac{1}{\lambda_2} \int_t^b v'(x)[G(x) - F(x)]dx \text{ for all } t \in [a, b].$$

Third, restricting risk aversion to remain below the benchmark by setting $\lambda_2 = 1$ gives $\tilde{x}$ $(v, \lambda_1, 1)$-BFSD $\tilde{y}$ if and only if

$$\frac{1}{\lambda_1} \int_a^t v'(x)[F(x) - G(x)]dx \leq \int_t^b v'(x)[G(x) - F(x)]dx \text{ for all } t \in [a, b].$$

3.2. Unifying Existing Rules

The BFSD brings together four existing notions of stochastic dominance in a single framework. Table 1 summarizes these rules. The first is the combined stochastic order proposed by Müller et al. (2017), referred to as $(1 + \gamma_{cv}, 1 + \gamma_{cx})$-stochastic dominance. It ranks prospects according to utility functions satisfying

$$\gamma_{cv} u'(y) \leq u'(x) \leq \frac{1}{\gamma_{cx}} u'(y) \text{ for all } x, y \in [a, b] \text{ with } x < y,$$

where $\gamma_{cv}, \gamma_{cx} \in (0, 1]$ bound the extent to which marginal utility may increase or decrease with wealth. This condition is a special case of (1) with $v(x) = x$, $\lambda_1 = 1/\gamma_{cv}$ and $\lambda_2 = 1/\gamma_{cx}$. Moreover, $u \in U_1^v(\lambda_1, \lambda_2)$ if and only if $u \circ v^{-1}$ satisfies the above inequality with $\gamma_{cv} = 1/\lambda_1$ and $\gamma_{cx} = 1/\lambda_2$. Consequently, $\tilde{x}$ $(v, \lambda_1, \lambda_2)$-BFSD $\tilde{y}$ holds if and only if the transformed prospects $v(\tilde{x})$ and $v(\tilde{y})$ exhibit $(1 + 1/\lambda_1, 1 + 1/\lambda_2)$-stochastic dominance.

The second notion, introduced by Meyer (1977b), is second-degree stochastic dominance with respect to a utility function. It represents the consensus ranking of utility functions with Arrow-Pratt risk aversion bounded from below, i.e.,

$$-\frac{u''(x)}{u'(x)} \geq -\frac{k''(x)}{k'(x)} \text{ for all } x \in [a, b],$$

where $k(x) \in U_1$ is a given utility function. By Proposition 1, this condition is equivalent to (1) with $v(x) = k(x)$, $\lambda_1 = 1$ and $\lambda_2 \to \infty$. Meyer also introduced a convex counterpart, which ranks prospects according to utility functions with lower Arrow-Pratt risk aversion than k, i.e.,

$$-\frac{u''(x)}{u'(x)} \leq -\frac{k''(x)}{k'(x)} \text{ for all } x \in [a, b].$$

This is equivalent to (1) with $v(x) = k(x)$, $\lambda_1 \to \infty$ and $\lambda_2 = 1$.

The third notion, almost stochastic dominance, broadens the set of rankable prospects by excluding pathological utility functions exhibiting extreme risk attitudes. The almost first-degree stochastic dominance proposed by Leshno and Levy (2002) requires

$$\sup_{x \in [a,b]} \{u'(x)\} \leq \inf_{x \in [a,b]} \{u'(x)\} \left(\frac{1}{\varepsilon} - 1 \right)$$

Table 1 Existing rules as special cases of benchmark-based first-degree stochastic dominance

v	$\lambda_1,\ \lambda_2$	rule	reference
A. Combined stochastic dominance			
$v(x)=x$	$\lambda_1=\frac{1}{\gamma_{cv}},\ \lambda_2=\frac{1}{\gamma_{cx}}$	$(1+\gamma_{cv},1+\gamma_{cx})$-SD	Müller et al. (2017)
B. Stochastic dominance with respect to a function			
general	$\lambda_1=1,\ \lambda_2\to\infty$	SSD with respect to v	Meyer (1977b)
general	$\lambda_1\to\infty,\ \lambda_2=1$	convex SSD with respect to v	Meyer (1977b)
C. Almost stochastic dominance			
$v(x)=x$	$\lambda_1=\lambda_2=\frac{1}{\varepsilon}-1$	almost FSD	Leshno and Levy (2002)
general	$\lambda_1=\lambda_2=\frac{1}{\varepsilon}-1$	weighted almost FSD	Tan (2015)
$v(x)=x$	$\lambda_1=1,\ \lambda_2=\frac{1}{\varepsilon}-1$	generalized almost SSD	Tsetlin et al. (2015)
$v(x)=x$	$\lambda_1=\frac{1}{\varepsilon}-1,\ \lambda_2=1$	generalized almost convex SSD	Tsetlin et al. (2015)
D. Fractional degree stochastic dominance			
$v(x)=x$	$\lambda_1=\frac{1}{\gamma},\ \lambda_2\to\infty$	$(1+\gamma)$-SD	Müller et al. (2017)
$v(x)=e^{(1/c-1)x}$	$\lambda_1=1,\ \lambda_2\to\infty$	$(1+c)$-SD	Huang et al. (2020)
$v(x)=x^{(1/c-1)}$	$\lambda_1=1,\ \lambda_2\to\infty$	$(1+c)$-SD	Mao et al. (2025)

Note: SD, FSD and SSD denote stochastic dominance, first-degree stochastic dominance and second-degree stochastic dominance, respectively. The "second-degree" case arises only when λ_1 or λ_2 equals 1, so that risk aversion deviates from the benchmark in a single direction. The parameters involved satisfy $\gamma,\gamma_{cv},\gamma_{cx}\in(0,1]$, $\varepsilon\in(0,1/2]$ and $c\in(0,1]$.

for some $\varepsilon\in(0,1/2]$. This condition is equivalent to (1) when $v(x)=x$ and $\lambda_1=\lambda_2=1/\varepsilon-1$. Observing that the above ranking fails to detect the preference for $\tilde{x}$ over $\tilde{y}$ when $\mathbb{E}\tilde{x}<\mathbb{E}\tilde{y}$ even in cases where $\tilde{x}$ is clearly more desirable, Tan (2015) introduces weighted almost first-degree stochastic dominance, defined as the consensus ranking of utility functions satisfying

$$\sqrt{\frac{\varepsilon}{1-\varepsilon}}\le\frac{u'(x)}{v'(x)}\le\sqrt{\frac{1}{\varepsilon}-1}\text{ for all }x\in[a,b],$$

where v is referred to as a canonical utility function. Normalizing $\inf_{x\in[a,b]}\{u'(x)/v'(x)\}=\sqrt{\varepsilon/(1-\varepsilon)}$, this condition is equivalent to

$$\sup_{x\in[a,b]}\left\{\frac{u'(x)}{v'(x)}\right\}\le\inf_{x\in[a,b]}\left\{\frac{u'(x)}{v'(x)}\right\}\left(\frac{1}{\varepsilon}-1\right),$$

which coincides with (1) when v is general and $\lambda_1=\lambda_2=1/\varepsilon-1$. Tsetlin et al. (2015) incorporate risk aversion into almost stochastic dominance by imposing

$$\sup_{x\in[a,b]}\{u'(x)\}\le\inf_{x\in[a,b]}\{u'(x)\}\left(\frac{1}{\varepsilon}-1\right)\text{ and }u''(x)\le 0\text{ for all }x\in[a,b],$$

which corresponds to (1) with $v(x)=x$, $\lambda_1=1$ and $\lambda_2=1/\varepsilon-1$. They also consider a convex counterpart for risk-loving preferences with

$$\sup_{x\in[a,b]}\{u'(x)\}\leq \inf_{x\in[a,b]}\{u'(x)\}\left(\frac{1}{\varepsilon}-1\right) \text{ and } u''(x)\geq 0 \text{ for all } x\in[a,b],$$

which is equivalent to (1) with $v(x)=x$, $\lambda_1=1/\varepsilon-1$ and $\lambda_2=1$.

The fourth notion, fractional degree stochastic dominance, provides an interpolation between adjacent integer-degree stochastic dominance rules. Müller et al. (2017) define stochastic dominance of degree $1+\gamma$ ($\gamma\in(0,1]$) as the consensus ranking of utility functions satisfying

$$0\leq \gamma u'(y)\leq u'(x) \text{ for all } x<y \text{ with } x,y\in[a,b].$$

This is equivalent to (1) with $v(x)=x$, $\lambda_1=1/\gamma$ and $\lambda_2\to\infty$. Huang et al. (2020) and Mao et al. (2025) propose stochastic dominance of degree $1+c$ ($c\in(0,1]$) as the consensus ranking of utility functions satisfying

$$-\frac{u''(x)}{u'(x)}\geq -\left(\frac{1}{c}-1\right) \text{ and } -\frac{xu''(x)}{u'(x)}\geq -\left(\frac{1}{c}-1\right) \text{ for all } x\in[a,b],$$

respectively. These are special cases of Meyer's condition and correspond to (1) with $v(x)=e^{(1/c-1)x}$ and $v(x)=x^{(1/c-1)}$, respectively, $\lambda_1=1$ and $\lambda_2\to\infty$.

Unifying previously disconnected stochastic dominance rules into a single framework facilitates their comparison and inspires the development of new rules. For instance, combining the ideas of Tan (2015) and Huang et al. (2020), one can construct a new rule, namely almost fractional degree stochastic dominance of degree $1+c$ ($c\in(0,1]$), by setting $v(x)=e^{(1/c-1)x}$, $\lambda_1=1$ and $\lambda_2=1/\varepsilon-1$ ($\varepsilon\in(0,1]$). In this rule, the Arrow-Pratt risk aversion is bounded below by $-(1/c-1)$, while extremely large risk aversion is ruled out by keeping λ_2 finite. This rule, which is also a special case of benchmark-based first-degree stochastic dominance, inherits the advantages of both fractional degree and almost stochastic dominance and exhibits the same level of tractability.

4. Bounding Deviations via Second Derivative

This section provides a meaningful approach to bounding deviations from the benchmark utility within the utility set underlying second-degree stochastic dominance (SSD), which characterizes the consensus of all increasing and concave utility functions in comparing random variables (Levy 1992). Formally, $\tilde{x}$ dominates $\tilde{y}$ by SSD, written as $\tilde{x}$ SSD $\tilde{y}$, if $\mathbb{E}u(\tilde{x})\geq\mathbb{E}u(\tilde{y})$ for all $u\in U_2$, where

$$U_2=\{u:\ u'(x)>0 \text{ and } u''(x)<0 \text{ for all } x\in[a,b]\}.$$

4.1. Definition

Given a benchmark utility $v \in U_2$, for any $u \in U_2$, $\kappa_1(x) = u'(x)/v'(x)$ and $\kappa_2(x) = u''(x)/v''(x)$ capture the deviations of the first and second derivatives of u from those of v, respectively. To parameterize deviations using $\kappa_1(x)$ and $\kappa_2(x)$, the constraints imposed on them should be independent, which is crucial for avoiding the analytical intractability arising from the inherent interdependence between the first and second derivatives.

An equivalent characterization for u representing the same preference as v (i.e., u is a positive affine transformation of v) is as follows: (i) $\kappa_2(x)$ is constant on $[a,b]$, and (ii) $\kappa_2(c) = \kappa_1(c)$ for some $c \in [a,b]$. Condition (i) implies that u' is a positive affine transformation of v', so u itself may differ from v by a positive affine transformation plus a linear term. Condition (ii) removes this linear term, thereby fully aligning the preferences represented by u and v. Thus, deviations from the benchmark preference v are captured by non-constancy in $\kappa_2(x)$ and the discrepancy between $\kappa_2(c)$ and $\kappa_1(c)$ at some point c. To control these deviations, we impose two constraints. First, we bound the relative variation of $\kappa_2(x)$ across wealth levels

$$\frac{1}{\lambda_1} \leq \frac{\kappa_2(x)}{\kappa_2(y)} \leq \lambda_2 \text{ for all } x, y \in [a,b] \text{ with } x < y,$$

where $\lambda_1, \lambda_2 \geq 1$. Second, we restrict the deviation between $\kappa_2(c)$ and $\kappa_1(c)$ at $c = b$

$$\frac{1}{\mu_1}\kappa_1(b) \leq \kappa_2(b) \leq \mu_2 \kappa_1(b).$$

The endpoint b is chosen because the positivity of $\kappa_1(b)$ ensures $\kappa_1(x) > 0$ throughout $[a,b]$ under $u'' < 0$. The first condition regulates the interior variation of the second derivative, while the second restricts the first derivative at the boundary. These two constraints operate independently, analogous to a second-order differential equation supplemented by an endpoint condition to uniquely determine a solution. When $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$, $U_2^v(\lambda_1, \lambda_2; \mu_1, \mu_2)$ reduces to the equivalence class of v under positive affine transformations, all describing the same preference as v. As these parameters tend to infinity, $U_2^v(\lambda_1, \lambda_2; \mu_1, \mu_2)$ expands to U_2. Substituting $\kappa_2(x) = u''(x)/v''(x)$ and $\kappa_1(b) = u'(b)/v'(b)$ yields an explicit characterization of controllable deviations within U_2.

Definition 3. Let $v \in U_2$ be a benchmark utility function and $\lambda_1, \lambda_2, \mu_1, \mu_2 \geq 1$ be finite constants. We say that $u \in U_2$ has its deviation bounded within $[1/\lambda_1, \lambda_2] \times [1/\mu_1, \mu_2]$, denoted by $u \in U_2^v(\lambda_1, \lambda_2; \mu_1, \mu_2)$, if it satisfies both

$$\frac{1}{\lambda_1}\frac{v''(x)}{v''(y)} \leq \frac{u''(x)}{u''(y)} \leq \lambda_2 \frac{v''(x)}{v''(y)} \text{ for all } x, y \in [a,b] \text{ with } x < y \text{ and} \tag{5}$$

$$\frac{1}{\mu_1}\left[-\frac{v''(b)}{v'(b)}\right] \leq \left[-\frac{u''(b)}{u'(b)}\right] \leq \mu_2 \left[-\frac{v''(b)}{v'(b)}\right]. \tag{6}$$

To clarify what conditions (5) and (6) reveal about risk attitudes, it is useful to relate them to the Arrow-Pratt indices of risk aversion. In addition to the second-order index, which characterizes attitudes towards mean-preserving spreads, we also need the third-order Arrow-Pratt index (originally termed the absolute prudence index (Kimball 1990)), which captures attitudes towards skewness and downside risks (Chiu 2005). Let $A_u(x) = -u''(x)/u'(x)$ and $A_v(x) = -v''(x)/v'(x)$ denote the second-order Arrow-Pratt indices for u and v as before, and let $P_u(x) = -u'''(x)/u''(x)$ and $P_v(x) = -v'''(x)/v''(x)$ denote the corresponding third-order indices.

Proposition 3. *Let $v \in U_2$ be a benchmark utility function and $\lambda_1, \lambda_2, \mu_1, \mu_2 \geq 1$ be finite constants. Under three-times differentiability, condition (5) is equivalent to*

$$-\log\lambda_1 \leq \int_x^y [P_u(t) - P_v(t)]dt \leq \log\lambda_2 \text{ for all } x, y \text{ such that } a \leq x < y \leq b,$$

and together with condition (6), it further implies

$$\frac{1}{\lambda_1\mu_1} \leq \frac{A_u(x)}{A_v(x)} \leq \lambda_2\mu_2 \text{ for all } x \in [a,b].$$

Given the infeasibility of deriving analytical ranking rules from bounds on *pointwise* differences of Arrow-Pratt indices (Meyer 1977a,b), we devise a different way to set bounds on utility deviations. Condition (5) can be read as placing a uniform bound on the *cumulative* differences between the third-order Arrow-Pratt indices, $\int_x^y [P_u(t) - P_v(t)]dt$, across all subintervals. When $\lambda_1 = 1$, the left inequality holds if and only if $P_u(x) \geq P_v(x)$ for all x, meaning that u is more prudent than v everywhere. Similarly, when $\lambda_2 = 1$, the right inequality requires u to be less prudent than v everywhere. Although condition (6), rewritten as $1/\mu_1 \leq A_u(b)/A_v(b) \leq \mu_2$, appears to restrict the relative magnitude of the second-order Arrow-Pratt indices only at the endpoint $x = b$, its effect propagates to the interior due to (5), resulting in a restriction on $A_u(x)/A_v(x)$ for every $x \in [a,b]$. Taken together, the combination of (5) and (6) constrains both the discrepancy between $P_u(x)$ and $P_v(x)$ and the ratio of $A_u(x)$ to $A_v(x)$ on $[a,b]$.

