Pricing the Catastrophe Reinsurance by Government

Abstract

In this paper, we analyze a catastrophe insurance market with many homogenous inhabitants, one monopolistic insurer, and one government as reinsurer. The market equilibrium features that all individuals choose to buy the catastrophe insurance, the monopolistic insurer earns positive profit from offering the coverage and buys catastrophe reinsurance from the government, and the government collects reinsurance premium from the insurer and charges poll tax from each individual. We prove that in the equilibrium, there exists reinsurance contracts and corresponding reinsurance premium that eliminate the default risk of insurer, and there exists primary insurance price and exogenous tax so that both inhabitants and the insurer are better off than the case of no government reinsurance program. Our theoretical predictions can be considered as an ex-ante reinsurance planning extension to Charpentier and Le Maux's (2014) ex-post government bail-out plan. All theoretical propositions are also illustrated by a numerical case based on the extant literature and market practice.

Introduction

The increasing frequency and severity of catastrophe events endanger the viability and sustainability of the insurance and reinsurance industry. In 2017, the insured losses from natural and man-made disasters worldwide were the highest ever recorded in a single year at USD 144 billion (Swiss Re, 2018)[1].

The concerns of availability and affordability of private catastrophe insurance market call for public solutions (Jaffee and Russell, 1997; Froot, 2001). Charpentier and Le Maux (2014) describe the public catastrophe solution as that the government bails-out the default of private insurer by charging all inhabitants an ex-post tax. In their model, an insurer's probability of insolvency depends on the risk distribution, the premium rate, and the amount of capital held by the insurer. Compared with the limited liability scenario where victims retain the residual loss of insurer default, the maximum willingness to pay and the expected utility of inhabitants increases if the ex-post government coverage is available.

Different from the ex-post bail-out plan, many countries use ex-ante public reinsurance to support the private insurance for catastrophes (Culter and Zeckhauser, 1999; Bruggerman et al., 2010). For example, after the 9/11, private reinsurers were reluctant to provide terrorism coverage. The U.S. Terrorism Risk Insurance Act (TRIA) created a transparent federal-level system of shared public and private compensation for certain insured losses resulting from a certified act of terrorism, which is in essence a type of non-proportional reinsurance. In France, insurance companies are required to transfer half of their catastrophe risk to Caisse Centrale de Réassurance (CCR) and the CCR covers unlimited catastrophe loss above primary insurers' retention. The Dutch government and the Dutch Association of Insurers establish an insurance pool to compensate damage to crop caused by heavy rainfall and frost damage. In return, insurance companies are required to pay a premium of 0.1% of the total insured value. To the best of our knowledge, no research has been done to model the equilibrium and pricing of the government reinsurance program for catastrophe risk.

[1]https://www.swissre.com/institute/research/sigma-research/sigma-2018-01.html.

In this paper, we develop a general equilibrium framework to fill this gap. Specifically, the catastrophe insurance market includes three parties: many homogenous inhabitants, a monopolistic insurer (Charpentier and Le Maux, 2014), and a government as reinsurer. Inhabitants maximize their expected utility by choosing to insure or not insure. The insurer maximizes its expected profit by offering catastrophe coverage and charging corresponding premium. The government maximizes the weighted utilities of inhabitants and the insurer by offering reinsurance coverage, charging corresponding reinsurance premium from the insurer and charging a poll tax from inhabitants. In the market equilibrium, we find that there exists reinsurance contracts and corresponding reinsurance premium that eliminate the default risk of the primary insurer, and there exists competitive primary premium and exogenous tax so that both inhabitants and the insurer are better off than the case of no government reinsurance program.

Our paper contributes to the extant government-backed catastrophe insurance market models. In Charpentier and Le Maux's (2014), the government spreads a negative pecuniary externality by taxing all inhabitants after the insurer default. We at first time model how the government supports catastrophe insurance by an ex-ante arrangement of reinsurance and poll tax. Previous models incorporate default risk into the insurance pricing decision and the expected utility of policyholders (see e.g. Charpentier and Le Maux, 2014). We, however, prove that the default risk of insurer can be eliminated by our proposed solution. Moreover, in our setup, the government concerns not only the inhabitants but also the insurer, a more comprehensive consideration than extant literature focusing on inhabitants' welfare.

Our paper also contributes to studies on government role in catastrophe insurance markets. Government as reinsurer stimulates private insurers' supply and keep catastrophe insurance affordable (Kunreuther, 2008). Among ways of government intervention in catastrophe insurance market, government as reinsurer is considered the least problematic scheme (Sugerman, 2007; Bruggerman et al., 2010). However, it remains an open question how the market equilibrium with government as reinsurer looks like.

Our paper also contributes to studies on reinsurance pricing. A conventional reinsurance market involves two parties:insurer and reinsurer, and the reinsurance pricing consider the underlying risk and the market competition (Bühlmann, 1980; Bernand and Vanduffel, 2014). In the catastrophe insurance market with government as reinsurer, public reinsurance pricing on the one hand cannot accurately estimate the catastrophe risk due to limited experience and correlation of risk units, on the other hand should also consider the tax revenues from the original policyholders (i.e. inhabitants). To the best of our knowledge, we build the first catastrophe reinsurance pricing framework involving three parties (inhabitants, insurer, and government as reinsurer) and incorporating the potential premium subsidies from general tax.

The rest of the paper is structured as follows. In Section 2, we set up the model and derive the market equilibrium. In Section 3, we present and prove the key propositions. With a numerical case, we illustrate our propositions in Section 4. We conclude in Section 5. All proofs are provided in the Appendix.

Model

We share inhabitants utility and loss distribution with Charpentier and Le Maux (2014). There are n inhabitants living in one region, facing a catastrophe event that will cause a certain loss l_0 to N habitants in this region. The share of disaster-hit population is $x = \frac{N}{n}, x \in [0, 1]$, whose distribution function, $F(x|p, \delta)$, is determined by two elements. One is the probability that each individual claims a loss, p and another is the correlation between individuals' risk, δ. The distribution function of x is given as below:

$$F(x|p,\delta) = \int_0^x f(t)dt \in [0,1], \tag{1}$$

with

$$\begin{array}{lll} (i) \int_0^1 xf(x)dx = p, & (ii) \dfrac{\partial F(x|p,\delta)}{\partial p} < 0 \ \forall x \in [0,1], & (iii) \dfrac{\partial f}{\partial p} > 0, \ \forall x > x^*, \\ (iv) \dfrac{\partial F(x|p,\delta)}{\partial \delta} < 0 \ \forall x > x^*, & (v) \dfrac{\partial f}{\partial \delta} > 0 \ \forall x > x^*, & (vi) \dfrac{\partial p}{\partial \delta} = 0, \end{array} \tag{2}$$

where any $x > x^*$ represents an extreme event and x^* is exogenous. In reality, $x > x^*$ takes place when catastrophes as earthquakes and hurricanes hit the region. Higher correlation of claims, δ, and the probability for any individual to be a victim, p, imply greater occurrence of extreme events. To simplify, inhabitants are assumed to be strictly identical with the same utility function

$$U'(\cdot) > 0, \ U''(\cdot) < 0, \ U(0) = 0. \tag{3}$$

All inhabitants decide simultaneously whether to buy full insurance coverage in order to maximize their expected utility.

Following the framework of Charpentier and Le Maux (2014), we consider an economy with n inhabitants, an insurer and a government. As well, inhabitants simultaneously make fully insured or not insured decision to maximize their expected utility. However, we describe the catastrophe insurance market in another way: The monopolistic insurer sets the price of catastrophe insurance to maximize expected profit. The government charges reinsurance premium from the insurer and taxes from inhabitants to run the catastrophic reinsurance program. The goal of the government is to maximize total social benefit. As discussed in prior studies (Litan, 2005; Kunreuther, 2006; Charpentier et al., 2014), implicit regulations by government prevent moral-hazard and time-inconsistency problems. Fortunately, the model is not trapped in demanding work of asymmetric information. Traditional adverse selection theory is not considered in the catastrophe insurance market(Kunreuther, 1984; Jaffee and Russell, 1997). Thus, the distribution of risks is exogenous with the insurance decision in our model.

The Demand for Insurance

Faced with catastrophe risks, each inhabitant has to decide whether to buy the catastrophe insurance with the given premium, α_0, to maximize his own utility, $U_c(\alpha_0, l_0, T_0)$. Besides, the government collects poll tax, T_0, from each inhabitant. For any inhabitant, if he purchases this insurance, loss caused by the catastrophe would be covered by the insurer, and the corresponding utility of the inhabitant is $U_c(\alpha_0, l_0, T_0) = U(-\alpha_0 - T_0)$. Otherwise, the inhabitant has to bear the loss on his own and the corresponding expected utility is $U_c(\alpha_0, l_0, T_0) = pU(-l_0 - T_0) + (1-p)U(-T_0)$. With given tax, T_0, inhabitants' insurance decisions depend on the premium. They prefer to buy the catastrophe insurance when the utility with insurance is no less than the utility without insurance, i.e.

$$U(-T_0 - \alpha_0) \geq pU(-T_0 - l_0) + (1-p)U(-T_0). \tag{4}$$

That is,

$$\alpha_0 \leq -T_0 - U^{-1}[pU(-T_0 - l_0) + (1-p)U(-T_0)]. \tag{5}$$

We denote

$$\alpha_0^*(T_0) = -T_0 - U^{-1}[pU(-T_0 - l_0) + (1-p)U(-T_0)] \tag{6}$$

This is the maximum willingness to pay of inhabitants for catastrophe insurance. Inhabitants decide simultaneously whether to purchase full insurance coverage to maximize their expected utility.

The Supply for Insurance

Insurance coverage is provided by a single insurer (Emons, 2001; Charpentier and Le Maux, 2014). Following Einav et al. (2010) and Charpentier and Le Maux (2014), we take the characteristics of insurance contracts as given: only the contract price is determined by insurer, not the coverage offered. The insurer sells catastrophe insurance by offering full insurance coverage for l_0 in exchange for a premium α_0 from each inhabitant. The total loss is $l = nxl_0$ and the total premium income of the insurer is $n\alpha_0$. Following the pattern of American TRIA, French CCR and Dutch Agriver introduced above, government will provide a CAT XL reinsurance treaty. The insurer will cede a premium M to the reinsurance program by the government[2]. In turn, the government as reinsurer will cover the loss above the insurer's deductible, K, up to a cover limit, $\overline{K}$. And thus, the insurer's total profit, Π, is

$$\Pi(\alpha_0, M, K, \overline{K}, p, \delta) = n\alpha_0 - M - E\left\{\max\{l - \overline{K}, \min\{l, K\}\}\right\}, \tag{7}$$

where $n\alpha_0$ is the total primary premium, M is the reinsurance premium and $E\left\{\max\{l - \overline{K}, \min\{l, K\}\}\right\}$ is the expectation of the part of loss covered by the insurer according to the non-proportional reinsurance treaty. The profit of the insurer may be positive or negative, determined by the total loss, $l = nxl_0$, the total premium income, $n\alpha_0$, and the reinsurance contract, $(\overline{K}, K, M)$. A more generous reinsurance treaty for the insurer increases the total profit of the insurer. In other words, Π increases with $\overline{K}$ and decreases with K and M. Besides, the insurer is better off with higher premium income but lower reinsurance premium.

Remark 1. *As for the expected profit of the insurer,* Π*, we have the following properties:*

$$\frac{\partial \Pi}{\partial \overline{K}} \geq 0, \ \frac{\partial \Pi}{\partial K} \leq 0,$$
$$\frac{\partial \Pi}{\partial \alpha} > 0 \ \frac{\partial \Pi}{\partial M} < 0.$$

Proof. See Appendix. □

The insurer will provide catastrophe insurance for inhabitants if it can earn a nonnegative profit from this business. This is the necessary constraint of the insurer. Thus, the optimization problem of the insurer is

$$\begin{aligned} &\max_{\alpha_0} \Pi(\alpha_0, M, K, \overline{K}, p, \delta) \\ &\textit{subjected to } \Pi(\alpha_0, M, K, \overline{K}, p, \delta) \geq 0 \end{aligned} \tag{8}$$

As is shown in Figure 1, the higher the premium α_0 is, the higher profit the insurer earns. However, once the premium exceeds the maximum willingness to pay of inhabitants, $\alpha_0^*(T_0)$, inhabitants will choose not to insure and the profit of the insurer will fall to zero. Thus, if the insurer chooses to provide catastrophe insurance in the market, it must sell the insurance at the highest price that inhabitants accept, i.e. the maximum willingness to pay of inhabitants, $\alpha_0^*(T_0)$.

[2]Generally, reinsurance premium absorbs both pure premium and a loading for the reinsurer's expenses. However, in the government reinsurance case, the government as reinsurer is not self-interested thus charges a reinsurance premium to maximize social welfare.