Example 2. Let $[a,b] \subset (0,\infty)$ and consider $v(x) = x^{1-\gamma_v}/(1-\gamma_v) + k_v x$ (with $v(x) = \log x + k_v x$ when $\gamma_v = 1$), whose Arrow-Pratt indices are $A_v(x) = \gamma_v/(x + k_v x^{1+\gamma_v})$ and $P_v(x) = \gamma_v/x$. For $u(x) = x^{1-\gamma_u}/(1-\gamma_u) + k_u x$ (with $u(x) = \log x + k_u x$ when $\gamma_u = 1$), we have $A_u(x) = \gamma_u/(x + k_u x^{1+\gamma_u})$ and $P_u(x) = \gamma_u/x$. Assuming $\gamma_u, \gamma_v, k_u, k_v$ all positive, $u \in U_2^v(\lambda_1, \lambda_2; \mu_1, \mu_2)$ if and only if

$$-\frac{\log\lambda_1}{\log(b/a)} \leq \gamma_u - \gamma_v \leq \frac{\log\lambda_2}{\log(b/a)} \text{ and } \frac{\gamma_v}{\mu_1(b^{-\gamma_v} + k_v)} \leq \frac{\gamma_u}{b^{-\gamma_u} + k_u} \leq \frac{\mu_2\gamma_v}{b^{-\gamma_v} + k_v}.$$

This example illustrates that (5) captures only the difference in risk attitudes arising from second-order and higher derivatives, while (6) takes into account the difference from the linear term.

The control over the second derivative via (5) governs the evolution of the first derivative from one point to the next, while the endpoint condition on the first derivative via (6) anchors its level. Together, the entire path of the first derivative across the domain is completely determined, suggesting that the refined bounds on deviations in U_2 would imply corresponding coarser bounds in U_1. Such a nesting relationship is formalized below.

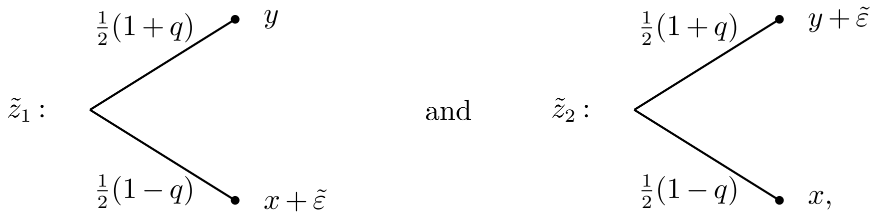
PROPOSITION 4. *Let $v \in U_2$ be a benchmark utility function and $\lambda_1, \lambda_2, \mu_1, \mu_2 \geq 1$ be finite constants. Then, any $u \in U_2^v(\lambda_1, \lambda_2; \mu_1, \mu_2)$ satisfies*

$$\frac{1}{\lambda_1 \mu_1} \frac{v'(x)}{v'(y)} \leq \frac{u'(x)}{u'(y)} \leq \lambda_2 \mu_2 \frac{v'(x)}{v'(y)} \quad \textit{for all } x, y \in [a,b] \textit{ with } x < y.$$

Accordingly, $U_2^v(\lambda_1, \lambda_2; \mu_1, \mu_2) \subset U_1^v(\lambda_1\mu_1, \lambda_2\mu_2)$, and thus the common ranking agreed upon by all utility functions in $U_1^v(\lambda_1\mu_1, \lambda_2\mu_2)$ also applies to all utility functions in $U_2^v(\lambda_1, \lambda_2; \mu_1, \mu_2)$.

4.2. Interpretation and Elicitation

As before, condition (5) admits a behavioral interpretation based on choices from pairs of meaningful lotteries, analogous to condition (1). Consider the choice between

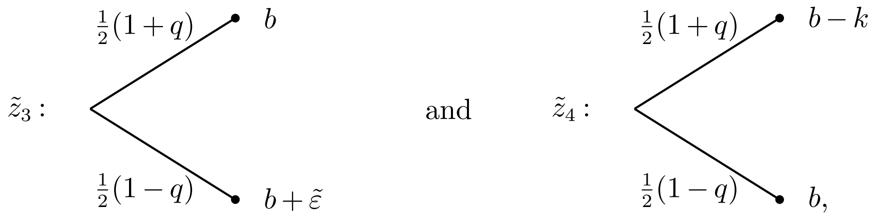


where $y > x$ are two wealth levels, $\tilde{\varepsilon}$ is a zero-mean risk, and all of x, y, $x+\tilde{\varepsilon}$ and $y+\tilde{\varepsilon}$ lie within $[a,b]$. When $q=-1$, risk-averse decision-makers always prefer $\tilde{z}_2$ to $\tilde{z}_1$, and when $q=1$, they always prefer $\tilde{z}_1$ to $\tilde{z}_2$. As q increases from -1 to 1, the difference between the expected utilities of $\tilde{z}_1$ and $\tilde{z}_2$ increases monotonically, transitioning from negative to positive. The unique switch point, which represents the probability premium required to compensate for the greater utility loss from adding $\tilde{\varepsilon}$ to x rather than to y, is given by

$$q_u(x, y; \tilde{\varepsilon}) = \frac{2}{\left[\frac{u(y) - \mathbb{E}u(y+\tilde{\varepsilon})}{u(x) - \mathbb{E}u(x+\tilde{\varepsilon})}\right] + 1} - 1.$$

It is determined by $[u(y) - \mathbb{E}u(y+\tilde{\varepsilon})]/[u(x) - \mathbb{E}u(x+\tilde{\varepsilon})]$ and ultimately by properties of $u''(x)/u''(y)$. Due to the structural similarity between $q_u(x,y;\tilde{\varepsilon})$ and the term $q_u(x,y;k)$ that underpins condition (1), condition (5) can be interpreted as requiring the probability premia $q_u(x,y;\tilde{\varepsilon})$ and $q_v(x,y;\tilde{\varepsilon})$ that are associated with u and v respectively to be close in value.

Regarding condition (6), consider the choice between



where $\tilde{\varepsilon}$ and $k>0$ are small enough. Under $u \in U_2$, there is a unique switch point at which the decision-maker is indifferent between $\tilde{z}_3$ and $\tilde{z}_4$, given by

$$q_u(b;-k,\tilde{\varepsilon}) = \frac{2}{\left[\frac{\mathbb{E}u(b)-\mathbb{E}u(b+\tilde{\varepsilon})}{\mathbb{E}u(b)-\mathbb{E}u(b-k)}\right]+1} - 1.$$

When u is twice continuously differentiable around $x=b$, the term $[\mathbb{E}u(b) - \mathbb{E}u(b+\tilde{\varepsilon})]/[\mathbb{E}u(b) - \mathbb{E}u(b-k)]$ is approximately proportional to $u''(b)/u'(b)$. Thus, condition (6) essentially requires $q_u(b;-k,\tilde{\varepsilon})$ and $q_v(b;-k,\tilde{\varepsilon})$ to be close in value.

Proposition 5. *Let $v \in U_2$ be a benchmark utility function and $\lambda_1,\lambda_2,\mu_1,\mu_2 \geq 1$ be finite constants. Under twice continuously differentiability, $u \in U_2$ satisfies condition (5) if and only if*

$$\xi(q_v(x,y;\tilde{\varepsilon}),1/\lambda_1) \leq q_u(x,y;\tilde{\varepsilon}) \leq \xi(q_v(x,y;\tilde{\varepsilon}),\lambda_2)$$

for all $a<x<y<b$ and all $\tilde{\varepsilon}$ bounded in absolute value by $\max\{x-a, b-y, (y-x)/2\}$, *and satisfies condition (6) if and only if*

$$\xi(q_v(b;-k,\tilde{\varepsilon}),1/\mu_1) \leq q_u(b;-k,\tilde{\varepsilon}) \leq \xi(q_v(b;-k,\tilde{\varepsilon}),\mu_2)$$

for all $\tilde{\varepsilon}$ and $k>0$ small enough, where $\xi(q_v,\eta)$ is defined as in Proposition 2.

By interpreting (5) and (6) in terms of simple lottery choices, the parameters $\lambda_1,\lambda_2,\mu_1,\mu_2$ can be directly elicited by applying the procedure described in Section 2.2 to those choices.

5. Benchmark-Based Second-Degree Stochastic Dominance

Benchmark-based second-degree stochastic dominance captures the consensus ranking of risky prospects among utility functions within $U_2^v(\lambda_1,\lambda_2;\mu_1,\mu_2)$.

Definition 4. Given a benchmark utility function $v \in U_2$ and finite constants $\lambda_1,\lambda_2 \geq 1$ and $\mu_1,\mu_2 \geq 1$, $\tilde{x}$ dominates $\tilde{y}$ by *benchmark-based second-degree stochastic dominance parameterized by* $(v,\lambda_1,\lambda_2,\mu_1,\mu_2)$, denoted by $\tilde{x}$ $(v,\lambda_1,\lambda_2,\mu_1,\mu_2)$-BSSD $\tilde{y}$, if $\mathbb{E}u(\tilde{x}) \geq \mathbb{E}u(\tilde{y})$ for all $u \in U_2^v(\lambda_1,\lambda_2;\mu_1,\mu_2)$.

5.1. Integral Condition

For expository purposes, we use $H^{(1)}(x) = \int_a^x H(t)dt$ as shorthand for the integral of a cdf H.

THEOREM 2. *Given a benchmark utility function $v \in U_2$ and four finite constants $\lambda_1, \lambda_2, \mu_1, \mu_2 \geq 1$, $\tilde{x}$ $(v, \lambda_1, \lambda_2, \mu_1, \mu_2)$-BSSD $\tilde{y}$ if and only if*

$$\begin{aligned}
&\min\left\{1, \frac{\lambda_2}{\lambda_1}\right\} \int_a^t [-v''(x)][F^{(1)}(x) - G^{(1)}(x)]_+ dx + \min\left\{1, \frac{\lambda_1}{\lambda_2}\right\} \int_t^b [-v''(x)][F^{(1)}(x) - G^{(1)}(x)]_+ dx \\
\leq &\frac{1}{\lambda_1} \int_a^t [-v''(x)][G^{(1)}(x) - F^{(1)}(x)]_+ dt + \frac{1}{\lambda_2} \int_t^b [-v''(x)][G^{(1)}(x) - F^{(1)}(x)]_+ dx \\
&+ \frac{1}{\lambda_2 \mu_2} v'(b)[G^{(1)}(b) - F^{(1)}(b)]_+ - \mu_1 \min\left\{1, \frac{\lambda_1}{\lambda_2}\right\} v'(b)[F^{(1)}(b) - G^{(1)}(b)]_+ \text{ for all } t \in [a,b], \qquad (7)
\end{aligned}$$

where F and G are the cdfs of $\tilde{x}$ and $\tilde{y}$, respectively.

Recall that the integral condition for $\tilde{x}$ SSD $\tilde{y}$ is $F^{(1)}(x) \leq G^{(1)}(x)$ for all $x \in [a,b]$ (Levy 1992). Using the weight function in (3), condition (7) can be rewritten as

$$\mathcal{V}(F,G,t) \leq \max\left\{\frac{1}{\lambda_1}, \frac{1}{\lambda_2}\right\} \mathcal{C}(F,G,t) \text{ for all } t \in [a,b], \qquad (8)$$

where

$$\begin{aligned}
\mathcal{V}(F,G,t) &= \int_a^b [-v''(x)][F^{(1)}(x) - G^{(1)}(x)]_+ w_{\lambda_1,\lambda_2}(x;t)dx + \mu_1 v'(b)[F^{(1)}(b) - G^{(1)}(b)]_+ w_{\lambda_1,\lambda_2}(b;t), \\
\mathcal{C}(F,G,t) &= \int_a^b [-v''(x)][G^{(1)}(x) - F^{(1)}(x)]_+ w_{\lambda_1,\lambda_2}(x;t)dx + \frac{1}{\mu_2} v'(b)[G^{(1)}(b) - F^{(1)}(b)]_+ w_{\lambda_1,\lambda_2}(b;t),
\end{aligned}$$

represent the overall weighted difference (i.e., weighted area plus endpoint gap) between F and G for the regions violating and satisfying SSD, respectively. Condition (8) asserts exactly that the overall weighted difference for the violation region is no greater than the fraction $\max\{1/\lambda_1, 1/\lambda_2\}$ of that for the compliance region. It reduces to $\mathbb{E}v(\tilde{x}) \geq \mathbb{E}v(\tilde{y})$ when $\lambda_1 = \lambda_2 = \mu_1 = \mu_2 = 1$, and approaches the condition for $\tilde{x}$ SSD $\tilde{y}$ as $\lambda_1, \lambda_2, \mu_1, \mu_2$ tend to infinity.

The BSSD rule involves four parameters in addition to the benchmark utility. By assigning special values that make economic sense, one can reduce the number of parameters that need to be estimated in practice. For λ_1 and λ_2, one may set $(\lambda_1, \lambda_2) = (\lambda, \lambda)$, $(1, \lambda)$ or $(\lambda, 1)$, which corresponds respectively to symmetric deviation, reinforcement or weakening of absolute prudence. For μ_1 and μ_2, note that given any pair of lotteries, only one parameter is relevant, since at most one of the terms $[G^{(1)}(b) - F^{(1)}(b)]_+$ and $[F^{(1)}(b) - G^{(1)}(b)]_+$ is nonzero. The relevant parameter can be set to 1, reflecting convergence in risk attitudes at the endpoint, or to ∞, ensuring robustness to any perturbations at that point. When these parameters take such special values, the integral condition for BSSD simplifies accordingly.

Table 2 Existing rules as special cases of benchmark-based second-degree stochastic dominance

v	$\lambda_1,\ \lambda_2,\ \mu_1,\ \mu_2$	rule	reference
A. Combined stochastic dominance			
$v(x)=kx-x^2$	$\lambda_1=\frac{1}{\gamma_{cv}},\ \lambda_2=\frac{1}{\gamma_{cx}}$ $\mu_1\to\infty,\ \mu_2\to\infty$	$(2+\gamma_{cv},2+\gamma_{cx})$-SD	Feng and Shanthikumar (2025)
B. Almost stochastic dominance			
$v(x)=kx-x^2$	$\lambda_1=\lambda_2=\frac{1}{\varepsilon}-1$ $\mu_1\to\infty,\ \mu_2\to\infty$	almost SSD	Tzeng et al. (2013)
C. Fractional degree stochastic dominance			
$v(x)=kx-x^2$	$\lambda_1=\frac{1}{\gamma},\ \lambda_2\to\infty$ $\frac{\mu_1}{\lambda_2}\to\infty,\ \mu_2\to\infty$	$(2+\gamma)$-SD	Bi and Zhu (2019)
$v(x)=kx-e^{(1/c-1)x}$	$\lambda_1=1,\ \lambda_2\to\infty$ $\frac{\mu_1}{\lambda_2}\to\infty,\ \mu_2\to\infty$	$(2+c)$-SD	Huang et al. (2020)
$v(x)=kx-x^{(1/c+1)}$	$\lambda_1=1,\ \lambda_2\to\infty$ $\frac{\mu_1}{\lambda_2}\to\infty,\ \mu_2\to\infty$	$(2+c)$-SD	Mao et al. (2025)

Note: SD and SSD stand for stochastic dominance and second-degree stochastic dominance, respectively. The parameter k is chosen so that $v'(x)>0$ for all $x\in[a,b]$. The remaining parameters satisfy $\gamma\in(0,1]$, $\varepsilon\in(0,1/2]$ and $c\in(0,1]$.

5.2. Unifying Existing Rules

The BSSD includes several existing stochastic dominance rules as special cases, see Table 2 for a summary. The first is the combined stochastic order, $(2+\gamma_{cv},2+\gamma_{cx})$-stochastic dominance, proposed in Feng and Shanthikumar (2025) as an extension of Müller et al. (2017) to the second-degree case. It ranks prospects according to utility functions satisfying

$$\gamma_{cv}[-u''(y)]\le[-u''(x)]\le\frac{1}{\gamma_{cx}}[-u''(y)] \text{ for all } x,y\in[a,b] \text{ with } x<y,$$

where $\gamma_{cv},\gamma_{cx}\in(0,1]$. This condition corresponds to a special case where $v(x)=kx-x^2$, $\lambda_1=1/\gamma_{cv}$ and $\lambda_2=1/\gamma_{cx}$ in condition (5) and $\mu_1,\mu_2\to\infty$ in condition (6). The clean equivalence between $\tilde{x}$ (v,λ_1,λ_2)-BFSD $\tilde{y}$ and $v(\tilde{x})$ $(1+1/\lambda_1,1+1/\lambda_2)$-SD $v(\tilde{y})$ breaks down in the second-degree case. This is because the chain rule inextricably links the second derivative of $u\circ v^{-1}$ to the first derivatives of both u and v, preventing it from satisfying the above condition for $(2+\gamma_{cv},2+\gamma_{cx})$-stochastic dominance. Hence, $\tilde{x}$ $(v,\lambda_1,\lambda_2,\mu_1,\mu_2)$-BSSD $\tilde{y}$ neither implies nor is implied by $v(\tilde{x})$ $(2+1/\lambda_1,2+1/\lambda_2)$-SD $v(\tilde{y})$, in stark contrast to the first-degree case.