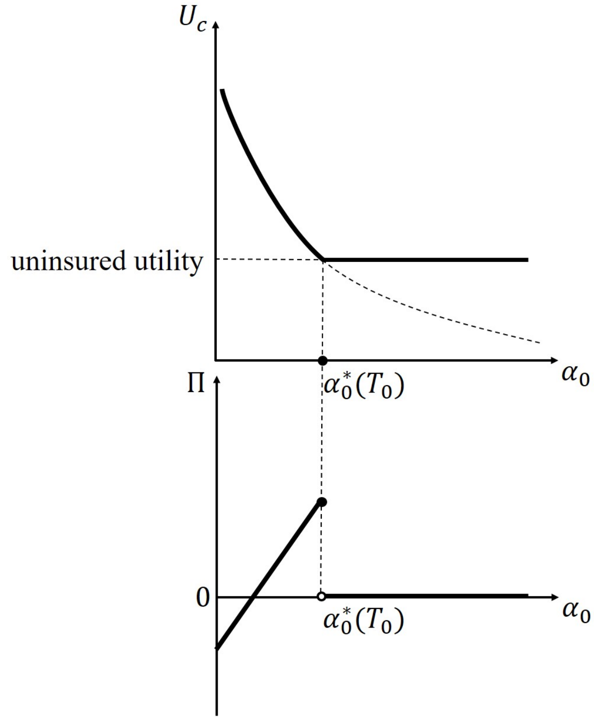


Figure 1: Decisions of inhabitants and the insurer

Note: This figure depicts the impact of the primary insurance premium, α_0, on the utility of inhabitants, U_c, and the profit of the insurer, Π. Utility of inhabitants decreases with α_0 when the premium is lower than the maximum willingness to pay of inhabitants, $\alpha_0^*(T_0)$. If the premium exceeds $\alpha_0^*(T_0)$, inhabitants prefer to bear the risk on their own and thus the utility of inhabitants is invarying. Correspondingly, if the premium is lower than $\alpha_0^*(T_0)$, profit of the insurer increases with α_0. If the premium exceeds $\alpha_0^*(T_0)$, the insurer loses the catastrophe insurance market with zero profit.

Governmental Intervention

As discussed above, the government acts as a reinsurer of last resort. Reinsurance premium, M, from the insurer would be ceded to the government reinsurance program. In return, the government will cover insurer's loss above the deductible, K, with a coverage, $\overline{K}$. The objective of the government is to maximize the weighted total utility of both inhabitants and the insurer by making its reinsurance pricing and tax decisions. Meanwhile, the government needs to provide CAT XL treaty. The government shall simulate existing market solutions of catastrophe insurance market. In other words, the governmental intervention has to ensure that the insurer is willing to supply catastrophe insurance at an acceptable price of inhabitants. Thus, the optimization problem of the government is as follows [3]:

$$\begin{aligned} \max_{M,T_0,K,\overline{K}} \quad & \lambda n U(-T_0 - \alpha_0^*(T_0)) + (1-\lambda)\Pi(\alpha_0^*(T_0), M, K, \overline{K}), \\ \textit{subjected to} \quad & M \geq 0,\ T_0 \geq 0, K \geq 0, \overline{K} \geq 0, \\ & K + \overline{K} \leq n l_0, \\ & \Pi(\alpha_0^*(T_0), M, K, \overline{K}, p, \delta) \geq 0, \\ & nT_0 + M \geq E\left\{\min\{\overline{K}, \max\{0, l - K\}\}\right\}, \end{aligned} \tag{9}$$

where λ is the weight of inhabitants utility and $1-\lambda$ is the weight of the insurer's utility. λ lies on (0,1) and captures the relative importance between inhabitants and the insurer. A beneficent government should care both inhabitants utility and the insurer's utility. Thus, λ should not be

[3] The insurer is assumed to be risk neutral with linear utility. The risk-neutral insurer buys reinsurance probably due to regulatory constraints (Bernard and Tian, 2009).

zero or one. Note that risk-neutral insurer has the equal expected profit and utility. When it comes to constraints, the first two constraints aim to ensure economic meaning of control variables. And the third constraint stands for non-negative profit of the insurer. The last constraint represents the budget constraint of the government where the sum of tax and reinsurance premium should be no less than the loss covered by the government.

The government makes reinsurance premium M, tax per capita T_0, the deductible of reinsurance K and the coverage of reinsurance $\overline{K}$ decisions to maximize the weighted sum utility of the insurer and inhabitants. To formalize the basic intuition, we illustrate the tradeoff of the government in the figure below. Reinsurance premium and tax are pockets to support the government reinsurance program. If the government aims to boost the total social utility, it shall reduce both reinsurance premium and tax. Obviously, if double reduction takes place, the government will have no sufficient fund to cover the loss and run the reinsurance program. For example, if both reinsurance premium M and tax per capita T_0 are equal to zero ,the government reinsurance program collapses immediately. Thus, the government has to tradeoff between tax and reinsurance program to maximize the weighted utilities and makes the optimal decisions, (T_0^*, M^*), i.e. point A in Figure 2.

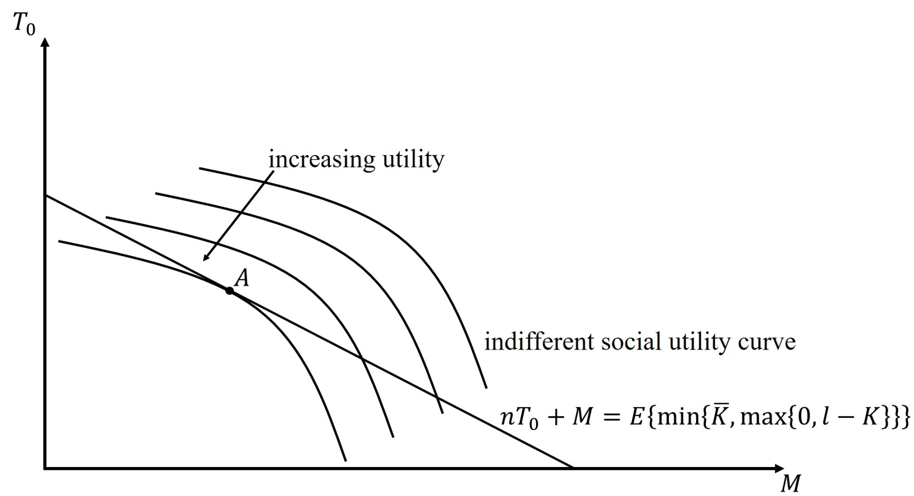


Figure 2: Tradeoff between tax and reinsurance premium

Note: This figure depicts the tradeoff between tax and reinsurance premium of the government. Curves represent indifferent social utility and the line represents the budget constraint of the government. The tangent point is the equilibrium.

Market Equilibrium

The market equilibrium can be characterized by optimal decisions of the government, the insurer and inhabitants. The government designs the optimal reinsurance contract $(K^*, \overline{K}^*, M^*)$ and decides the optimal tax T_0^*. The insurer prices the primary insurance $\alpha^*(T_0)$ with given tax and decides whether to purchase government reinsurance with given reinsurance contract. Inhabitants choose to insure or not insure depending on the premium of the primary insurance.