The second rule is almost second-degree stochastic dominance introduced by Leshno and Levy (2002) and revisited by Tzeng et al. (2013), which requires $u\in U_2$ and

$$\sup_{x\in[a,b]}\{-u''(x)\}\le\inf_{x\in[a,b]}\{-u''(x)\}\left(\frac{1}{\varepsilon}-1\right)$$

for some $\varepsilon \in (0,1/2]$. This condition is a special case of (5) with the benchmark utility being a quadratic function and $\lambda_1 = \lambda_2 = 1/\varepsilon - 1$, and a special case of (6) with $\mu_1, \mu_2 \to \infty$.

The third rule is fractional degree stochastic dominance lying between the second and third degrees. Bi and Zhu (2019) introduce stochastic dominance of degree $2+\gamma$ ($\gamma \in (0,1]$) as the consensus ranking of utility functions satisfying $u \in U_2$ and

$$0 \le \gamma[-u''(y)] \le [-u''(x)] \text{ for all } x < y \text{ with } x, y \in [a,b].$$

This is equivalent to (5) and (6) with v being a quadratic utility function, $\lambda_1 = 1/\gamma$ and $\lambda_2 \to \infty$ and $\mu_1, \mu_2 \to \infty$. Huang et al. (2020) and Mao et al. (2025) define stochastic dominance of degree $2+c$ ($c \in (0,1]$) as the consensus ranking of utility functions satisfying $u \in U_2$ and

$$-\frac{u'''(x)}{u''(x)} \ge -\left(\frac{1}{c} - 1\right) \text{ and } -\frac{xu'''(x)}{u''(x)} \ge -\left(\frac{1}{c} - 1\right) \text{ for all } x \in [a,b],$$

respectively. These conditions are equivalent to (5) and (6) with $v(x) = kx - e^{(1/c-1)x}$ and $v(x) = kx - x^{(1/c+1)}$, respectively, $\lambda_1 = 1$, $\lambda_2 \to \infty$ and $\mu_1, \mu_2 \to \infty$.

Bringing together previously separate stochastic dominance rules within a unified framework can lead to the creation of new rules. For instance, by integrating the insights from Tzeng et al. (2013) and Tan (2015), one can construct a new rule, namely weighted almost second-degree stochastic dominance, by letting v be general and setting $\lambda_1 = \lambda_2 = 1/\varepsilon - 1$ ($\varepsilon \in (0,1]$), to serve as the second-degree counterpart of weighted almost first-degree stochastic dominance (Tan 2015).

6. Benchmark-Based Higher-Degree Stochastic Dominance

The above method for establishing benchmark-based second-degree stochastic dominance is structured and coherent, allowing for a systematic generalization to defining benchmark-based stochastic dominance of arbitrary higher degrees. For any integer $N \ge 2$ and $n = 1, ..., N$, let $u^{(n)}$ denote the nth derivative of u, and define

$$U_N = \left\{u: \ (-1)^{n+1} u^{(n)}(x) > 0 \text{ for all } x \in [a,b],\ n = 1, ..., N\right\}$$

as the set of utility functions with odd-order derivatives positive and even-order derivatives negative up to order N. The condition $(-1)^{n+1}u^{(n)} > 0$ for all $n \in \mathbb{N}$ is known as mixed risk aversion, since utility functions with this property can be expressed as mixtures of exponential utilities (Caballé and Pomansky 1996). We say that $\tilde{x}$ dominates $\tilde{y}$ by Nth-degree stochastic dominance (NSD), written as $\tilde{x}$ NSD $\tilde{y}$, if $\mathbb{E}u(\tilde{x}) \ge \mathbb{E}u(\tilde{y})$ for all $u \in U_N$.

Definition 5. For $N\geq 2$, let $v\in U_N$ be a benchmark utility function and $\lambda_1,\lambda_2,\mu_1,...,\mu_{2N-2}\geq 1$ be finite constants. We say that $u\in U_N$ has its deviation bounded within $[1/\lambda_1,\lambda_2]\times[1/\mu_1,\mu_2]\times\cdots\times[1/\mu_{2N-3},\mu_{2N-2}]$, denoted by $u\in U_N^v(\lambda_1,\lambda_2;\mu_1,...,\mu_{2N-2})$, if it satisfies

$$\frac{1}{\lambda_1}\frac{v^{(N)}(x)}{v^{(N)}(y)}\leq\frac{u^{(N)}(x)}{u^{(N)}(y)}\leq\lambda_2\frac{v^{(N)}(x)}{v^{(N)}(y)}\text{ for all }x,y\in[a,b]\text{ with }x<y\text{ and}\tag{9}$$

$$\frac{1}{\mu_{2n-1}}\left[(-1)^{N-n}\frac{v^{(N)}(b)}{v^{(n)}(b)}\right]\leq\left[(-1)^{N-n}\frac{u^{(N)}(b)}{u^{(n)}(b)}\right]\leq\mu_{2n}\left[(-1)^{N-n}\frac{v^{(N)}(b)}{v^{(n)}(b)}\right]\text{ for }n=1,...,N-1.\tag{10}$$

We say that $\tilde{x}$ dominates $\tilde{y}$ by *benchmark-based Nth-degree stochastic dominance parameterized by* $(v,\lambda_1,\lambda_2,\mu_1,...,\mu_{2N-2})$, denoted by $\tilde{x}$ $(v,\lambda_1,\lambda_2,\mu_1,...,\mu_{2N-2})$-BNSD $\tilde{y}$, if $\mathbb{E}u(\tilde{x})\geq\mathbb{E}u(\tilde{y})$ for all $u\in U_N^v(\lambda_1,\lambda_2;\mu_1,...,\mu_{2N-2})$.

6.1. On the Preference Condition

For $u\in U_N$ and $n=1,...,N$, $\kappa_n(x)=u^{(n)}(x)/v^{(n)}(x)$ captures the deviation of the nth derivative of u from that of v. For u to represent the same preference as v, it is necessary and sufficient that $\kappa_N(x)$ is constant on $[a,b]$ and that $\kappa_N(b)=\kappa_n(b)$ for all $n=1,...,N-1$. Condition (9) bounds the relative variation of $\kappa_N(x)$ over wealth, analogous to (5). Condition (10) extends the endpoint matching by imposing $N-1$ constraints to ensure $\kappa_N(b)$ remains close to each lower-order $\kappa_n(b)$, paralleling the role of (6). Together, conditions (9) and (10) govern the deviation of u from v across all derivatives, thereby extending conditions (5) and (6) to higher degrees.

Caballé and Pomansky (1996) and Denuit and Eeckhoudt (2010) generalize the Arrow-Pratt index of risk aversion (Pratt 1964) and prudence (Kimball 1990) to higher orders, collectively known as higher-order Arrow-Pratt indices, which play a key role in various applications (Deck et al. 2024). For $n\in\mathbb{N}$, let $A_u^{(n+1)}(x)=-u^{(n+1)}(x)/u^{(n)}(x)$ and $A_v^{(n+1)}(x)=-v^{(n+1)}(x)/v^{(n)}(x)$ denote the $(n+1)$th-order Arrow-Pratt indices for u and v, respectively. The following result connects conditions (9) and (10) to these indices.

Proposition 6. *For $N\geq 2$, let $v\in U_N$ be a benchmark utility function and $\lambda_1,\lambda_2,\mu_1,...,\mu_{2N-2}\geq 1$ be finite constants. Under $(N+1)$ times differentiability, condition (9) is equivalent to*

$$-\log\lambda_1\leq\int_x^y\left[A_u^{(N+1)}(t)-A_v^{(N+1)}(t)\right]dt\leq\log\lambda_2\textit{ for all }x,y\textit{ such that }a\leq x<y\leq b,$$

and together with condition (10), it further implies

$$\frac{1}{\lambda_1\mu_{2N-3}}\leq\frac{A_u^{(N)}(x)}{A_v^{(N)}(x)}\leq\lambda_2\mu_{2N-2}\textit{ and}$$

$$\frac{1}{\lambda_1\mu_{2n-3}}\prod_{k=n}^{N-1}\frac{1}{\mu_{2k-1}\mu_{2k}}\leq\frac{A_u^{(n)}(x)}{A_v^{(n)}(x)}\leq\lambda_2\mu_{2n-2}\prod_{k=n}^{N-1}\mu_{2k-1}\mu_{2k}\textit{ when }n=2,...,N-1$$

for all $x\in[a,b]$.

The interior control of the Nth derivative via (9), combined with the endpoint conditions on derivatives up to order $N-1$ via (10), propagates the interior constraint down to the $(N-1)$th derivative, inducing a nested property $U_N^v(\lambda_1, \lambda_2; \mu_1, ..., \mu_{2N-2}) \subset U_{N-1}^v\left(\hat{\lambda}_1, \hat{\lambda}_2; \hat{\mu}_1, ..., \hat{\mu}_{2N-4}\right)$, where $\hat{\lambda}_1 = \lambda_1\mu_{2N-3}$, $\hat{\lambda}_2 = \lambda_2\mu_{2N-2}$, $\hat{\mu}_{2n-1} = \mu_{2n-1}\mu_{2N-2}$ and $\hat{\mu}_{2n} = \mu_{2n}\mu_{2N-3}$ for $n = 1, ..., N-2$.

PROPOSITION 7. *For $N \geq 2$, let $v \in U_N$ be a benchmark utility function and $\lambda_1, \lambda_2, \mu_1, ..., \mu_{2N-2} \geq 1$ be finite constants. Then, any $u \in U_N^v(\lambda_1, \lambda_2; \mu_1, ..., \mu_{2N-2})$ also satisfies*

$$\frac{1}{\lambda_1\mu_{2N-3}} \frac{v^{(N-1)}(x)}{v^{(N-1)}(y)} \leq \frac{u^{(N-1)}(x)}{u^{(N-1)}(y)} \leq \lambda_2\mu_{2N-2} \frac{v^{(N-1)}(x)}{v^{(N-1)}(y)} \text{ for all } x, y \in [a, b] \text{ with } x < y \text{ and}$$

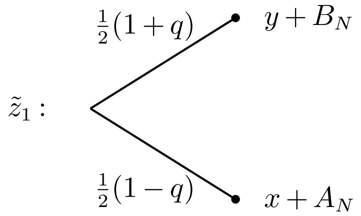
$$\frac{1}{\mu_{2n-1}\mu_{2N-2}} \left[(-1)^{N-n-1} \frac{v^{(N-1)}(b)}{v^{(n)}(b)}\right] \leq \left[(-1)^{N-n-1} \frac{u^{(N-1)}(b)}{u^{(n)}(b)}\right] \leq \mu_{2n}\mu_{2N-3} \left[(-1)^{N-n-1} \frac{v^{(N-1)}(b)}{v^{(n)}(b)}\right]$$

for $n = 1, ..., N-2$.

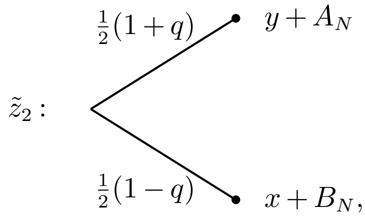
Conditions (9) and (10) can be interpreted as constraining how much the probability premia induced by a utility function u in simple lottery choices can deviate from those implied by the benchmark v. The design of the lottery choices is based on Eeckhoudt and Schlesinger (2006). Denote by $[x; y]$ a lottery with a 50-50 chance of receiving either outcome x or y, where x and y themselves may also be lotteries. Let $k > 0$ be a constant and $\{\tilde{\varepsilon}_i\}$ be an indexed set of mutually independent zero-mean risks. Define $A_1 = -k$, $B_1 = 0$, $A_2 = \tilde{\varepsilon}_1$, $B_2 = 0$, and for $n \geq 3$ iteratively

$$A_n = \left[A_{n-2} + \tilde{\varepsilon}_{\text{Int}(n/2)}; B_{n-2}\right], \quad B_n = \left[A_{n-2}; B_{n-2} + \tilde{\varepsilon}_{\text{Int}(n/2)}\right],$$

where $\text{Int}(n/2)$ denotes the greatest integer not exceeding $n/2$. Economically, A_n attaches an independent zero-mean risk to A_{n-2} and B_n attaches that risk to B_{n-2}. Consequently, A_n represents an increase in nth-degree risk relative to B_n, and the choice between them reveals the sign of $u^{(n)}$. Indeed, Eeckhoudt and Schlesinger (2006) show that $(-1)^{n+1}u^{(n)}(x) > 0$ if and only if B_n is always preferred to A_n for any k and collection $\{\tilde{\varepsilon}_i\}$. To interpret (9), consider the choice between



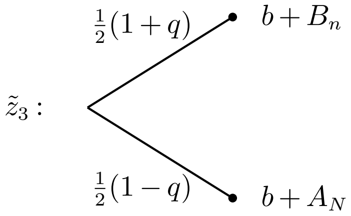
and



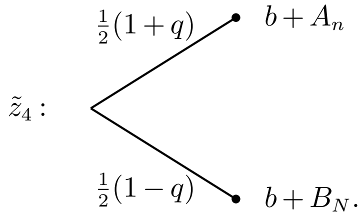
where $y > x$ are two wealth levels, and all outcomes $x + A_N$, $x + B_N$, $y + A_N$, and $y + B_N$ are contained within $[a, b]$. Under $(-1)^{N+1}u^{(N)}(x) > 0$, there is a unique switch point given by

$$q_u(x, y; A_N, B_N) = \frac{2}{\left[\frac{\mathbb{E}u(y+B_N) - \mathbb{E}u(y+A_N)}{\mathbb{E}u(x+B_N) - \mathbb{E}u(x+A_N)}\right] + 1} - 1,$$

which depends on $[\mathbb{E}u(y+B_N)-\mathbb{E}u(y+A_N)]\,/\,[\mathbb{E}u(x+B_N)-\mathbb{E}u(x+A_N)]$ and ultimately on $u^{(N)}(y)/u^{(N)}(x)$. To interpret condition (10), consider the choice between



and



Under $u \in U_N$, there is again a unique switch point given by

$$q_u(b;A_N,B_N,A_n,B_n)=\frac{2}{\left[\frac{\mathbb{E}u(b+B_N)-\mathbb{E}u(b+A_N)}{\mathbb{E}u(b+B_n)-\mathbb{E}u(b+A_n)}\right]+1}-1,$$

in which the term $[\mathbb{E}u(b+B_N)-\mathbb{E}u(b+A_N)]\,/\,[\mathbb{E}u(b+B_n)-\mathbb{E}u(b+A_n)]$ is approximately proportional to $u^{(N)}(b)/u^{(n)}(b)$ when the payoffs of A_N, B_N, A_n and B_n are sufficiently small.

PROPOSITION 8. *For $N \geq 2$, let $v \in U_N$ be a benchmark utility function $\lambda_1,\lambda_2,\mu_1,\ldots,\mu_{2N-2} \geq 1$ be finite constants. Suppose u and v have continuous derivatives up to order N. Then, $u \in U_N$ satisfies condition (9) if and only if*

$$\xi(q_v(x,y;A_N,B_N),1/\lambda_1) \leq q_u(x,y;A_N,B_N) \leq \xi(q_v(x,y;A_N,B_N),\lambda_2)$$

for all $a<x<y<b$ and all A_N,B_N bounded in absolute value by $\max\{x-a,b-y,(y-x)/2\}$, and satisfies condition (10) if and only if

$$\xi(q_v(b;A_N,B_N,A_n,B_n),1/\mu_{2n-1}) \leq q_u(b;A_N,B_N,A_n,B_n) \leq \xi(q_v(b;A_N,B_N,A_n,B_n),\mu_{2n})$$

for all A_N,B_N,A_n,B_n with sufficiently small payoffs, where $\xi(q_v,\eta)$ is defined as in Proposition 2.

6.2. On the Integral Condition

For a generic cdf H, define $H^{(0)}(x)=H(x)$ and $H^{(n)}(x)=\int_a^x H^{(n-1)}(t)dt$ iteratively for $n \geq 1$.

THEOREM 3. *For $N \geq 2$, given a benchmark utility function $v \in U_N$ and $2N$ finite constants $\lambda_1,\lambda_2,\mu_1,\ldots,\mu_{2N-2} \geq 1$, $\tilde{x}$ $(v,\lambda_1,\lambda_2,\mu_1,\ldots,\mu_{2N-2})$-BNSD $\tilde{y}$ if and only if*

$$\begin{aligned}
&\min\left\{1,\frac{\lambda_2}{\lambda_1}\right\}\int_a^t(-1)^{N+1}v^{(N)}(x)[F^{(N-1)}(x)-G^{(N-1)}(x)]_+dx\\
&+\min\left\{1,\frac{\lambda_1}{\lambda_2}\right\}\int_t^b(-1)^{N+1}v^{(N)}(x)[F^{(N-1)}(x)-G^{(N-1)}(x)]_+dx\\
\leq&\frac{1}{\lambda_1}\int_a^t(-1)^{N+1}v^{(N)}(x)[G^{(N-1)}(x)-F^{(N-1)}(x)]_+dt\\
&+\frac{1}{\lambda_2}\int_t^b(-1)^{N+1}v^{(N)}(x)[G^{(N-1)}(x)-F^{(N-1)}(x)]_+dx\\
&+\sum_{n=1}^{N-1}\frac{1}{\lambda_2\mu_{2n}}(-1)^{n+1}v^{(n)}(b)[G^{(n)}(b)-F^{(n)}(b)]_+\\
&-\sum_{n=1}^{N-1}\mu_{2n-1}\min\left\{1,\frac{\lambda_1}{\lambda_2}\right\}(-1)^{n+1}v^{(n)}(b)[F^{(n)}(b)-G^{(n)}(b)]_+ \text{ for all } t\in[a,b],
\end{aligned}\tag{11}$$

where F and G are the cdfs of $\tilde{x}$ and $\tilde{y}$, respectively.

Recall that the integral condition for $\tilde{x}$ NSD $\tilde{y}$ is $F^{(n)}(b) \le G^{(n)}(b)$ for $n = 1, ..., N-1$ together with $F^{(N-1)}(x) \le G^{(N-1)}(x)$ for all $x \in [a, b]$ (Levy 1992). Using the weight function in (3), condition (11) can be rewritten as

$$\mathcal{V}(F, G, t) \le \max\left\{\frac{1}{\lambda_1}, \frac{1}{\lambda_2}\right\} \mathcal{C}(F, G, t) \text{ for all } t \in [a, b], \tag{12}$$

where

$$\begin{aligned}
\mathcal{V}(F, G, t) = & \int_a^b (-1)^{N+1} v^{(N)}(x) [F^{(N-1)}(x) - G^{(N-1)}(x)]_+ w(x; t) dx \\
& + \sum_{n=1}^{N-1} \mu_{2n-1} (-1)^{n+1} v^{(n)}(b) [F^{(n)}(b) - G^{(n)}(b)]_+ w(b; t), \\
\mathcal{C}(F, G, t) = & \int_a^b (-1)^{N+1} v^{(N)}(x) [G^{(N-1)}(x) - F^{(N-1)}(x)]_+ w(x; t) dx \\
& + \sum_{n=1}^{N-1} \frac{1}{\mu_{2n}} (-1)^{n+1} v^{(n)}(b) [G^{(n)}(b) - F^{(n)}(b)]_+ w(b; t).
\end{aligned}$$

Thus, condition (12) admits the same interpretation as (8): the overall weighted difference on the region violating NSD is no greater than the fraction $\max\{1/\lambda_1, 1/\lambda_2\}$ of that on the region satisfying NSD. It reduces to $\mathbb{E}v(\tilde{x}) \ge \mathbb{E}v(\tilde{y})$ when $\lambda_1 = \lambda_2 = \mu_1 = \cdots = \mu_{2N-2} = 1$, and approaches the condition for $\tilde{x}$ NSD $\tilde{y}$ as $\lambda_1, \lambda_2, \mu_1, ..., \mu_{2N-2}$ tend to infinity.

To facilitate the use of BNSD, one can reduce the number of parameters that need to be estimated by assigning special values. The pair (λ_1, λ_2) can take values (λ, λ), $(1, \lambda)$ or $(\lambda, 1)$, reflecting symmetric deviation, reinforcement or weakening of $(N+1)$th-order absolute risk aversion. For each $1 \le n \le N-1$, only one of μ_{2n-1} or μ_{2n} is relevant for a given pair of lotteries, as only one of $[G^{(n)}(b) - F^{(n)}(b)]_+$ or $[F^{(n)}(b) - G^{(n)}(b)]_+$ is nonzero. The relevant parameter may be set to 1, reflecting consensus at $x = b$, or to ∞, ensuring robustness to perturbations there. With these choices, the corresponding integral condition for BNSD simplifies accordingly. In addition to these simpler specifications, BNSD generalizes many known stochastic dominance rules. Table 3 summarizes these cases, showing how existing rules are recovered under specific choices of benchmark utilities and deviation parameters. It directly extends Table 2 to higher degrees.

It is worth noting that the almost Nth-degree stochastic dominance shown in Table 3 lacks the hierarchy property. To overcome this drawback, Tsetlin et al. (2015) introduce generalized almost stochastic dominance (GASD), which successively imposes

$$\sup_{x \in [a,b]} \left\{(-1)^{n+1} u^{(n)}(x)\right\} \le \inf_{x \in [a,b]} \left\{(-1)^{n+1} u^{(n)}(x)\right\} \left(\frac{1}{\varepsilon_n} - 1\right).$$

for all $n = 1, ..., N$. This condition for order n implicitly takes the nth-degree polynomial utility as the benchmark, resulting in different benchmark utility functions being used across orders.

Table 3 Existing rules as special cases of benchmark-based Nth-degree ($N \geq 2$) stochastic dominance

v	$\lambda_1,\ \lambda_2,\ \mu_1,...,\mu_{2N-2}$	rule	reference
A. Combined stochastic dominance			
$v(x) = (-1)^{N+1}x^N + \sum_{n=1}^{N-1} k_n x^n$	$\lambda_1 = \frac{1}{\gamma_{cv}},\ \lambda_2 = \frac{1}{\gamma_{cx}}$; $\mu_1,...,\mu_{2N-2} \to \infty$	$(N+\gamma_{cv}, N+\gamma_{cx})$-SD	Feng and Shanthikumar (2025)
B. Almost stochastic dominance			
$v(x) = (-1)^{N+1}x^N + \sum_{n=1}^{N-1} k_n x^n$	$\lambda_1 = \lambda_2 = \frac{1}{\varepsilon} - 1$; $\mu_1,...,\mu_{2N-2} \to \infty$	almost NSD	Tsetlin et al. (2015)
C. Fractional degree stochastic dominance			
$v(x) = (-1)^{N+1}x^N + \sum_{n=1}^{N-1} k_n x^n$	$\lambda_1 = \frac{1}{\gamma},\ \lambda_2 \to \infty$; $\frac{\mu_{2n-1}}{\lambda_2} \to \infty,\ \mu_{2n} \to \infty$	$(N+\gamma)$-SD	Bi and Zhu (2019)
$v(x) = (-1)^{N+1}e^{(1/c-1)x} + \sum_{n=1}^{N-1} k_n x^n$	$\lambda_1 = 1,\ \lambda_2 \to \infty$; $\frac{\mu_{2n-1}}{\lambda_2} \to \infty,\ \mu_{2n} \to \infty$	$(N+c)$-SD	Huang et al. (2020)
$v(x) = (-1)^{N+1}x^{N+1/c-1} + \sum_{n=1}^{N-1} k_n x^n$	$\lambda_1 = 1,\ \lambda_2 \to \infty$; $\frac{\mu_{2n-1}}{\lambda_2} \to \infty,\ \mu_{2n} \to \infty$	$(N+c)$-SD	Mao et al. (2025)

Note: SD and NSD stand for stochastic dominance and Nth-degree stochastic dominance, respectively. The parameters k_n $(n = 1,...,N-1)$ are chosen so that $v(x)$ satisfies $(-1)^{n+1}v^{(n)}(x) > 0$ for all $x \in [a,b]$ and $n = 1,...,N-1$. The remaining parameters satisfy $\gamma \in (0,1]$, $\varepsilon \in (0,1/2]$ and $c \in (0,1]$.

Such implicit substitution substantially reduces tractability and makes the corresponding integral condition for GASD cumbersome. By contrast, our benchmark-based stochastic dominance retains the same benchmark utility across all orders, not only exhibiting a hierarchy property in that $(v, \hat{\lambda}_1, \hat{\lambda}_2, \hat{\mu}_1, ..., \hat{\mu}_{2N-4})$-B(N-1)SD implies $(v, \lambda_1, \lambda_2, \mu_1, ..., \mu_{2N-2})$-BNSD (c.f., Proposition 7), but also being significantly more tractable than GASD.

Finally, we have established benchmark-based stochastic dominance rules for utility functions exhibiting mixed risk aversion, which implies an intuitive preference for "combining good with bad" in the context of risk apportionment (Eeckhoudt and Schlesinger 2006, Eeckhoudt et al. 2009). Recently, increasing attention has been given to preferences for "combining good with good" and "bad with bad" (Crainich et al. 2013, Denuit et al. 2013, Ebert 2013). These opposite preferences require utility functions to satisfy $u^{(n)} > 0$ for all $n \in \mathbb{N}$, a property termed mixed risk lovingness or convex preferences (Crainich et al. 2013). Benchmark-based stochastic dominance can likewise be defined for convex preferences, and the corresponding results are provided in the Appendix.

7. Application

To demonstrate the advantage of benchmark-based stochastic dominance, we employ it to rank stocks and bonds for different investment horizons. This analysis helps determine whether the opti-

Table 4 Descriptive statistics for stock and bond portfolios

	1 month	6 months	12 months	24 months	48 months	60 months
A. Stock portfolio						
mean	1.01	1.06	1.12	1.25	1.54	1.72
std. dev.	0.05	0.14	0.21	0.30	0.49	0.58
skew	0.13	0.05	0.17	-0.22	0.36	0.21
kurt	7.37	3.40	3.20	0.72	1.57	0.50
min	0.71	0.52	0.34	0.21	0.25	0.34
max	1.39	1.99	2.56	2.45	3.91	4.49
B. Bond portfolio						
mean	1.00	1.02	1.03	1.07	1.14	1.19
std. dev.	0.00	0.02	0.03	0.06	0.13	0.17
skew	1.10	1.05	1.05	1.08	1.12	1.10
kurt	1.41	1.08	1.00	0.94	0.89	0.74
min	1.00	1.00	1.00	1.00	1.00	1.00
max	1.01	1.08	1.15	1.29	1.56	1.70

Note: This table reports the means, standard deviations, skewness, kurtosis, minimal value and maximum value for gross returns on bond portfolios (Panel A) and stock portfolios (Panel B) with different holding periods.

mal allocation between stocks and bonds ought to vary with the investment horizon, an important issue related to retirement planning and life-cycle investing (Leshno and Levy 2002, Levy 2009, Bali et al. 2009). To facilitate comparison, we adopt the empirical setting of Bali et al. (2009), who ranked stocks and bonds across multiple horizons using almost stochastic dominance.

Denote the one-period rate of return on stocks and bonds by $\tilde{r}_t^{\mathrm{S}}$ and $\tilde{r}_t^{\mathrm{B}}$, respectively. The gross returns of portfolios invested 100% in stocks and 100% in bonds over T consecutive periods, denoted by $\tilde{x}_T$ and $\tilde{y}_T$, can be written as $\tilde{x}_T = \prod_{t=1}^{T}(1+\tilde{r}_t^{\mathrm{S}})$ and $\tilde{y}_T = \prod_{t=1}^{T}(1+\tilde{r}_t^{\mathrm{B}})$. As in Bali et al. (2009), we compare the returns of stock and bond portfolios over holding periods of 1, 6, 12, 24, 48 and 60 months. The monthly returns on stocks are measured by value-weighted returns of all U.S. firms listed on the NYSE, AMEX or NASDAQ, and the monthly returns on bonds are measured by the one-month Treasury bill rate. Both data series are retrieved from the Kenneth French data library, covering the period from July 1926 to December 2024. Table 4 reports the descriptive statics for stock and bond portfolios with different holding periods. For all investment horizons, the mean and standard deviation of stock returns are greater than those of bond returns. Both the mean and standard deviation of stock and bond returns increase with the investment horizon. The

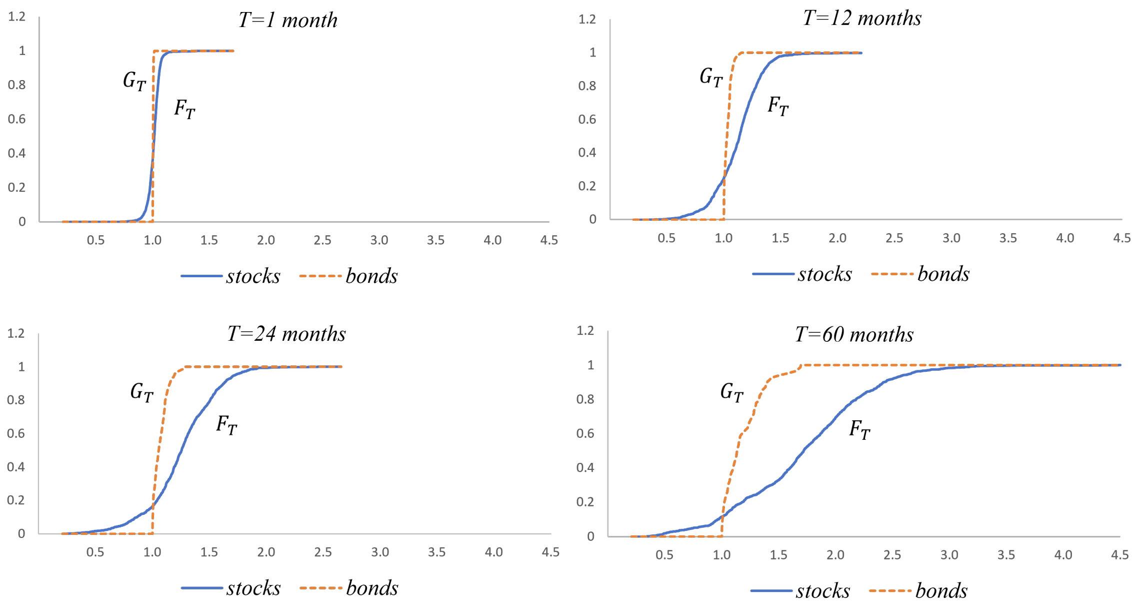


Figure 2 **The empirical cdfs of stock and bond portfolios for different investment horizons.**

skewness and excess kurtosis are far different from zero in most cases, implying that the empirical distributions of stock and bond returns do not generally follow a normal distribution.

To construct the empirical cdf for each portfolio, we first identify the minimum and maximum returns and divide the return range into equal-sized bins, each with a width of 0.1%. The empirical cdf is then generated by computing the proportion of portfolio returns that fall below the boundary value of each bin. Figure 2 displays the empirical cdfs of the stock and bond portfolios for investment horizons of 1, 12, 24 and 60 months. In each case, the cdf of the stock portfolio intersects that of the bond portfolio exactly once from above, indicating that stocks do not dominate bonds by FSD. As the investment horizon increases, the cdfs of both portfolios shift to the right, reflecting higher returns. Meanwhile, the area where the stock cdf lies above the bond cdf shrinks, while the area where it lies below expands. This suggests that the violation area of the FSD relationship in which stocks are thought to dominate bonds diminishes with the investment horizon.

Following Bali et al. (2009), we base the comparison between the stock portfolio $\tilde{x}_T$ and the bond portfolio $\tilde{y}_T$ on their empirical cdfs, which are denoted as F_T and G_T, respectively. We adopt the widely used iso-power utility function

$$v(x) = \frac{1}{1-\gamma_v}(x^{1-\gamma_v} - 1)$$

as the benchmark utility function, with the Arrow-Pratt relative risk aversion index γ_v taking values from $\{-2, -1, 0, 1, 2, ..., 8\}$. The range spanned by these values covers the majority of relative risk aversion levels observed in empirical and experimental studies (Halek and Eisenhauer 2001, Deck et al. 2024), and both risk averse and risk seeking preferences are considered. As our characterization of deviations depends on the wealth domain, we normalize the initial wealth to one and fix $[a,b] = [0.21, 4.49]$ throughout the analysis, with the endpoints set to the minimum and maximum returns across all portfolios. This ensures that utility functions are defined on the same interval, regardless of which portfolios are compared.

When applying benchmark-based stochastic dominance to rank $\tilde{x}_T$ and $\tilde{y}_T$, we can derive three results in sequence: i) which prospect is more favorable under the benchmark utility function; ii) the maximum allowable deviation of the utility function from the benchmark that preserves the ranking; iii) the critical probability premium in simple lottery choices corresponding to this maximum deviation. Among them, ii) is derived from the cdfs of $\tilde{x}_T$ and $\tilde{y}_T$, and iii) helps the decision-maker assess whether the allowable deviation accommodates their natural variation in choice behavior. If it does, the ranking remains robust even if their preference fluctuates; otherwise, the ranking may be overturned due to larger-than-allowed preference changes. Notably, under benchmark-based stochastic dominance, either $\tilde{x}_T$ or $\tilde{y}_T$ may dominate the other, depending on which prospect is more favorable under the benchmark utility function.