In the equilibrium, government's optimal tax and reinsurance contract are as follows:
If $\frac{\lambda}{1-\lambda} > (U^{-1})'[pU(-l_0)]$,

$$T_0^* = 0, \tag{10}$$

$$(K^*, \overline{K}^*) \in \{(K, \overline{K}) | K \geq 0, \overline{K} \geq 0, K + \overline{K} \leq nl_0\}, \tag{11}$$

$$M^* = E\left\{\min\{\overline{K}^*, \max\{0, l - K^*\}\}\right\}, \tag{12}$$

If $\frac{\lambda}{1-\lambda} \le (U^{-1})'[pU(-l_0)]$,

$$(U^{-1})'[pU(-l_0 - T_0^*) + (1-p)U(-T_0^*)] = \frac{\lambda}{1-\lambda}, \tag{13}$$

$$(K^*, \overline{K}^*) \in \{(K, \overline{K}) | K \ge 0, \overline{K} \ge 0, K + \overline{K} \le nl_0, E\left\{\min\{\overline{K}, \max\{0, l-K\}\}\right\} \ge nT_0^*\}, \tag{14}$$

$$M^* = E\left\{\min\{\overline{K}^*, \max\{0, l-K^*\}\}\right\} - nT_0^*, \tag{15}$$

Government decisions depend on the relative weight of inhabitants' utility $\frac{\lambda}{1-\lambda}$. If the government weights inhabitants more, it will charge no tax and set reinsurance premium equal to expected total loss covered by the government. In this case, the government reinsurance program is only backed up by reinsurance premium.

Otherwise, the government will simultaneously choose optimal tax and reinsurance price. Optimal tax is determined by (13), which implies the substitution effect between the total utility of inhabitants and the profit of the insurer. If the government charges a unit of extra tax, $\triangle T_0$, inhabitants will undergo decrease in utility by $n\triangle T_0 \left[-pU'(-T_0 - l_0) - (1-p)U'(-T_0)\right]$. Since the government is not self-interested, i.e. $nT_0 + M = E\left\{\min\{\overline{K}, \max\{0, l-K\}\}\right\}$, reinsurance premium decreases by $n\triangle T_0$ and as a result, the insurer earns $n\triangle T_0$ extra profit from the decreasing reinsurance price and $n\triangle T_0 {\alpha^*}'(T_0)$ from the change in the maximum willingness to pay of inhabitants. And the optimal tax should satisfy $\lambda n\triangle T_0 \left[-pU'(-T_0^* - l_0) - (1-p)U'(-T_0^*)\right] = -(1-\lambda)\left[n\triangle T_0 + n\triangle T_0 {\alpha^*}'(T_0^*)\right]$, i.e. $(U^{-1})'[pU(-l_0 - T_0^*) + (1-p)U(-T_0^*)] = \frac{\lambda}{1-\lambda}$. The reinsurance premium should be nonnegative so the last inequality in Equ. (11) must hold. In Equ. (12), the sum of tax and reinsurance premium equals the expected loss covered by the government.

Remark 2. *In the equilibrium state, we have the following properties of the government budget constraint and reinsurance premium:*

$$nT_0^* + M^* = E\left\{\min\{\overline{K}^*, \max\{0, l-K^*\}\}\right\},$$

$$M^* \le E\left\{\min\{\overline{K}^*, \max\{0, l-K^*\}\}\right\}.$$

Proof. See Appendix. □

The government breaks even in the equilibrium to maximize the social utility. In other words, the budget constraint of the government, $nT_0 + M \ge E\left\{\min\{\overline{K}, \max\{0, l-K\}\}\right\}$ must be binding in the equilibrium. Our optimal reinsurance premium M^* is risk-based moreover less than actuarially fair premium, $E\left\{\min\{\overline{K}, \max\{0, l-K\}\}\right\}$, since the government uses not only reinsurance premium but also tax to finance the catastrophe reinsurance program.

Remark 3. *In the equilibrium state, we have the following properties of reinsurance contract* $(K^*, \overline{K}^*)$*:*

1. *The optimal reinsurance contract* $(K^*, \overline{K}^*)$ *is not unique.*
2. *The optimal tax* T_0^* *is irrelevant with* $(K^*, \overline{K}^*)$.
3. $\frac{\partial M^*}{\partial K^*} \le 0$; $\frac{\partial M^*}{\partial \overline{K}^*} \ge 0$.

Proof. See Appendix. □

The optimal reinsurance contract is actually not unique. Thus, the government has flexibility to provide appropriate reinsurance contract, which is applicable to the practice. The last two properties in Remark 3 claim the relationship between the optimal reinsurance contract, $(K^*, \overline{K}^*)$ and another two decisions of the government, i.e. tax and reinsurance premium. First, the optimal tax T_0^* is irrelevant with $(K^*, \overline{K}^*)$. According to Remark 2, we insert $M = E\left\{\min\{\overline{K}, \max\{0, l - K\}\}\right\} - nT_0$ into objective function of the government and yield

$$\lambda n\left[pU(-l_0 - T_0) + (1-p)U(-T_0)\right] + (1-\lambda)[n\alpha_0^*(T_0) + nT_0 - pnl_0],$$

It is obvious that the weighted social welfare is irrelevant with the reinsurance treaty when the government charges tax. Thus, the optimal tax T_0^* is not related to the optimal deductible and coverage the reinsurance by government but all of them are related to reinsurance premium. Second, the optimal reinsurance premium decreases with the deductible, K^*, and increases with the coverage, $\overline{K}^*$. It is reasonable because the lower (higher) the deductible (coverage) is, the more risk the government covers, and the more expensive the reinsurance should be. This proposition sheds light on the pricing procedure of the catastrophe reinsurance by government. Once we determine optimal tax T_0^*, deductible K^* and coverage $\overline{K}^*$, we can give an appropriate reinsurance premium in tune with risk sharing by the government.With optimal tax, T_0^*, and reinsurance contract, $(K^*, \overline{K}^*, M^*)$, the government achieves the largest social welfare, i.e.

$$\begin{aligned}\lambda U_c(\alpha_0^*(T_0^*), l_0, T_0^*) &+ (1-\lambda)\Pi(\alpha_0^*(T_0^*), M^*, K^*, \overline{K}^*, p, \delta) \\ &= \lambda n\left[pU(-l_0 - T_0^*) + (1-p)U(-T_0^*)\right] + (1-\lambda)[n\alpha_0^*(T_0^*) + nT_0^* - pnl_0].\end{aligned}$$

The insurer makes two decisions in the equilibrium. First, it sets the primary premium with given tax. As is shown in Figure 1, with given optimal tax of the government, T_0^*, the higher the primary premium is, the higher profit the insurer earns if the primary premium does not exceed the maximum willingness to pay of inhabitants, $\alpha_0^*(T_0^*)$. Thus, as a monopoly, the insurer sets the primary insurance premium at $\alpha_0^*(T_0^*) = -T_0^* - U^{-1}[pU(-T_0^* - l_0) + (1-p)U(-T_0^*)]$. Second, the optimal reinsurance decision of the insurer depends on the reinsurance treaty. With optimal contract of the government, $(K^*, \overline{K}^*, M^*)$, the insurer prefers to purchase reinsurance from the government since the optimal reinsurance of the government increases the profit of the insurer. The reason is that the government is not self-interested and thus has no incentive to earn profit through the reinsurance premium. Moreover, the capital source of the public reinsurance program is wider including reinsurance premium as well as tax (from Remark 2).

Remark 4. *In the equilibrium state, we have the following property of the expected profit of the insurer:*

$$\Pi(\alpha_0^*(T_0^*), M^*, K^*, \overline{K}^*, p, \delta) > 0,$$
$$M^* < n\alpha_0^*(T^*).$$

Proof. See Appendix. □

We prove that the profit of the insurer is positive in the equilibrium thus the insurer is willing to enter the catastrophe market. Since the insurer earns positive profit in the equilibrium, i.e. $n\alpha_0^*(T_0^*) - M^* - E\left\{\max\{l - \overline{K}^*, \min\{l, K^*\}\}\right\} > 0$, it is obvious that the reinsurance premium is lower than the primary premium, i.e. $M^* < n\alpha_0^*(T^*)$. With the optimal primary premium and reinsurance program by government, the insurer earns profit of

$$\Pi(\alpha_0^*(T_0^*), M^*, K^*, \overline{K}^*, p, \delta) = n\alpha_0^*(T_0^*) + nT_0^* - pnl_0, \tag{16}$$

where T_0^* is the optimal tax of the government and $(K^*, \overline{K}^*, M^*)$ is an optimal reinsurance contract designed by the government.

In the equilibrium, inhabitants choose to purchase the full-coverage insurance from the insurer if the primary insurance premium is no more than their maximum willingness to pay. We assume that inhabitants choose to purchase insurance at their maximum willingness to pay with utility of

$$U_c(\alpha_0^*(T_0^*), l_0, T_0^*) = pU(-T_0^* - l_0) + (1-p)U(-T_0^*), \tag{17}$$

where T_0^* is the optimal tax determined by the government. Note that in the equilibrium, inhabitants with no surplus are indifferent between insured and not insured.

Propositions

The solution to our model has some important economic implications which give insights to policy on catastrophe risk management.

Proposition 1. *The government can always solve the insolvency problem of the insurer by appropriate reinsurance treaty, i.e.* $\exists(K^*, \overline{K}^*)$ *and corresponding* (T_0^*, M^*) *such that*

$$Prob\left(\max\{l - \overline{K}^*, \min\{l, K^*\}\} > n\alpha_0^*(T_0^*) + C - M^*\right) = 0.$$

where $\max\{l - \overline{K}, \min\{l, K\}\}$ *is the loss covered by the insurer,* C *is the initial capital of the insurer and total loss is* $l = nxl_0$*.*

Proof. See Appendix. □

The default risk of the insurer can be avoided by a proper reinsurance contract, which is still an optimal reinsurance contract for the government. That is, no matter how large the loss is, the government is able to design a treaty including $(K^*, \overline{K}^*, M^*)$ and tax T_0^* to ensure the solvency ability of the insurer. This is one of the edges of public catastrophe program (Bruggeman et al., 2012). Actually, due to unpredicted nature of catastrophe risk, private reinsurer may charge a price that many times the expected loss to meet the solvency requirement. Thus, the insurer tends to retain instead of sharing large-event risks (Froot, 2001; Kousky and Cooke, 2012). This causes the failure of coverage in catastrophe reinsurance market. On the contrary, in practice, the government offers reinsurance cheaper than the expected loss thus lower than private reinsurer (Remark 2). This advantage benefits from costless capital and the ability to borrow intertemporally.

A thread of literatures on optimal policies aims to identify the policies that maximize welfare, trading off the distortionary costs of public insurance programs with the benefits they provide in reducing exposure to risk (Eckstein et al., 1985; Bound et al.,2004; Chetty and Looney, 2006; Chetty, 2009; Einav et al., 2010). Next, we compare our solution with a free market without government reinsurance.

The market equilibrium without government reinsurance will be characterized by decisions of the insurer and inhabitants as follows:
(1) The monopolistic insurer offers full catastrophe coverage and set the primary insurance premium at the maximum willingness to pay of inhabitants,
(2) Inhabitants choose to insure if the premium is no more than their maximum willingness to pay.

In the equilibrium without government reinsurance, inhabitants choose to purchase the insurance at their maximum willingness to pay. Thus the weighted social utility without the government reinsurance in the equilibrium is:

$$V_1 = \lambda n\left[pU(-l_0 - T_1) + (1-p)U(-T_1)\right] + (1-\lambda)[n\alpha_0^*(T_1) - pnl_0], \tag{18}$$

where T_1 is a exogenous tax in this case. Note that $0 \leq T_1 \leq T_0^*$ and reinsurance by government no longer exists. Denote $U_{c1} = pU(-l_0 - T_1) + (1-p)U(-T_1)$ and $\Pi_1 = n\alpha_0^*(T_1) - npl_0$. The weighted social utility with the government reinsurance in the equilibrium is:

$$V_2 = \lambda n \left[pU(-l_0 - T_0^*) + (1-p)U(-T_0^*)\right] + (1-\lambda)[n\alpha_0^*(T_0^*) + nT_0^* - pnl_0]. \tag{19}$$

Proposition 2. *In the equilibrium, the weighted social welfare is improved by the government reinsurance, i.e.*

$$V_1 \leq V_2.$$

Proof. See Appendix. □

It is clear from the perspective of the whole society, public catastrophe reinsurance improves the total benefit. The government enhances social welfare by providing an ex-ante risk-transfer solution as reinsurance financed by both premium and poll tax. From the form of Equ. (18) and (19), tax acts like subsidy to the insurer with reinsurance by government, this "subsidy" do not exist without government reinsurance. Moreover, the government choose the optimal tax to achieve the highest social utility in the case with government reinsurance, while T_1 may not be optimal.

Though public catastrophe reinsurance improves the total welfare, it is not for sure that both inhabitants and the insurer are better off. If the improvement of social welfare solely benefits from the enhancing insurer's profit, it may deviate from the wish of social planner. To achieve Pareto improvement, the primary insurance premium should be lower than the maximum willingness to pay of the inhabitants.

Proposition 3. *There exists a primary insurance premium, α_1, to ensure that both inhabitants and the insurer are better off with government reinsurance.*

Proof. See Appendix. □

In reality, to achieve such a primary insurance premium, the government can impose price regulation on the insurer. Besides, when the catastrophe market resonates oligopoly characteristics (Cummins and Zi, 1998; Sonnenholzner and Wambach, 2006; Chiappori et al., 2006), this Pareto improvement could also be obtained without price regulation.