7.1. Results Ranked by BFSD

To apply BFSD to compare stock and bond portfolios, we first set $\lambda_1 = \lambda_2 = \lambda$. According to Theorem 1, when $\mathbb{E}v(\tilde{x}_T) > \mathbb{E}v(\tilde{y}_T)$, the maximum value of λ for $\tilde{x}_T$ (v, λ, λ)-BFSD $\tilde{y}_T$ is given by

$$\bar{\lambda} = \frac{\int_a^b v'(x)[G_T(x) - F_T(x)]_+ dx}{\int_a^b v'(x)[F_T(x) - G_T(x)]_+ dx},$$

and when $\mathbb{E}v(\tilde{y}_T) > \mathbb{E}v(\tilde{x}_T)$, the maximum value of λ for $\tilde{y}_T$ (v, λ, λ)-BFSD $\tilde{x}_T$ is given by

$$\bar{\lambda}^{\dagger} = \frac{\int_a^b v'(x)[F_T(x) - G_T(x)]_+ dx}{\int_a^b v'(x)[G_T(x) - F_T(x)]_+ dx},$$

where a superscript $\dagger$ is used to indicate that bonds dominate stocks.

The estimates of $\bar{\lambda}$ and $\bar{\lambda}^{\dagger}$ based on empirical cdfs F_T and G_T are reported in Panel A of Table 5. When $\gamma_v \leq 3$ regardless of the investment horizon, or when $\gamma_v = 4$ and the horizon is one month or six months, stocks dominate bonds under the corresponding $\bar{\lambda}$. For all $\gamma_v \leq 3$, $\bar{\lambda}$ increases with the horizon, implying that the set of utility functions that favor stocks over bonds grows larger, meaning that the consensus that stocks are superior to bonds becomes increasingly broad. When $\gamma_v = 4$ and the horizon is twelve months or longer, or when $\gamma_v \geq 5$ regardless of the horizon, bonds dominate stocks under the corresponding $\bar{\lambda}^{\dagger}$. However, for any given $\gamma_v \geq 5$, $\bar{\lambda}^{\dagger}$ does not vary monotonically with the investment horizon; instead, it first rises and then declines as the horizon lengthens. This finding suggests that, when the benchmark utility function exhibits high risk aversion, an increasing number of deviating utility functions initially prefer the bond

Table 5 Maximum allowable deviation for BFSD to hold

Investment horizon	γ_v										
	-2	-1	0	1	2	3	4	5	6	7	8
A. The maximum value of λ for (v,λ,λ)-BFSD to hold											
1 month	1.71	1.58	1.46	1.35	1.25	1.15	1.06	$1.03^{\dagger}$	$1.13^{\dagger}$	$1.23^{\dagger}$	$1.36^{\dagger}$
6 months	3.74	3.05	2.48	2.01	1.62	1.29	1.02	$1.27^{\dagger}$	$1.67^{\dagger}$	$2.25^{\dagger}$	$3.11^{\dagger}$
12 months	6.41	4.72	3.47	2.53	1.81	1.25	$1.20^{\dagger}$	$1.94^{\dagger}$	$3.38^{\dagger}$	$6.45^{\dagger}$	$13.52^{\dagger}$
24 months	14.94	9.53	6.00	3.66	2.10	1.09	$2.03^{\dagger}$	$5.34^{\dagger}$	$16.63^{\dagger}$	$59.06^{\dagger}$	$230.08^{\dagger}$
48 months	53.15	26.93	13.77	6.90	3.26	1.41	$1.86^{\dagger}$	$5.51^{\dagger}$	$18.02^{\dagger}$	$63.52^{\dagger}$	$235.25^{\dagger}$
60 months	90.67	41.65	19.60	9.30	4.38	2.02	$1.12^{\dagger}$	$2.61^{\dagger}$	$6.40^{\dagger}$	$16.28^{\dagger}$	$42.88^{\dagger}$
B. The maximum value of λ_2 for $(v,1,\lambda_2)$-BFSD to hold											
1 month	1.71	1.58	1.46	1.35	1.25	1.15	1.06	$\infty^{\dagger}$	$\infty^{\dagger}$	$\infty^{\dagger}$	$\infty^{\dagger}$
6 months	3.74	3.05	2.48	2.01	1.62	1.29	1.02	$\infty^{\dagger}$	$\infty^{\dagger}$	$\infty^{\dagger}$	$\infty^{\dagger}$
12 months	6.41	4.72	3.47	2.53	1.81	1.25	$\infty^{\dagger}$	$\infty^{\dagger}$	$\infty^{\dagger}$	$\infty^{\dagger}$	$\infty^{\dagger}$
24 months	14.94	9.53	6.00	3.66	2.10	1.09	$\infty^{\dagger}$	$\infty^{\dagger}$	$\infty^{\dagger}$	$\infty^{\dagger}$	$\infty^{\dagger}$
48 months	53.15	26.93	13.77	6.90	3.26	1.41	$\infty^{\dagger}$	$\infty^{\dagger}$	$\infty^{\dagger}$	$\infty^{\dagger}$	$\infty^{\dagger}$
60 months	90.67	41.65	19.60	9.30	4.38	2.02	$\infty^{\dagger}$	$\infty^{\dagger}$	$\infty^{\dagger}$	$\infty^{\dagger}$	$\infty^{\dagger}$
C. The maximum value of λ_1 for $(v,\lambda_1,1)$-BFSD to hold											
1 month	∞	∞	∞	∞	∞	∞	∞	$1.03^{\dagger}$	$1.13^{\dagger}$	$1.23^{\dagger}$	$1.36^{\dagger}$
6 months	∞	∞	∞	∞	∞	∞	∞	$1.27^{\dagger}$	$1.67^{\dagger}$	$2.25^{\dagger}$	$3.11^{\dagger}$
12 months	∞	∞	∞	∞	∞	∞	$1.20^{\dagger}$	$1.94^{\dagger}$	$3.38^{\dagger}$	$6.45^{\dagger}$	$13.52^{\dagger}$
24 months	∞	∞	∞	∞	∞	∞	$2.03^{\dagger}$	$5.34^{\dagger}$	$16.63^{\dagger}$	$59.06^{\dagger}$	$230.08^{\dagger}$
48 months	∞	∞	∞	∞	∞	∞	$1.86^{\dagger}$	$5.51^{\dagger}$	$18.02^{\dagger}$	$63.53^{\dagger}$	$235.25^{\dagger}$
60 months	∞	∞	∞	∞	∞	∞	$1.12^{\dagger}$	$2.61^{\dagger}$	$6.40^{\dagger}$	$16.28^{\dagger}$	$42.88^{\dagger}$

Note: Numbers without the superscript † correspond to stocks dominating bonds, and numbers with a superscript † correspond to bonds dominating stocks.

portfolio as the investment horizon grows. However, when the horizon becomes sufficiently long, stocks become more appealing to some of these utility functions, thereby reducing the set of utility functions that continue to favor bonds over stocks.

In the special case of $\gamma_v = 0$, (v,λ,λ)-BFSD coincides with the notion of almost FSD developed by Leshno and Levy (2002), who introduced a parameter ε to eliminate extreme risk preferences within the utility set used to define stochastic dominance. By converting the critical value $\bar{\lambda}$ at $\gamma_v = 0$ to the corresponding critical ε via $\bar{\varepsilon} = 1/(1+\bar{\lambda})$, we obtain $\bar{\varepsilon} = 40.7\%, 28.7\%, 22.4\%, 14.3\%, 6.8\%, 4.9\%$ for investment horizons of 1, 6, 12, 24, 48 and 60 months, respectively. Based on the experimental results of Levy et al. (2010), who estimated $\bar{\varepsilon} = 5.9\%$ for a group of subjects from simple lottery choices, we conclude that the stock portfolio dominates the bond portfolio by almost FSD only for the 60-month investment horizon. It is important to note that almost FSD, which uses $v(x) = x$ as the sole benchmark, excludes many economically relevant preferences from the analysis, a viewpoint first demonstrated by Levy (2009). Indeed, according to our Proposition 1, an iso-power utility function $u(x) = x^{1-\gamma_u}/(1-\gamma_u)$ falls within $U_1^v(\lambda_1, \lambda_2)$ if and only if

$$-\frac{\log \lambda_1}{\log(b/a)} \leq \gamma_u - \gamma_v \leq \frac{\log \lambda_2}{\log(b/a)}.$$

Substituting $a = 0.21$, $b = 4.49$, $\gamma_v = 0$, $\lambda_1 = \lambda_2 = 19.36$ into the above formula, we obtain $-0.97 \leq \gamma_u \leq 0.97$. This means that if $v(x) = x$ is used as the benchmark, iso-power utility functions with $\gamma_u \leq -1$ or $\gamma_u \geq 1$ are excluded from the underlying utility set when comparing stocks and bonds over horizons up to 60 months. Therefore, allowing users to specify different utility functions as the benchmark enables them to incorporate a broader range of economically relevant preferences in their evaluation. Overall, the results in Panel A of Table 5 echo the finding of Levy (2009) that popular preferences do not necessarily support long-run investment in equities. In fact, for utility functions with relative risk aversion of 4 or more, bonds may be preferred to stocks for the long run while stocks are preferred for shorter horizons. The robustness of these assertions to preference perturbations can be measured by the critical value $\bar{\lambda}$ or $\bar{\lambda}^\dagger$ in the table.

The symmetric deviation assumption in (v, λ, λ)-BFSD treats increases and decreases in risk aversion equally. To explore how widely the ranking by the benchmark utility function is accepted when risk aversion only goes up or down, $(v, 1, \lambda_2)$-BFSD and $(v, \lambda_1, 1)$-BFSD can be applied. According to Theorem 1, when $\mathbb{E}v(\tilde{x}_T) > \mathbb{E}v(\tilde{y}_T)$, the maximum value of λ_2 for $\tilde{x}_T$ $(v, 1, \lambda_2)$-BFSD $\tilde{y}_T$ is $\bar{\lambda}_2 = \infty$ if $\Omega_2 = \{t \in [a,b] : \int_a^t v'(x)[F_T(x) - G_T(x)]dx > 0\} = \emptyset$ and is given by

$$\bar{\lambda}_2 = \inf_{t \in \Omega_2} \frac{\int_t^b v'(x)[G_T(x) - F_T(x)]dx}{\int_a^t v'(x)[F_T(x) - G_T(x)]dx}$$

otherwise; when $\mathbb{E}v(\tilde{y}_T) > \mathbb{E}v(\tilde{x}_T)$, the maximum value of λ_2 for $\tilde{y}_T$ $(v, 1, \lambda_2)$-BFSD $\tilde{x}_T$ is $\bar{\lambda}_2^\dagger = \infty$ if $\Omega_2^\dagger = \{t \in [a,b] : \int_a^t v'(x)[G_T(x) - F_T(x)]dx > 0\} = \emptyset$ and is given by

$$\bar{\lambda}_2^\dagger = \inf_{t \in \Omega_2^\dagger} \frac{\int_t^b v'(x)[F_T(x) - G_T(x)]dx}{\int_a^t v'(x)[G_T(x) - F_T(x)]dx}$$

otherwise. Similarly, when $\mathbb{E}v(\tilde{x}_T) > \mathbb{E}v(\tilde{y}_T)$, the maximum value of λ_1 for $\tilde{x}_T$ $(v, \lambda_1, 1)$-BFSD $\tilde{y}_T$ is $\bar{\lambda}_1 = \infty$ if $\Omega_1 = \{t \in [a,b] : \int_t^b v'(x)[F_T(x) - G_T(x)]dx > 0\} = \emptyset$ and is given by

$$\bar{\lambda}_1 = \inf_{t \in \Omega_1} \frac{\int_a^t v'(x)[G_T(x) - F_T(x)]dx}{\int_t^b v'(x)[F_T(x) - G_T(x)]dx}$$

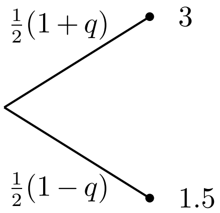
otherwise; when $\mathbb{E}v(\tilde{y}_T) > \mathbb{E}v(\tilde{x}_T)$, the maximum value of λ_1 for $\tilde{y}_T$ $(v, \lambda_1, 1)$-BFSD $\tilde{x}_T$ is $\bar{\lambda}_1^\dagger = \infty$ if $\Omega_1^\dagger = \{t \in [a,b] : \int_t^b v'(x)[G_T(x) - F_T(x)]dx > 0\} = \emptyset$ and is given by

$$\bar{\lambda}_1^\dagger = \inf_{t \in \Omega_1^\dagger} \frac{\int_a^t v'(x)[F_T(x) - G_T(x)]dx}{\int_t^b v'(x)[G_T(x) - F_T(x)]dx}$$

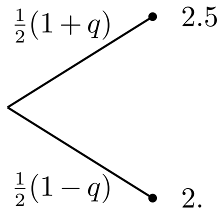
otherwise.

Given that in our data F_T crosses G_T exactly once from above, the theoretical analysis (see Claim A1 in the Appendix) shows that when $\mathbb{E}v(\tilde{x}_T) > \mathbb{E}v(\tilde{y}_T)$, $\bar{\lambda}_2 = \bar{\lambda}$ and $\bar{\lambda}_1 = \infty$, whereas when $\mathbb{E}v(\tilde{y}_T) > \mathbb{E}v(\tilde{x}_T)$, $\bar{\lambda}_2^\dagger = \infty$ and $\bar{\lambda}_1^\dagger = \bar{\lambda}$. The numerical estimates of these parameters, reported in Panels B and C of Table 5, are in exact agreement with this theoretical result. In Panel B, where risk aversion is allowed to increase only, the ranking that bonds dominate stocks is always preserved, regardless of how risk-averse the decision-maker becomes, whereas the ranking that stocks dominate bonds is maintained only when risk aversion increases within a limited range, with the critical threshold $\bar{\lambda}_2$ exactly matching the critical threshold $\bar{\lambda}$ in Panel A. In Panel C, when risk aversion moves in the opposite direction, the results show a diametrically opposed trend. Taken together, Table 5 demonstrates that the preference for stocks is affected only when risk aversion increases, while the preference for bonds is affected only when risk aversion decreases.

Finally, by mapping the parameters defining the maximum allowable deviation to bounds on probability premia in simple lottery choices, the decision-maker can assess whether their observed preference fluctuations fall within the range permitted by the ranking. For illustration, consider the case where the benchmark utility function has $\gamma_v = 2$, the investment horizon is 24 months, and deviations from the benchmark are symmetric. With the initial wealth normalized to one, the decision-maker makes choices between



and



The benchmark utility function implies a probability premium of 0.43. According to Proposition 2, the lower and upper bounds on probability premia corresponding to $\bar{\lambda} = 2.05$ are 0.10 and 0.67, respectively. Thus, an observed probability premium within $[0.10, 0.67]$ suggests that the decision-maker's preference variation is within the permitted range, ensuring the ranking's robustness to their preference uncertainty. Conversely, a probability premium outside this range invalidates robustness. The decision-maker can elicit their preference variation more reliably by making a series of such choices, as demonstrated in Section 2.2, and then check the robustness of the ranking by comparing the elicited λ with $\bar{\lambda}$.

7.2. Results Ranked by BSSD

To demonstrate the application of BSSD, we focus the analysis on $\lambda_1 = \lambda_2 = \lambda$ for simplicity. The fact $\mathbb{E}\tilde{x}_T > \mathbb{E}\tilde{y}_T$ implies $F_T^{(1)}(b) < G_T^{(1)}(b)$. Substituting this into Theorem 2, we obtain that when $\mathbb{E}v(\tilde{x}_T) > \mathbb{E}v(\tilde{y}_T)$, the maximum value of λ for $\tilde{x}_T$ $(v, \lambda, \lambda, \mu_1, \mu_2)$-BSSD $\tilde{y}_T$ is

$$\bar{\lambda} = \frac{\int_a^b [-v''(x)][G_T^{(1)}(x) - F_T^{(1)}(x)]_+ dx + \frac{v'(b)}{\mu_2}[G_T^{(1)}(b) - F_T^{(1)}(b)]_+}{\int_a^b [-v''(x)][F_T^{(1)}(x) - G_T^{(1)}(x)]_+ dx},$$

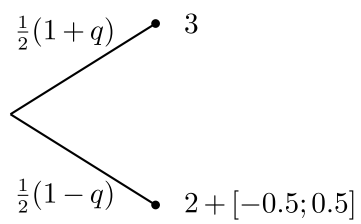
and when $\mathbb{E}v(\tilde{y}_T) > \mathbb{E}v(\tilde{x}_T)$, the maximum value of λ for $\tilde{y}_T$ $(v, \lambda, \lambda, \mu_1, \mu_2)$-BSSD $\tilde{x}_T$ is

$$\bar{\lambda}^\dagger = \frac{\int_a^b [-v''(x)][F_T^{(1)}(x) - G_T^{(1)}(x)]_+ dx}{\int_a^b [-v''(x)][G_T^{(1)}(x) - F_T^{(1)}(x)]_+ dx + \mu_1 v'(b)[G_T^{(1)}(b) - F_T^{(1)}(b)]_+}, \text{ provided } \bar{\lambda}^\dagger \geq 1.$$

Otherwise, if $\bar{\lambda}^\dagger < 1$, μ_1 is so large that no valid λ exists for $\tilde{y}_T$ $(v, \lambda, \lambda, \mu_1, \mu_2)$-BSSD $\tilde{x}_T$.