Numerical Illustration

In this section, we present numerical examples for our reinsurance pricing model and propositions above. At first, we follow parameters in the one-region model of Charpentier and Le Maux (2014) and give list of parameters as below.

Table 1: Model Parameters

Parameter	Variable	Value
Number of inhabitants	n	1000
Loss per capita	l_0	0.5
Correlation among individual risk	δ	0.4
Probability of catastrophe	p^*	0.05
Individual probability of claiming loss	p	0.25
Utility function of inhabitants	$U(Y)$	$U(Y) = 1 - e^{-2Y}$
Distribution function of event size	$F(x)$	$F(x) = F(n, p^*, p, \delta)$

The number of inhabitants n is set to 1000, probability of catastrophe p^* to 0.05 and correlation among individual risk δ to 0.4, the same as Charpentier and Le Maux (2014).For graphical convenience, we modify some parameters of Charpentier and Le Maux (2014). Our utility function is set to $U(Y) = 1 - e^{-2Y}$, individual probability of claiming loss p to 0.25 and loss per capita l_0 to 0.5 [4]. We share the density and cumulative distribution function of x (the share of inhabitants that claim a loss) with Charpentier and Le Maux (2014). The distribution function is actually a mixture of two binomial distributions depending on number of inhabitants n, probability of catastrophep^*, individual probability of claiming loss$p$ and correlation among individual risk δ . In Appendix B, we illustrate the cumulative distribution function F and density function f with $n = 1000$, $p^* = 0.05$, $p = 0.25$ and $\delta = 0.4$.

The threshold of the relative weight between inhabitants and the insurer, $(U^{-1})'[pU(-l_0)]$, is 0.3798 which implies that from the perspective of the government, if the relative weight of inhabitants to the insurer exceeds 0.3798, the government will charge no tax ($T_0^* = 0$) and use reinsurance program only to finance the program. The corresponding λ is 0.2753. If the absolute weight of inhabitants λ is over 0.2753, the government will choose no tax. Otherwise, the government will charge both tax and reinsurance premium to back up the program.

Table 2 reports 12 cases of government decisions including optimal tax per capita T_0^*, reinsurance premium M^*, deductible K^* and coverage $\overline{K}^*$ with different weights. In the latter two columns, λ is over 0.2753 thus the government charges no tax. When λ is equal to 0.25, the government uses tax and reinsurance premium to finance the public reinsurance program. It is clear from the table that optimal deductible K^* and coverage $\overline{K}^*$ are highly related with reinsurance premium M^* but irrelevant with optimal tax per capita T_0^*. The lower deductible and higher coverage take the cost of higher reinsurance premium.

[4]In Charpentier and Le Maux (2014), the utility function is set to $U(Y) = 100(1 - e^{-2Y})$, individual probability of claiming loss to 0.3, loss per capita to 1.

Table 2: Cases of government decisions

	$\lambda = 0.25$		$\lambda = 0.5$		$\lambda = 0.75$	
Case 1	$K^* = 50$	$\overline{K}^* = 100$	$K^* = 50$	$\overline{K}^* = 100$	$K^* = 50$	$\overline{K}^* = 100$
	$T^* = 0.024$	$M^* = 47.48$	$T^* = 0$	$M^* = 72.41$	$T^* = 0$	$M^* = 72.41$
Case 2	$K^* = 50$	$\overline{K}^* = 200$	$K^* = 50$	$\overline{K}^* = 200$	$K^* = 50$	$\overline{K} = 200$
	$T^* = 0.024$	$M^* = 50.98$	$T^* = 0$	$M^* = 75$	$T^* = 0$	$M^* = 75$
Case 3	$K^* = 100$	$\overline{K}^* = 100$	$K^* = 100$	$\overline{K}^* = 100$	$K^* = 100$	$\overline{K}^* = 100$
	$T^* = 0.024$	$M^* = 2.148$	$T^* = 0$	$M^* = 24.8$	$T^* = 0$	$M^* = 24.8$
Case 4	$K^* = 100$	$\overline{K}^* = 200$	$K^* = 100$	$\overline{K}^* = 200$	$K^* = 100$	$\overline{K}^* = 200$
	$T^* = 0.024$	$M^* = 2.684$	$T^* = 0$	$M^* = 25$	$T^* = 0$	$M^* = 25$

To start with Proposition 1, we want to verify if our reinsurance by government solves the insolvency risk of the insurer. We plot difference between the insurer's capital including initial capital and primary premium net reinsurance premium and loss covered by the insurer. In Figure 3, the x axis stands for total loss l and the box plot stands for the distribution of the difference above. The weight λ is set to 0.25 and thus the optimal tax per capital T_0^* is equal to 0.024. It is clear from the box plot that for any loss amount, we are able to find optimal deductible K^*, coverage $\overline{K}^*$ and reinsurance premium M^* to make sure that the capital the insurer can use is larger than the loss covered by the insurer. When the loss amount is less than 250, the box is totally above zero implying the solvency is always the case. Otherwise, the insolvency probability increases with the loss amount. It is clear from Figure 3 that with appropriate reinsurance treaty by government, the insurer has enough capital to settle down all claims regardless of loss amount. Thus, the default risk of the insurer is eliminated by our public reinsurance program.

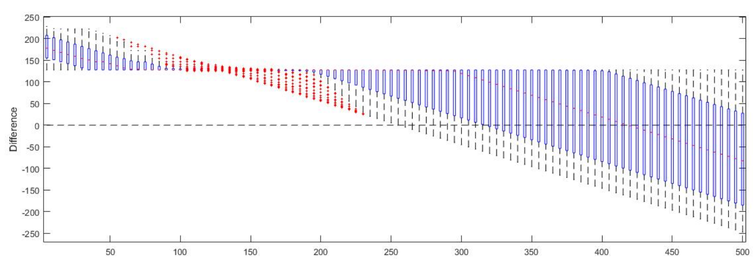
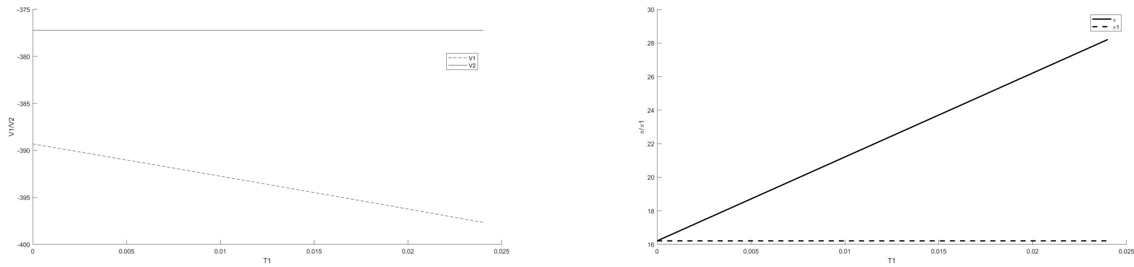


Figure 3: The difference between the capital of insurer and the loss covered by the insurer($\lambda = 0.25$)

When it comes to the social utility with reinsurance by government or without reinsurance by government in Proposition 2, we fix $\lambda = 0.25$ at first and depict two types of utility with different exogenous tax T_1 in non-government reinsurance case in Figure 4a. Note that exogenous tax T_1 is lower than $T_0^* = 0.024$. In the equilibrium, the social utility without reinsurance by government is V_1. The social utility with reinsurance by government is V_2. The x axis stands for exogenous tax and the y axis stands for the social utility V_1 and V_2.It is clear from Figure 4a that regardless of exogenous tax T_1, total social utility with reinsurance by government V_2 is always above the counterpart with reinsurance by government V_1. Thus, as Proposition 2 put, the weighted social utility is improved by our public reinsurance.