Table 6 presents the estimates of $\bar{\lambda}$ and $\bar{\lambda}^\dagger$, considering only risk-averse benchmark utilities with $\gamma_v \in \{1, 2, ..., 8\}$. Panel A corresponds to the case where $\mu_1 = \mu_2 = 1$, which reveals a pattern similar to that observed in Table 5. Specifically, when the benchmark utility has low relative risk aversion and favors the stock portfolio, $\bar{\lambda}$ increases broadly with the investment horizon. Conversely, for benchmark utilities with high relative risk aversion and preferring the bond portfolio, $\bar{\lambda}$ initially increases with the investment horizon but declines once the horizon exceeds a certain length. Panel B sets $\mu_1 = \mu_2 = \infty$, implying that the risk aversion level at $x = b$ can take any arbitrary positive value. Under this condition, $\bar{\lambda}$ becomes smaller than its counterpart in Panel A, indicating that relaxing the endpoint constraint entails tightening the constraint on the interior. Meanwhile, $\bar{\lambda}^\dagger$ is not definable, because the preference for bonds reverses as the allowable deviating function $u = v + kx$ becomes risk-neutral when $k > 0$ tends to infinity. Panel C estimates $\bar{\lambda}$ and $\bar{\lambda}^\dagger$ for the hypothetical scenario where the decision-maker's relative risk aversion at $x = b$ is elicited to be within $[1, 10]$. This translates to $\mu_1 = \gamma_v$ and $\mu_2 = 10/\gamma_v$, both of which are finite. Then both $\bar{\lambda}$ and $\bar{\lambda}^\dagger$ are valid and smaller than their counterparts in the case where $\mu_1 = \mu_2 = 1$.

The parameters defining the maximum allowable deviation in Table 6 can also be mapped to bounds on probability premia in simple lottery choices, allowing the decision-maker to easily assess whether their observed preference fluctuations fall within the permitted range. Consider again the case with a benchmark utility function of $\gamma_v = 2$ and a 24-month investment horizon. With the initial wealth normalized to one, the decision-maker makes choices between



and

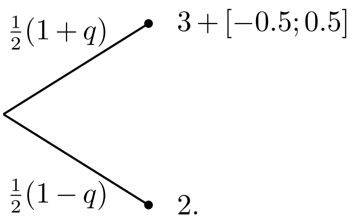


Table 6 Maximum allowable deviation for BSSD to hold

Investment horizon	γ_v							
	1	2	3	4	5	6	7	8
A. The maximum value of λ for $(v, \lambda, \lambda, 1, 1)$-BSSD to hold								
1 month	6.87	3.09	1.85	1.25	$1.12^{\dagger}$	$1.50^{\dagger}$	$1.96^{\dagger}$	$2.52^{\dagger}$
6 months	8.23	3.30	1.75	1.04	$1.56^{\dagger}$	$2.45^{\dagger}$	$3.82^{\dagger}$	$5.98^{\dagger}$
12 months	8.79	3.17	1.48	$1.34^{\dagger}$	$2.65^{\dagger}$	$5.39^{\dagger}$	$11.59^{\dagger}$	$26.61^{\dagger}$
24 months	10.53	3.13	1.13	$2.46^{\dagger}$	$7.49^{\dagger}$	$25.60^{\dagger}$	$97.61^{\dagger}$	$404.88^{\dagger}$
48 months	19.47	4.91	1.53	$2.08^{\dagger}$	$6.94^{\dagger}$	$24.73^{\dagger}$	$93.71^{\dagger}$	$372.14^{\dagger}$
60 months	28.00	7.18	2.41	$1.14^{\dagger}$	$3.07^{\dagger}$	$8.33^{\dagger}$	$23.16^{\dagger}$	$66.17^{\dagger}$
B. The maximum value of λ for $(v, \lambda, \lambda, \infty, \infty)$-BSSD to hold								
1 month	5.21	2.91	1.83	1.24	-	-	-	-
6 months	6.11	3.09	1.72	1.03	-	-	-	-
12 months	6.39	2.94	1.45	-	-	-	-	-
24 months	7.43	2.86	1.10	-	-	-	-	-
48 months	12.82	4.36	1.47	-	-	-	-	-
60 months	17.74	6.26	2.30	-	-	-	-	-
C. The maximum value of λ for $(v, \lambda, \lambda, \gamma_v, \frac{10}{\gamma_v})$-BSSD to hold								
1 month	5.38	2.95	1.83	1.24	$1.12^{\dagger}$	$1.50^{\dagger}$	$1.96^{\dagger}$	$2.52^{\dagger}$
6 months	6.32	3.13	1.73	1.03	$1.55^{\dagger}$	$2.45^{\dagger}$	$3.82^{\dagger}$	$5.98^{\dagger}$
12 months	6.63	2.98	1.46	$1.32^{\dagger}$	$2.64^{\dagger}$	$5.38^{\dagger}$	$11.59^{\dagger}$	$26.61^{\dagger}$
24 months	7.74	2.91	1.11	$2.41^{\dagger}$	$7.43^{\dagger}$	$25.53^{\dagger}$	$97.52^{\dagger}$	$404.77^{\dagger}$
48 months	13.49	4.47	1.49	$2.01^{\dagger}$	$6.84^{\dagger}$	$24.60^{\dagger}$	$93.54^{\dagger}$	$371.91^{\dagger}$
60 months	18.77	6.44	2.33	$1.09^{\dagger}$	$3.01^{\dagger}$	$8.27^{\dagger}$	$23.10^{\dagger}$	$66.11^{\dagger}$

Note: Numbers without the superscript † correspond to stocks dominating bonds, and numbers with a superscript † correspond to bonds dominating stocks. The symbol - indicates that valid values do not exist.

The benchmark utility implies a probability premium of 0.56. According to Proposition 8, the lower and upper bounds on probability premia corresponding to the critical value $\bar{\lambda} = 2.91$ in Panel C are 0.09 and 0.82, respectively. Thus, if the relative risk aversion at $x = b$ lies within $[1, 10]$, then an observed probability premium within $[0.09, 0.82]$ suggests that the decision-maker's preference variation is within the permitted range, indicating the ranking's robustness. Conversely, premium fluctuations outside this range invalidate robustness. The decision-maker can make a series of similar choices as demonstrated in Section 2.2 to better infer their preference variation.

8. Conclusion

Utility function and stochastic dominance are two widely used tools for comparing risky prospects. They represent two extreme ends of the expected utility framework: the former yields a total order but is sensitive to model specification, while the latter is model-free but often too narrow in scope to be practically useful. To enhance flexibility, we propose an integrated approach—benchmark-based stochastic dominance—as a smooth interpolation between these two extremes. It uses a single utility function as a benchmark to describe an individual's preference while incorporating the principle of stochastic dominance by allowing deviations from the benchmark within a defined range. The bounds on utility deviations are deliberately designed to be both interpretable and elicitable, facilitating a closed-form expression of the consensus on the ranking of risky prospects. Benchmark-based stochastic dominance not only enables users to rank risky prospects according to the benchmark utility function but also evaluates the robustness of the ranking to varying levels of preference uncertainty. A case study comparing stocks and bonds for different investment horizons illustrates its application. As a product of this integration, benchmark-based stochastic dominance encompasses existing rules of almost stochastic dominance and fractional degree stochastic dominance as special cases, providing a unified perspective on these rules.

The expected utility framework has spawned diverse applications over decades, from constructing efficient portfolios (Levy and Sarnat 1984), to studying risky behavior (Eeckhoudt and Schlesinger 2008), to solving optimization problems (Zhang and Homem-de Mello 2017). Yet in these applications, utility function and stochastic dominance have typically been used separately, often without meaningful integration. Benchmark-based stochastic dominance preserves the precision of utility function in capturing key patterns of individual preferences, while leveraging the non-parametric strength of stochastic dominance to handle personalized preference variations. We therefore expect that benchmark-based stochastic dominance, as a novel semi-parametric approach, can facilitate the integration of utility-function-based and stochastic-dominance-based analyses to deliver more comprehensive insights across various applications.

While we have briefly introduced the procedure for eliciting the critical parameters in benchmark-based stochastic dominance using simple lottery choices in laboratory experiments, an important task for future research is to develop methods for estimating these parameters from real-world data, possibly building on Huang et al. (2021), who estimated the parameters in almost stochastic dominance from insurance deductible choices. Finally, our study is situated within the univariate expected utility framework. Extending our approach to multivariate utility frameworks (Müller et al. 2025) or non-expected utility frameworks (Cerreia-Vioglio et al. 2016) could be promising avenues for further investigation.

Appendix

In the proofs, we rely on Lebesgue-Stieltjes integrals, which generalize Riemann integrals and coincide with them under standard regularity conditions.

A. Auxiliary Lemmas

Lemma A1. *Let $[a,b] \subset \mathbb{R}$ be a bounded interval. Suppose f and g are right-continuous of bounded variation on $[a,b]$. Then the Lebesgue-Stieltjes integrals $\int_a^b f(x)dg(x)$ and $\int_a^b g(x)df(x)$ are well-defined and finite. If f or g is continuous, then*

$$\int_a^b f(x)dg(x) + \int_a^b g(x)df(x) = f(b)g(b) - f(a)g(a).$$

Proof. The stated integration-by-parts identity appears verbatim in Theorem 3.36 of Folland (1999). A more general version that allows for discontinuities in both f and g, along with a complete proof, is provided in Carter and van Brunt (2000). □

Lemma A2. *Let $[a,b] \subset \mathbb{R}$ be a bounded interval. Suppose α is a monotone decreasing function and β is a Lebesgue integrable function. If there exits a finite number $m \leq 0$ such that $\int_a^t \beta(x)dx \geq m$ for all $t \in [a,b]$, then $\int_a^b \alpha(x)\beta(x)dx \geq \alpha(b)(\int_a^b \beta(x)dx - m) + \alpha(a)m$.*

Proof. The set of discontinuities of α is at most countable. Define $\overline{\alpha}(x) = \alpha(x+) = \lim_{y \to x+} \alpha(y)$ for $x \in [a,b)$, and set $\overline{\alpha}(b) = \alpha(b)$. Then $\overline{\alpha}$ is right-continuous and decreasing. Moreover, both α and $\overline{\alpha}$ are differentiable almost everywhere, and satisfy $\alpha'(x) = \overline{\alpha}'(x)$ almost everywhere (Folland 1999, Theorem 3.23). Since β is Lebesgue integrable, the function $x \mapsto \int_a^x \beta(y)dy$ is continuous and of bounded variation. Applying Lemma A1 with $f(x) = \overline{\alpha}(x)$ and $g(x) = \int_a^x \beta(y)dy - m$ yields

$$\int_a^b \alpha(x)\beta(x)dx = \int_a^b \overline{\alpha}(x)\beta(x)dx = \int_a^b \overline{\alpha}(x)dg(x) = \overline{\alpha}(b)g(b) + \overline{\alpha}(a)m - \int_a^b g(x)d\overline{\alpha}(x).$$

By the Lebesgue-Radon-Nikodym theorem, the signed Borel measure associated with $d\overline{\alpha}$ admits a unique decomposition into an absolutely continuous part and a singular part. This yields

$$\int_a^b g(x)d\overline{\alpha}(x) = \int_a^b g(x)\overline{\alpha}'(x)dx + \sum_{x_i \in \Omega} g(x_i)(\overline{\alpha}(x_i) - \overline{\alpha}(x_i-)) \leq 0,$$

where Ω is the countable set of discontinuities of $\overline{\alpha}$. The proof is completed by substituting $\overline{\alpha}(a) \leq \alpha(a)$, $\overline{\alpha}(b) = \alpha(b)$, $m \leq 0$ and $\int_a^b g(x)d\overline{\alpha}(x) \leq 0$ into the integration-by-parts identity above. □

Lemma A3. *Let $[a,b] \subset \mathbb{R}$ be a bounded interval, and $\delta \in (0,1), k \in \mathbb{R}$ be constants. Suppose β is a Lebesgue integrable function. If*

$$\int_a^t \beta(x)dx + \delta \int_t^b \beta(x)dx + \delta k \geq 0 \textit{ for all } t \in [a,b],$$

then $\int_a^b \alpha(x)\beta(x)dx + \alpha(b)k \geq 0$ for all monotone decreasing functions α with $\alpha(a) = 1$ and $\alpha(b) \geq \delta$.

Proof. Substituting $\int_t^b \beta(x)dx = \int_a^b \beta(x)dx - \int_a^t \beta(x)dx$ into the condition gives $\int_a^t \beta(x)dx \geq m$ for all $t \in [a,b]$, where $m = -\frac{\delta}{1-\delta}(\int_a^b \beta(x)dx + k)$. Applying Lemma A2, we obtain

$$\begin{aligned}\int_a^b \alpha(x)\beta(x)dx + \alpha(b)k \geq& \alpha(b)\left(\int_a^b \beta(x)dx - m\right) + m + \alpha(b)k = \alpha(b)\left(\int_a^b \beta(x)dx + k\right) + m(1-\alpha(b))\\ \geq& \delta\left(\int_a^b \beta(x)dx + k\right) + m(1-\delta) = 0,\end{aligned}$$

where the final inequality holds because $m \leq 0$, which is implied by the assumed condition. □

Claim A1. *Let $v \in U_1$ be a benchmark utility function, and let the cdf of $\tilde{x}_T$, denoted as F_T, cross the cdf of $\tilde{y}_T$, denoted as G_T, exactly once from above. When using BFSD to rank $\tilde{x}_T$ and $\tilde{y}_T$, if $\mathbb{E}v(\tilde{x}_T) > \mathbb{E}v(\tilde{y}_T)$, then $\bar{\lambda}_1 = \infty$ and $\bar{\lambda}_2 = \bar{\lambda}$; if $\mathbb{E}v(\tilde{y}_T) > \mathbb{E}v(\tilde{x}_T)$, then $\bar{\lambda}_1^\dagger = \bar{\lambda}$ and $\bar{\lambda}_2^\dagger = \infty$.*

Proof. If $\mathbb{E}v(\tilde{x}_T) > \mathbb{E}v(\tilde{y}_T)$, then $\int_a^b v'(x)[G_T(x) - F_T(x)]dx > 0$. By the single-crossing property, $\Omega_1 = \{t \in [a,b] : \int_t^b v'(x)[F_T(x) - G_T(x)]dx > 0\} = \emptyset$ and $\bar{\lambda}_1 = \infty$; $\Omega_2 = \{t \in [a,b] : \int_a^t v'(x)[F_T(x) - G_T(x)]dx > 0\} \neq \emptyset$ and $\bar{\lambda}_2 = \inf_{t \in \Omega_2} \frac{\int_t^b v'(x)[G_T(x) - F_T(x)]dx}{\int_a^t v'(x)[F_T(x) - G_T(x)]dx} > 1$. Moreover,

$$\frac{d}{dt}\log\bar{\lambda}_2 = [F_T(t) - G_T(t)]\left(\frac{1}{\int_t^b v'(x)[G_T(x) - F_T(x)]dx} - \frac{1}{\int_a^t v'(x)[F_T(x) - G_T(x)]dx}\right),$$

indicating that $\bar{\lambda}_2$ first decreases and then increases, attaining its minimum at the crossing point. Consequently, $\bar{\lambda}_2 = \frac{\int_a^b v'(x)[G_T(x) - F_T(x)]_+ dx}{\int_a^b v'(x)[F_T(x) - G_T(x)]_+ dx} = \bar{\lambda}$. The case $\mathbb{E}v(\tilde{y}_T) > \mathbb{E}v(\tilde{x}_T)$ follows by symmetry. □

B. Proofs of Main Results

Proof of Proposition 1. The proposition follows by substituting $u'(x)/u'(y) = \exp(\int_x^y A_u(t)dt)$ and $v'(x)/v'(y) = \exp(\int_x^y A_v(t)dt)$ into condition (1). □

Proof of Proposition 2. Since u' and v' are continuous, the analysis can be restricted to $x, y \in (a,b)$. Abbreviate $q_w(x,y;k)$ as q_w. For $w = u, v$, $(1+q_w)[w(y) - w(y-k)] = (1-q_w)[w(x) - w(x-k)]$. Dividing the corresponding terms on both sides yields

$$\frac{1+q_u}{1+q_v}\left[\frac{u(y)-u(y-k)}{v(y)-v(y-k)}\right] = \frac{1-q_u}{1-q_v}\left[\frac{u(x)-u(x-k)}{v(x)-v(x-k)}\right].$$