(a) (b)

Figure 4: Comparison between case with public reinsurance and case without($\lambda = 0.25$)

In Proposition 3, we claim if the market resonates oligopoly characteristics or the government imposes price regulation on the primary insurer, the price of catastrophe insurance will be lower than maximum willingness to pay of inhabitants and Pareto improvement could be fulfilled. Note that in our model, inhabitants have no surplus in the equilibrium since the monopolistic insurer sets price at the maximum willingness to pay of inhabitants. If the new primary premium α_1 ,competitive or regulated, makes sure that inhabitants are better off with reinsurance by government, we need to illustrate this premium also makes the insurer better in Figure 4b. In Figure 4b, we depict the profit of the insurer with government reinsurance Π and without government reinsurance Π_1 with different exogenous tax T_1 in non-government reinsurance case. As well, exogenous tax T_1 is lower than $T_0^* = 0.024$. The x axis stands for exogenous tax and the y axis stands for the profit of the insurer . We find the profit with government reinsurance Π is always higher than the counterpart without government reinsurance Π_1 regardless of exogenous tax. That is, Pareto improvement of inhabitants and the insurer could be achieved through more competitive or regulated primary premium.

Conclusion

We consider the market equilibrium and pricing of catastrophe reinsurance by the government in the context of Charpentier and Le Maux (2014). Subsequently, we set up a one-region model with utility-maximization inhabitants and the profit-maximization primary insurer. The government aims to maximize the weighted sum of inhabitants' utility and the insurer's profit by charging reinsurance premiums from the insurer and taxing from inhabitants. We then derive optimal primary premium α_0^*, tax per capita T_0^*, reinsurance premium M^*, reinsurance deductible K^* and reinsurance coverage $\overline{K}^*$ in the market equilibrium. The optimal policy for the government is flexible with matching reinsurance premium, deductible of reinsurance and coverage of reinsurance. Also, we find that our ex-ante reinsurance treaty by government is able to eliminate the default risk of the insurer. Besides, we compare the equilibrium above with the equilibrium without government reinsurance and find that the government reinsurance program enhances social utility. If the primary premium is regulated or the oligopolistic catastrophe insurance market exists, Pareto improvement of inhabitants and the insurer could be achieved. Finally, we conduct numerical studies to verify propositions of model.

Our results have profound implications. One common characteristic of public program on catastrophe risk management is that the government must strike a balance between ensuring zero default risk and limited tax exposure. In other words, the government must distribute risk and wealth between inhabitants and the primary insurer well. In our model, for simplicity, we assume that the government charges reinsurance premium from the primary insurer and tax from inhabitants. Moreover, the public reinsurance program ensures the insurer's solvency. Solution to model shows catastrophe reinsurance provided by the government could well stimulate existing market solutions. That is, the insurer always earns positive profit and inhabitants accept the price. Naturally, the government shall charge a risk-based premium and tax based on risk per capita (the probability that each individual claims a loss and loss). Further, when it comes to treaty design, parameters of reinsurance treaty (deductible and coverage) are only related to price of the reinsurance but irrelevant with tax per capital. Our findings show that the government can enhance the efficiency of catastrophe insurance markets by acting as a reinsurer of last resort for truly large losses given the government's special ability to bear timing risk through budgetary actions.

However, some potential limitations to our study must be considered. For instance, private reinsurers play an important role in catastrophe insurance market but are not considered in our model. The crowd-out impact of government as reinsurer on private reinsurers is not discussed then (Kousky and Cooke, 2012). In addition, the government as the last resort of large events may cause moral hazard problems. Specifically, the public reinsurance program may discourage individuals in regions at risk from taking protective measures (Kunreuther, 2006; Goodspeed and Haughwout, 2012). As well, construction in hazard-prone areas may increase with the additional risk layer by government (Kunreuther, 2006; Wildasin, 2008). Besides, the government reinsurance program frees up capital of the insurer and some risky investment may arise (Chen and Steiner, 2001; Ho et al., 2013; Filipović et al., 2015). In this case, it would be interesting to evaluate the appropriate premium regarding avoidance of crowd-out of private reinsurers and moral hazard.

Appendix

The table below shows all variables used in our model. In "Appendix A", we provide the detailed solution to the optimization problem of the government and proofs of our remarks and propositions. In "Appendix B", we present the cumulative distribution function and density function of event size x.

Table 3: Model Variables

Variable	Concept	Definition
n	population of the whole region	
N	disaster-hit population	
x	share of disaster-hit population	$x = \frac{N}{n}$
p	probability that each individual claims a loss	
δ	correlation between individual's risk	
$F(x\|p, \delta)$	distribution function of x	
$f(x)$	probability density function of x	$F(x\|p, \delta) = \int_0^x f(t)dt$
$U(\cdot)$	utility function of inhabitants	$U'(\cdot) > 0,\ U''(\cdot) < 0,\ U(0) = 0$
l_0	loss of each disaster-hit inhabitant	
l	total loss of the region	$l = nxl_0$
α_0	primary insurance premium	
T_0	per capita tax	
$\alpha_0^*(T_0)$	maximum willingness to pay of inhabitants with given tax T_0	$\alpha_0^*(T_0) = -T_0 - U^{-1}[pU(-T_0 - l_0) + (1 - p)U(-T_0)]$
Π	profit of the insurer	$\Pi(\alpha_0, M, K, \overline{K}, p, \delta) = n\alpha_0 - M - E\left\{\max\{l - \overline{K}, \min\{l, K\}\}\right\}$
M	reinsurance premium	
K	deductible of the reinsurance	
$\overline{K}$	coverage of the reinsurance	
λ	weight of sum utility of inhabitants in social welfare	
C	initial wealth of the insurer	
T_1	per capita tax in the case without government reinsurance	
U_{c1}	utility of inhabitants in the equilibrium without government reinsurance	$U_{c1} = pU(-l_0 - T_1) + (1 - p)U(-T_1)$
Π_1	profit of the insurer in the equilibrium without government reinsurance	$\Pi_1 = n\alpha_0^*(T_1) - pnl_0$
V_1	weighted social welfare in the equilibrium without government reinsurance	$V_1 = \lambda n\left[pU(-l_0 - T_1) + (1 - p)U(-T_1)\right] + (1 - \lambda)[n\alpha_0^*(T_1) - pnl_0]$
V_2	weighted social welfare in the equilibrium with government reinsurance	$V_2 = \lambda n\left[pU(-l_0 - T_0^*) + (1 - p)U(-T_0^*)\right] + (1 - \lambda)[n\alpha_0^*(T_0^*) + nT_0^* - pnl_0]$