Using the mean value theorem and rearranging the terms, we obtain for $\theta_1, \theta_2 \in (0,1)$ that

$$\frac{(1+q_u)/(1-q_u)}{(1+q_v)/(1-q_v)} = \frac{u'(x-\theta_1 k)/u'(y-\theta_2 k)}{v'(x-\theta_1 k)/v'(y-\theta_2 k)}. \tag{A1}$$

Based on this equation, we claim that $u \in U_1^v(\lambda_1, \lambda_2)$ if and only if

$$\frac{1}{\lambda_1} \leq \frac{(1+q_u)/(1-q_u)}{(1+q_v)/(1-q_v)} \leq \lambda_2 \tag{A2}$$

for all x, y, k such that $a \leq x-k < x < y-k < y \leq b$. The "if" part can be seen by letting $k \to 0$, which implies (1) through (A1). The "only if" part follows by applying (1) with $x - \theta_1 k < y - \theta_2 k$ to (A1). The proof is completed by noting that (A2) is equivalent to $\xi(q_v, \frac{1}{\lambda_1}) \leq q_u \leq \xi(q_v, \lambda_2)$. □

Proof of Theorem 1. To prove the "only if" part, let $\hat{u}(x)$ be a specific utility function satisfying

$$\frac{\hat{u}'(x)}{v'(x)}=\begin{cases}\min\{1,\lambda_2/\lambda_1\}, & \text{if } x<t \text{ and } F(x)\geq G(x),\\ 1/\lambda_1, & \text{if } x<t \text{ and } F(x)<G(x),\\ \min\{1,\lambda_1/\lambda_2\}, & \text{if } x\geq t \text{ and } F(x)\geq G(x),\\ 1/\lambda_2, & \text{if } x\geq t \text{ and } F(x)<G(x).\end{cases}$$

Then, $\hat{u}\in U_1^v(\lambda_1,\lambda_2)$ and condition (2) is equivalent to $\mathbb{E}\hat{u}(\tilde{x})\geq\mathbb{E}\hat{u}(\tilde{y})$ for all $\hat{u}$ as t varies over the entire interval $[a,b]$. To prove the "if" part, we normalize $\sup\{\frac{u'(y)}{v'(y)}:a\leq y\leq b\}=1$ and assume $\lambda_1\leq\lambda_2$. Let $\phi_u(x)=\sup\{\frac{u'(y)}{v'(y)}:x\leq y\leq b\}$ and $\varphi_u(x)=\sup\{\frac{u'(y)}{v'(y)}:a\leq y\leq x\}$. Then, $\phi_u(x)$ is decreasing in x, $\varphi_u(x)$ is increasing in x, and condition (1) is equivalent to $u'(x)/v'(x)\geq\max\{\frac{1}{\lambda_1}\phi_u(x),\frac{1}{\lambda_2}\varphi_u(x)\}$ for all $x\in[a,b]$. Noting that $\max\{\phi_u(x),\varphi_u(x)\}\equiv\sup\{\frac{u'(y)}{v'(y)}:a\leq y\leq b\}$ is independent of x, $\psi_u(x)=\max\{\phi_u(x),\frac{\lambda_1}{\lambda_2}\varphi_u(x)\}$ is decreasing in x with $\psi_u(a)=1$ and $\psi_u(b)\geq\frac{\lambda_1}{\lambda_2}$, and condition (1) becomes

$$\frac{1}{\lambda_1}\psi_u(x)\leq\frac{u'(x)}{v'(x)}\leq\psi_u(x)\text{ for all }x\in[a,b].$$

By integration by parts, we obtain

$$\begin{aligned}\mathbb{E}u(\tilde{x})-\mathbb{E}u(\tilde{y})&=\int_a^b\frac{u'(x)}{v'(x)}[G(x)-F(x)]v'(x)dx\\&\geq\int_a^b\psi_u(x)\left(\frac{1}{\lambda_1}[G(x)-F(x)]_+-[F(x)-G(x)]_+\right)v'(x)dx.\end{aligned}$$

Applying Lemma A3 with $\delta=\frac{\lambda_1}{\lambda_2}$, $k=0$, $\alpha(x)=\psi_u(x)$ and $\beta(x)=(\frac{1}{\lambda_1}[G(x)-F(x)]_+-[F(x)-G(x)]_+)v'(x)$ to the last integral in this inequality yields $\mathbb{E}u(\tilde{x})\geq\mathbb{E}u(\tilde{y})$ under condition (1). The alternative case $\lambda_1>\lambda_2$ can be handled symmetrically. □

Proof of Proposition 3. The first result follows by substituting $u''(x)/u''(y)=\exp(\int_x^y P_u(t)dt)$ and $v''(x)/v''(y)=\exp(\int_x^y P_v(t)dt)$ into condition (5). By the mean value theorem, we have

$$\frac{u'(x)}{v'(x)}=\left[\frac{v'(b)}{v'(x)}\right]\frac{u'(b)}{v'(b)}+\left[1-\frac{v'(b)}{v'(x)}\right]\frac{u'(x)-u'(b)}{v'(x)-v'(b)}=\left[\frac{v'(b)}{v'(x)}\right]\frac{u'(b)}{v'(b)}+\left[1-\frac{v'(b)}{v'(x)}\right]\frac{u''(\zeta)}{v''(\zeta)}$$

for some $\zeta\in(x,b)$. Hence,

$$\frac{u'(x)/u''(x)}{v'(x)/v''(x)}=\left[\frac{v'(b)}{v'(x)}\right]\frac{[u'(b)/u''(b)][u''(b)/u''(x)]}{[v'(b)/v''(b)][v''(b)/v''(x)]}+\left[1-\frac{v'(b)}{v'(x)}\right]\frac{u''(\zeta)/u''(x)}{v''(\zeta)/v''(x)}.$$

If u satisfies (5) and (6), then $\frac{1}{\mu_2}\leq\frac{u'(b)/u''(b)}{v'(b)/v''(b)}\leq\mu_1$, $\frac{1}{\lambda_2}\leq\frac{u''(b)/u''(x)}{v''(b)/v''(x)}\leq\lambda_1$ and $\frac{1}{\lambda_2}\leq\frac{u''(\zeta)/u''(x)}{v''(\zeta)/v''(x)}\leq\lambda_1$. Substituting these bounds into the previous expression establishes the second result. □

Proof of Proposition 4. By the mean value theorem, we have

$$\frac{u'(x)}{v'(x)}=\left[\frac{v'(y)}{v'(x)}\right]\frac{u'(y)}{v'(y)}+\left[1-\frac{v'(y)}{v'(x)}\right]\frac{u'(x)-u'(y)}{v'(x)-v'(y)}=\left[\frac{v'(y)}{v'(x)}\right]\frac{u'(y)}{v'(y)}+\left[1-\frac{v'(y)}{v'(x)}\right]\frac{u''(\zeta_1)}{v''(\zeta_1)},\tag{A3}$$

$$\frac{u'(y)}{v'(y)}=\left[\frac{v'(b)}{v'(y)}\right]\frac{u'(b)}{v'(b)}+\left[1-\frac{v'(b)}{v'(y)}\right]\frac{u'(y)-u'(b)}{v'(y)-v'(b)}=\left[\frac{v'(b)}{v'(y)}\right]\frac{u'(b)}{v'(b)}+\left[1-\frac{v'(b)}{v'(y)}\right]\frac{u''(\zeta_2)}{v''(\zeta_2)},\tag{A4}$$

where $\zeta_1 \in (x,y)$ and $\zeta_2 \in (y,b)$. By (A4), $\frac{u'(y)}{v'(y)}$ lies in between $\frac{u'(b)}{v'(b)}$ and $\frac{u''(\zeta_2)}{v''(\zeta_2)}$. Comparing $\frac{u''(\zeta_1)}{v''(\zeta_1)}$ with $\frac{u'(b)}{v'(b)}$, we obtain $\frac{1}{\lambda_1\mu_1}\frac{u'(b)}{v'(b)} \le \frac{1}{\lambda_1}\frac{u''(b)}{v''(b)} \le \frac{u''(\zeta_1)}{v''(\zeta_1)} \le \lambda_2\frac{u''(b)}{v''(b)} \le \lambda_2\mu_2\frac{u'(b)}{v'(b)}$ from (5) and (6). Comparing $\frac{u''(\zeta_1)}{v''(\zeta_1)}$ with $\frac{u''(\zeta_2)}{v''(\zeta_2)}$, we obtain $\frac{1}{\lambda_1}\frac{u''(\zeta_2)}{v''(\zeta_2)} \le \frac{u''(\zeta_1)}{v''(\zeta_1)} \le \lambda_2\frac{u''(\zeta_2)}{v''(\zeta_2)}$ from (5). Taken together, we have $\frac{1}{\lambda_1\mu_1}\frac{u'(y)}{v'(y)} \le \frac{u''(\zeta_1)}{v''(\zeta_1)} \le \lambda_2\mu_2\frac{u'(y)}{v'(y)}$. Substituting it into (A3), we conclude that $\frac{1}{\lambda_1\mu_1}\frac{u'(y)}{v'(y)} \le \frac{u'(x)}{v'(x)} \le \lambda_2\mu_2\frac{u'(y)}{v'(y)}$. ☐

Proof of Theorem 2. To prove the "only if" part, let $\hat{u}(x)$ be a specific utility function satisfying

$$\frac{\hat{u}''(x)}{v''(x)} = \begin{cases} \min\{1,\lambda_2/\lambda_1\}, \text{ if } x<t \text{ and } F^{(1)}(x) \ge G^{(1)}(x), \\ 1/\lambda_1, \text{ if } x<t \text{ and } F^{(1)}(x) < G^{(1)}(x), \\ \min\{1,\lambda_1/\lambda_2\}, \text{ if } x \ge t \text{ and } F^{(1)}(x) \ge G^{(1)}(x), \\ 1/\lambda_2, \text{ if } x \ge t \text{ and } F^{(1)}(x) < G^{(1)}(x), \end{cases}$$

with $\hat{u}'(b)/v'(b) = \mu_1\hat{u}''(b)/v''(b) = \mu_1\min\{1,\lambda_1/\lambda_2\}$ if $F^{(1)}(b) \ge G^{(1)}(b)$ and $\hat{u}'(b)/v'(b) = (1/\mu_2)\hat{u}''(b)/v''(b) = 1/(\lambda_2\mu_2)$ otherwise. Then, $\hat{u} \in U_2^v(\lambda_1,\lambda_2;\mu_1,\mu_2)$ and condition (7) is equivalent to $\mathbb{E}\hat{u}(\tilde{x}) \ge \mathbb{E}\hat{u}(\tilde{y})$ for all $\hat{u}$ as t varies over the entire interval $[a,b]$. To prove the "if" part, assume $\sup\{\frac{u''(y)}{v''(y)} : a \le y \le b\} = 1$ and $\lambda_1 \le \lambda_2$. Let $\phi_u(x) = \sup\{\frac{u''(y)}{v''(y)} : x \le y \le b\}$ and $\varphi_u(x) = \sup\{\frac{u''(y)}{v''(y)} : a \le y \le x\}$. By an argument similar to that in the proof of Theorem 1, $\psi_u(x) = \max\{\phi_u(x), \frac{\lambda_1}{\lambda_2}\varphi_u(x)\}$ is decreasing in x with $\psi_u(a) = 1$ and $\psi_u(b) \ge \frac{\lambda_1}{\lambda_2}$. Condition (5) becomes

$$\frac{1}{\lambda_1}\psi_u(x) \le \frac{u''(x)}{v''(x)} \le \psi_u(x) \text{ for all } x \in [a,b].$$

By integration by parts, we obtain

$$\mathbb{E}u(\tilde{x}) - \mathbb{E}u(\tilde{y}) = \left[\frac{u'(b)}{v'(b)}\right] v'(b)[G^{(1)}(b) - F^{(1)}(b)] + \int_a^b \frac{u''(x)}{v''(x)}[-v''(x)][G^{(1)}(x) - F^{(1)}(x)]dx.$$

Invoking condition (6) and substituting $\frac{1}{\lambda_1\mu_2}\psi_u(b) \le \frac{1}{\mu_2}\frac{u''(b)}{v''(b)} \le \frac{u'(b)}{v'(b)} \le \mu_1\frac{u''(b)}{v''(b)} \le \mu_1\psi_u(b)$ gives

$$\begin{aligned}\mathbb{E}u(\tilde{x}) - \mathbb{E}u(\tilde{y}) \ge& \psi_u(b)v'(b)\left(\frac{1}{\lambda_1\mu_2}[G^{(1)}(b) - F^{(1)}(b)]_+ - \mu_1[F^{(1)}(b) - G^{(1)}(b)]_+\right) \\ &+ \int_a^b \psi_u(x)[-v''(x)]\left(\frac{1}{\lambda_1}[G^{(1)}(x) - F^{(1)}(x)]_+ - [F^{(1)}(x) - G^{(1)}(x)]_+\right)dx.\end{aligned}$$

Applying Lemma A3 with $\delta = \frac{\lambda_1}{\lambda_2}$, $k = v'(b)(\frac{1}{\lambda_1\mu_2}[G^{(1)}(b) - F^{(1)}(b)]_+ - \mu_1[F^{(1)}(b) - G^{(1)}(b)]_+)$, $\alpha(x) = \psi_u(x)$ and $\beta(x) = [-v''(x)](\frac{1}{\lambda_1}[G^{(1)}(x) - F^{(1)}(x)]_+ - [F^{(1)}(x) - G^{(1)}(x)]_+)$, we obtain $\mathbb{E}u(\tilde{x}) \ge \mathbb{E}u(\tilde{y})$. The alternative case $\lambda_1 > \lambda_2$ can be handled symmetrically. ☐

Proof of Proposition 6. The first result follows by substituting $u^{(N)}(x)/u^{(N)}(y) = \exp(\int_x^y A_u^{(N+1)}(t)dt)$ and $v^{(N)}(x)/v^{(N)}(y) = \exp(\int_x^y A_v^{(N+1)}(t)dt)$ into condition (9). Condition (10) implies $\frac{1}{\mu_{2N-3}} \le \frac{A_u^{(N)}(b)}{A_v^{(N)}(b)} \le \mu_{2N-2}$, and $\frac{1}{\mu_{2n-3}\mu_{2n}} \le \frac{A_u^{(n)}(b)}{A_v^{(n)}(b)} \le \mu_{2n-2}\mu_{2n-1}$ for $n = 2,...,N-1$. For $u \in U_N$, $\frac{v^{(n-1)}(b)}{v^{(n-1)}(x)} \in (0,1)$ for $x < b$ and by the mean value theorem, we have

$$\begin{aligned}\frac{u^{(n-1)}(x)}{v^{(n-1)}(x)} &= \left[\frac{v^{(n-1)}(b)}{v^{(n-1)}(x)}\right]\frac{u^{(n-1)}(b)}{v^{(n-1)}(b)} + \left[1 - \frac{v^{(n-1)}(b)}{v^{(n-1)}(x)}\right]\frac{u^{(n-1)}(x) - u^{(n-1)}(b)}{v^{(n-1)}(x) - v^{(n-1)}(b)} \\ &= \left[\frac{v^{(n-1)}(b)}{v^{(n-1)}(x)}\right]\frac{u^{(n-1)}(b)}{v^{(n-1)}(b)} + \left[1 - \frac{v^{(n-1)}(b)}{v^{(n-1)}(x)}\right]\frac{u^{(n)}(\zeta)}{v^{(n)}(\zeta)}\end{aligned}$$

for some $\zeta \in (x, b)$. Hence,

$$\frac{A_v^{(n)}(x)}{A_u^{(n)}(x)} = \left[\frac{v^{(n-1)}(b)}{v^{(n-1)}(x)}\right]\left[\frac{A_v^{(n)}(b)}{A_u^{(n)}(b)}\right]\frac{u^{(n)}(b)/u^{(n)}(x)}{v^{(n)}(b)/v^{(n)}(x)} + \left[1 - \frac{v^{(n-1)}(b)}{v^{(n-1)}(x)}\right]\frac{u^{(n)}(\zeta)/v^{(n)}(x)}{v^{(n)}(\zeta)/v^{(n)}(x)}$$

If u satisfies (9) and (10), substituting $\frac{1}{\mu_{2N-2}} \le \frac{A_v^{(N)}(b)}{A_u^{(N)}(b)} \le \mu_{2N-3}$, $\frac{1}{\lambda_2} \le \frac{u^{(N)}(b)/u^{(N)}(x)}{v^{(N)}(b)/v^{(N)}(x)} \le \lambda_1$ and $\frac{1}{\lambda_2} \le \frac{u^{(N)}(\zeta)/u^{(N)}(x)}{v^{(N)}(\zeta)/v^{(N)}(x)} \le \lambda_1$ into the above identity establishes the second result for $n = N$. For $n \le N-1$, we appeal to the results stated in Proposition 7. Substituting $\frac{1}{\mu_{2N-4}\mu_{2N-3}} \le \frac{A_v^{(N-1)}(b)}{A_u^{(N-1)}(b)} \le \mu_{2N-5}\mu_{2N-2}$, $\frac{1}{\lambda_2\mu_{2N-2}} \le \frac{u^{(N-1)}(b)/u^{(N-1)}(x)}{v^{(N-1)}(b)/v^{(N-1)}(x)} \le \lambda_1\mu_{2N-3}$ and $\frac{1}{\lambda_2\mu_{2N-2}} \le \frac{u^{(N-1)}(\zeta)/u^{(N-1)}(x)}{v^{(N-1)}(\zeta)/v^{(N-1)}(x)} \le \lambda_1\mu_{2N-3}$ establishes the result for $n = N-1$, and substituting $\frac{1}{\mu_{2n-2}\mu_{2n-1}} \le \frac{A_v^{(n)}(b)}{A_u^{(n)}(b)} \le \mu_{2n-3}\mu_{2n}$, $\frac{1}{\lambda_2\mu_{2n}}\prod_{k=n+1}^{N-1}\frac{1}{\mu_{2k-1}\mu_{2k}} \le \frac{u^{(n)}(b)/u^{(n}(x)}{v^{(n)}(b)/v^{(n)}(x)} \le \lambda_1\mu_{2n-1}\prod_{k=n+1}^{N-1}\mu_{2k-1}\mu_{2k}$ and $\frac{1}{\lambda_2\mu_{2n}}\prod_{k=n+1}^{N-1}\frac{1}{\mu_{2k-1}\mu_{2k}} \le \frac{u^{(n)}(\zeta)/u^{(n}(x)}{v^{(n)}(\zeta)/v^{(n)}(x)} \le \lambda_1\mu_{2n-1}\prod_{k=n+1}^{N-1}\mu_{2k-1}\mu_{2k}$ establishes the results for $n = 2, ..., N-2$. $\square$

Proof of Proposition 7. The argument in Proposition 4 shows that $u \in U_N^v(\lambda_1, \lambda_2; \mu_1, ..., \mu_{2N-2})$ implies $\frac{1}{\lambda_1\mu_{2N-3}}\frac{u^{(N-1)}(y)}{v^{(N-1)}(y)} \le \frac{u^{(N-1)}(x)}{v^{(N-1)}(x)} \le \lambda_2\mu_{2N-2}\frac{u^{(N-1)}(y)}{v^{(N-1)}(y)}$. Moving to next lower order, we derive from condition (10) that $-\frac{1}{\mu_{2N-5}\mu_{2N-2}}\frac{v^{(N-1)}(b)}{v^{(N-2)}(b)} \le -\frac{u^{(N-1)}(b)}{u^{(N-2)}(b)} \le -\mu_{2N-4}\mu_{2N-3}\frac{v^{(N-1)}(b)}{v^{(N-2)}(b)}$. Applying the argument in Proposition 4 again, we obtain $\frac{1}{\lambda_1\mu_{2N-5}\mu_{2N-3}\mu_{2N-2}}\frac{u^{(N-2)}(y)}{v^{(N-2)}(y)} \le \frac{u^{(N-2)}(x)}{v^{(N-2)}(x)} \le \lambda_2\mu_{2N-4}\mu_{2N-3}\mu_{2N-2}\frac{u^{(N-2)}(y)}{v^{(N-2)}(y)}$. The result for all lower orders follows by induction. $\square$

Proof of Proposition 8. Let H_{A_N} and H_{B_N} denote the cdfs of A_N and B_N, respectively, and let q_w be short for $q_w(x, y; A_N, B_N)$. To verify the behavioral characterization of (9), note that

$$\frac{1+q_u}{1+q_v}\left[\frac{\mathbb{E}u(y+B_N) - \mathbb{E}u(y+A_N)}{\mathbb{E}v(y+B_N) - \mathbb{E}v(y+A_N)}\right] = \frac{1-q_u}{1-q_v}\left[\frac{\mathbb{E}u(x+B_N) - \mathbb{E}u(x+A_N)}{\mathbb{E}v(x+B_N) - \mathbb{E}v(x+A_N)}\right].$$

Since $u^{(N)}$ and $v^{(N)}$ are continuous, the analysis can be restricted to $x, y \in (a, b)$. By integration by parts, $\mathbb{E}u(y+B_N) - \mathbb{E}u(y+A_N) = (-1)^{N+1}\int_0^{b-a} u^{(N)}(y+z)[H_{A_N}^{(N-1)}(z) - H_{B_N}^{N-1}(z)]dz$, and similar expressions hold when u is replaced by v and y is replaced by x. Using the mean value theorem and rearranging the terms, we obtain for some z_1, z_2 that

$$\frac{(1+q_u)/(1-q_u)}{(1+q_v)/(1-q_v)} = \frac{u^{(N)}(x+z_1)/u^{(N)}(y+z_2)}{v^{(N)}(x+z_1)/v^{(N)}(y+z_2)}. \tag{A5}$$

Based on this equation, we claim that $u \in U_N$ satisfies condition (9) if and only if

$$\frac{1}{\lambda_1} \le \frac{(1+q_u)/(1-q_u)}{(1+q_v)/(1-q_v)} \le \lambda_2 \tag{A6}$$

for all $a < x < y < b$ and all A_N, B_N bounded by $\min\{x-a, b-y, (y-x)/2\}$. The "if" part follows by letting $k \to 0$, which implies (9) through (A5). The "only if" part follows by applying (9) with $x + z_1 < y + z_2$ to (A5). The characterization of (9) is completed by noting that (A6) is equivalent to $\xi(q_v, \frac{1}{\lambda_1}) \le q_u \le \xi(q_v, \lambda_2)$. The characterization of (10) can be verified in a similar manner. $\square$

Proof of Theorem 3. To prove the "only if" part, let $\hat{u}(x)$ be a specific utility function satisfying

$$\frac{\hat{u}^{(N)}(x)}{v^{(N)}(x)} = \begin{cases} \min\{1, \lambda_2/\lambda_1\}, \text{ if } x < t \text{ and } F^{(N-1)}(x) \geq G^{(N-1)}(x), \\ 1/\lambda_1, \text{ if } x < t \text{ and } F^{(N-1)}(x) < G^{(N-1)}(x), \\ \min\{1, \lambda_1/\lambda_2\}, \text{ if } x \geq t \text{ and } F^{(N-1)}(x) \geq G^{(N-1)}(x), \\ 1/\lambda_2, \text{ if } x \geq t \text{ and } F^{(N-1)}(x) < G^{(N-1)}(x), \end{cases}$$

together with $\hat{u}^{(n)}(b)/v^{(n)}(b) = \mu_{2n-1}\hat{u}^{(N)}(b)/v^{(N)}(b) = \mu_{2n-1}\min\{1, \lambda_1/\lambda_2\}$ if $F^{(n)}(b) \geq G^{(n)}(b)$ and $\hat{u}^{(n)}(b)/v^{(n)}(b) = (1/\mu_{2n})\hat{u}^{(n)}(b)/v^{(n)}(b) = 1/(\lambda_2\mu_{2n})$ otherwise for $n = 1, ..., N-1$. Then, $\hat{u} \in U_N^v(\lambda_1, \lambda_2; \mu_1, ..., \mu_{2N-2})$ and condition (11) is equivalent to $\mathbb{E}\hat{u}(\tilde{x}) \geq \mathbb{E}\hat{u}(\tilde{y})$ for all $\hat{u}$ as t varies over the entire interval $[a, b]$. To prove the "if" part, assume $\sup\{\frac{u^{(N)}(y)}{v^{(N)}(y)} : a \leq y \leq b\} = 1$ and $\lambda_1 \leq \lambda_2$. Let $\phi_u(x) = \sup\{\frac{u^{(N)}(y)}{v^{(N)}(y)} : x \leq y \leq b\}$ and $\varphi_u(x) = \sup\{\frac{u^{(N)}(y)}{v^{(N)}(y)} : a \leq y \leq x\}$. By an argument similar to that in the proof of Theorem 1, $\psi_u(x) = \max\{\phi_u(x), \frac{\lambda_1}{\lambda_2}\varphi_u(x)\}$ is decreasing in x with $\psi_u(a) = 1$ and $\psi_u(b) \geq \frac{\lambda_1}{\lambda_2}$. Condition (9) becomes

$$\frac{1}{\lambda_1}\psi_u(x) \leq \frac{u^{(N)}(x)}{v^{(N)}(x)} \leq \psi_u(x) \text{ for all } x \in [a, b].$$

By integration by parts, we obtain

$$\begin{aligned} \mathbb{E}u(\tilde{x}) - \mathbb{E}u(\tilde{y}) = & \sum_{n=1}^{N-1} (-1)^{n+1} \left[\frac{u^{(n)}(b)}{v^{(n)}(b)}\right] v^{(n)}(b)[G^{(n)}(b) - F^{(n)}(b)] \\ & + \int_a^b \left[\frac{u^{(N)}(x)}{v^{(N)}(x)}\right] (-1)^{N+1} v^{(N)}(x) \left[G^{(N-1)}(x) - F^{(N-1)}(x)\right] dx. \end{aligned}$$

Invoking condition (10) and substituting $\frac{1}{\lambda_1\mu_{2n}}\psi_u(b) \leq \frac{1}{\mu_{2n}}\frac{u^{(N)}(b)}{v^{(N)}(b)} \leq \frac{u^{(n)}(b)}{v^{(n)}(b)} \leq \mu_{2n-1}\frac{u^{(N)}(b)}{v^{(N)}(b)} \leq \mu_{2n-1}\psi_u(b)$ for $n = 1, ..., N-1$ into the above gives

$$\begin{aligned} & \mathbb{E}u(\tilde{x}) - \mathbb{E}u(\tilde{y}) \\ \geq & \psi_u(b) \sum_{n=1}^{N-1} (-1)^{n+1} v^{(n)}(b) \left(\frac{1}{\lambda_1\mu_{2n}}[G^{(n)}(b) - F^{(n)}(b)]_+ - \mu_{2n-1}[F^{(n)}(b) - G^{(n)}(b)]_+\right) \\ & + \int_a^b \psi_u(x)(-1)^{N+1} v^{(N)}(x) \left(\frac{1}{\lambda_1}[G^{(N-1)}(x) - F^{(N-1)}(x)]_+ - [F^{(N-1)}(x) - G^{(N-1)}(x)]_+\right) dx. \end{aligned}$$

Applying Lemma A3 with $\delta = \frac{\lambda_1}{\lambda_2}$, $k = \sum_{n=1}^{N-1}(-1)^{n+1}v^{(n)}(b)(\frac{1}{\lambda_1\mu_{2n}}[G^{(n)}(b) - F^{(n)}(b)]_+ - \mu_{2n-1}[F^{(n)}(b) - G^{(n)}(b)]_+)$, $\alpha(x) = \psi_u(x)$ and $\beta(x) = (-1)^{N+1}v^{(N)}(x)(\frac{1}{\lambda_1}[G^{(N-1)}(x) - F^{(N-1)}(x)]_+ - [F^{(N-1)}(x) - G^{(N-1)}(x)]_+)$, we obtain $\mathbb{E}u(\tilde{x}) \geq \mathbb{E}u(\tilde{y})$. The alternative case $\lambda_1 > \lambda_2$ can be handled symmetrically. $\square$

C. Benchmark-Based Convex Stochastic Dominance

For $N \geq 2$, utility functions exhibiting convex preferences form the set

$$\bar{U}_N = \left\{ u: \ u^{(n)}(x) > 0 \text{ for all } x \in [a,b],\ n = 1, ..., N \right\}.$$

Definition A1. For $N \geq 2$, let $v \in \bar{U}_N$ be a benchmark utility function and $\lambda_1, \lambda_2, \mu_1, ..., \mu_{2N-2} \geq 1$ be finite constants. We say that $u \in \bar{U}_N$ belongs to $\bar{U}_N^v(\lambda_1, \lambda_2; \mu_1, ..., \mu_{2N-2})$, meaning that its deviation from v is bounded by $[1/\lambda_1, \lambda_2] \times [1/\mu_1, \mu_2] \times \cdots \times [1/\mu_{2N-3}, \mu_{2N-2}]$, if it satisfies

$$\frac{1}{\lambda_1} \frac{v^{(N)}(x)}{v^{(N)}(y)} \leq \frac{u^{(N)}(x)}{u^{(N)}(y)} \leq \lambda_2 \frac{v^{(N)}(x)}{v^{(N)}(y)} \text{ for all } x, y \in [a,b] \text{ with } x < y \text{ and}$$
$$\frac{1}{\mu_{2n-1}} \frac{v^{(N)}(a)}{v^{(n)}(a)} \leq \frac{u^{(N)}(a)}{u^{(n)}(a)} \leq \mu_{2n} \frac{v^{(N)}(a)}{v^{(n)}(a)} \text{ for } n = 1, ..., N-1.$$

We say that $\tilde{x}$ dominates $\tilde{y}$ by *benchmark-based Nth-degree convex stochastic dominance parameterized by* $(v, \lambda_1, \lambda_2, \mu_1, ..., \mu_{2N-2})$, denoted by $\tilde{x}$ $(v, \lambda_1, \lambda_2, \mu_1, ..., \mu_{2N-2})$-BNCSD $\tilde{y}$, if $\mathbb{E}u(\tilde{x}) \geq \mathbb{E}u(\tilde{y})$ for all $u \in \bar{U}_N^v(\lambda_1, \lambda_2; \mu_1, ..., \mu_{2N-2})$.

The formulation of deviations parallels that within U_N, with the endpoint conditions now imposed at $x = a$. The results below follow from the same arguments as for concave preferences.

Proposition A1. *For $N \geq 2$, let $v \in \bar{U}_N$ be a benchmark utility function and $\lambda_1, \lambda_2, \mu_1, ..., \mu_{2N-2} \geq 1$ be finite constants. Then, any $u \in \bar{U}_N^v(\lambda_1, \lambda_2; \mu_1, ..., \mu_{2N-2})$ also satisfies*

$$\frac{1}{\lambda_1 \mu_{2N-2}} \frac{v^{(N-1)}(x)}{v^{(N-1)}(y)} \leq \frac{u^{(N-1)}(x)}{u^{(N-1)}(y)} \leq \lambda_2 \mu_{2N-3} \frac{v^{(N-1)}(x)}{v^{(N-1)}(y)} \textit{ for all } x, y \in [a,b] \textit{ with } x < y \textit{ and}$$
$$\frac{1}{\mu_{2n-1} \mu_{2N-2}} \frac{v^{(N-1)}(a)}{v^{(n)}(a)} \leq \frac{u^{(N-1)}(a)}{u^{(n)}(a)} \leq \mu_{2n} \mu_{2N-3} \frac{v^{(N-1)}(a)}{v^{(n)}(a)} \textit{ for } n = 1, ..., N-1.$$

For a generic cdf H, define $\bar{H}^{(0)}(x) = H(x)$ and $\bar{H}^{(n)}(x) = \int_x^b \bar{H}^{(n-1)}(t)dt$ iteratively for $n \geq 1$.

Theorem A1. *For $N \geq 2$, given a benchmark utility function $v \in U_N$ and $2N$ finite constants $\lambda_1, \lambda_2, \mu_1, ..., \mu_{2N-2} \geq 1$, $\tilde{x}$ $(v, \lambda_1, \lambda_2, \mu_1, ..., \mu_{2N-2})$-BNCSD $\tilde{y}$ if and only if*

$$\begin{aligned}
&\min\left\{1, \frac{\lambda_2}{\lambda_1}\right\} \int_a^t v^{(N)}(x)[\bar{F}^{(N-1)}(x) - \bar{G}^{(N-1)}(x)]_+ dx \\
&+ \min\left\{1, \frac{\lambda_1}{\lambda_2}\right\} \int_t^b v^{(N)}(x)[\bar{F}^{(N-1)}(x) - \bar{G}^{(N-1)}(x)]_+ dx \\
\leq &\frac{1}{\lambda_1} \int_a^t v^{(N)}(x)[\bar{G}^{(N-1)}(x) - \bar{F}^{(N-1)}(x)]_+ dt + \frac{1}{\lambda_2} \int_t^b v^{(N)}(x)[\bar{G}^{(N-1)}(x) - \bar{F}^{(N-1)}(x)]_+ dx \\
&+ \sum_{n=1}^{N-1} \frac{1}{\lambda_1 \mu_{2n}} v^{(n)}(a)[\bar{G}^{(n)}(a) - \bar{F}^{(n)}(a)]_+ - \sum_{n=1}^{N-1} \mu_{2n-1} \min\left\{1, \frac{\lambda_2}{\lambda_1}\right\} v^{(n)}(a)[\bar{F}^{(n)}(a) - \bar{G}^{(n)}(a)]_+
\end{aligned}$$

for all $t \in [a,b]$, where F and G are the cdfs of $\tilde{x}$ and $\tilde{y}$, respectively.

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