Appendix A

Proof of Remark 1

The profit of the insurer is

$$\Pi(\alpha_0, M, K, \overline{K}, p, \delta) = n\alpha_0 - M - E\left\{\max\{l - \overline{K}, \min\{l, K\}\}\right\}, \tag{20}$$

where

$$E\left\{\max\{l - \overline{K}, \min\{l, K\}\}\right\} = \int_0^{\frac{K}{nl_0}} nxl_0 f(x)dx + \int_{\frac{K}{nl_0}}^{\frac{\overline{K}+K}{nl_0}} Kf(x)dx + \int_{\frac{\overline{K}+K}{nl_0}}^{1} (nxl_0 - \overline{K})f(x)dx. \tag{21}$$

Carrying out some algebra, we have that

$$\begin{aligned}
\frac{\partial \Pi}{\partial \alpha} &= n > 0, \\
\frac{\partial \Pi}{\partial M} &= -1 < 0, \\
\frac{\partial \Pi}{\partial \overline{K}} &= 1 - F\left(\frac{\overline{K}+K}{nl_0}|p, \delta\right) \geq 0, \\
\frac{\partial \Pi}{\partial K} &= F\left(\frac{K}{nl_0}|p, \delta\right) - F\left(\frac{\overline{K}+K}{nl_0}|p, \delta\right) \leq 0.
\end{aligned}$$

Proof of the solution to the maximization problem of the government, Remark 2 to 4

It is useful to simplify the maximization problem of the government first. The monopolistic insurer sets the primary insurance premium at the maximum willingness to pay of inhabitants, and thus, $U(-\alpha_0^*(T_0) - T_0) = pU(-T_0 - l_0) + (1-p)U(-T_0)$. Besides, the government is not self-interest, implying that the budget constraint of the government, $nT_0 + M \geq E\left\{\min\{\overline{K}, \max\{0, l - K\}\}\right\}$ should be binding (Remark 2). With this binding constraint, we prove Remark 4 as follows.

$$\begin{aligned}
\Pi(\alpha_0^*(T_0^*), M^*, K^*, \overline{K}^*, p, \delta) &= n\alpha_0^*(T_0^*) - M^* - E\left\{\max\{l - \overline{K}^*, \min\{l, K^*\}\}\right\} \\
&= n\alpha_0^*(T_0^*) + nT_0^* - E\left\{\min\{\overline{K}^*, \max\{0, l - K^*\}\}\right\} - E\left\{\max\{l - \overline{K}^*, \min\{l, K^*\}\}\right\} \\
&= -nT_0^* - nU^{-1}[pU(-T_0^* - l_0) + (1-p)U(-T_0^*)] + nT_0^* - npl_0 \\
&= -nU^{-1}[pU(-T_0^* - l_0) + (1-p)U(-T_0^*)] - npl_0 \\
&> -nU^{-1}[U(-T_0^* - pl_0)] - npl_0 \\
&= nT_0^* \geq 0.
\end{aligned} \tag{22}$$

The binding budget constraint of the government ensures that the profit of the insurer is non-negative. Thus, $\Pi(\alpha_0^*(T_0), M, K, \overline{K}, p, \delta) \geq 0$ is a redundant constraint. The maximization problem of the government is rewritten as

$$\begin{aligned}
&\max_{T_0, K, \overline{K}} \lambda n[pU(-T_0 - l_0) + (1-p)U(-T_0)] + (1-\lambda)[n\alpha_0^*(T_0) + nT_0 - pnl_0], \\
&\textit{subjected to } 0 \leq nT_0 \leq E\left\{\min\{\overline{K}, \max\{0, l - K\}\}\right\}, \\
&\qquad K \geq 0, \overline{K} \geq 0, \\
&\qquad K + \overline{K} \leq nl_0,
\end{aligned} \tag{23}$$

where the objective function is irrelevant with K and $\overline{K}$. We denote $W(T_0) := \lambda n[pU(-T_0 - l_0) + (1-p)U(-T_0)] + (1-\lambda)[n\alpha_0^*(T_0) + nT_0 - pnl_0]$. Then, with some algebra, we have

$$W'(T_0) = -n[pU'(-T_0-l_0)+(1-p)U'(-T_0)]\{\lambda-(1-\lambda)(U^{-1})'[pU(-T_0-l_0)+(1-p)U(-T_0)]\}. \tag{24}$$

Let, $W'(\hat{T}_0) = 0$, i.e. $(U^{-1})'[pU(-\hat{T}_0 - l_0) + (1-p)U(-\hat{T}_0)] = \frac{\lambda}{1-\lambda}$. We have, $W''(\hat{T}_0) < 0$. Thus, if $\hat{T}_0 \geq 0$, i.e. $W'(0) \geq 0$, we have $\frac{\lambda}{1-\lambda} \leq (U^{-1})'[pU(-l_0)]$, and $T_0^* = \hat{T}_0$ is the optimal tax. To ensure that $nT_0^* \leq E\left\{\min\{\overline{K}, \max\{0, l-K\}\}\right\}$, the optimal reinsurance contract should satisfy

$$(K^*, \overline{K}^*) \in \{(K, \overline{K}) | K \geq 0, \overline{K} \geq 0, K + \overline{K} \leq nl_0, E\left\{\min\{\overline{K}, \max\{0, l-K\}\}\right\} \geq nT_0^*\}. \tag{25}$$

If $\hat{T}_0 < 0$, i.e. $W'(0) < 0$, we have $\frac{\lambda}{1-\lambda} > (U^{-1})'[pU(-l_0)]$, and the optimal tax is $T_0^* = 0$. And the optimal reinsurance contract should satisfy

$$(K^*, \overline{K}^*) \in \{(K, \overline{K}) | K \geq 0, \overline{K} \geq 0, K + \overline{K} \leq nl_0\}. \tag{26}$$

In conclusion, the solution to the maximization problem of the government is as follows. If $\frac{\lambda}{1-\lambda} > (U^{-1})'[pU(-l_0)]$,

$$T_0^* = 0, \tag{27}$$

$$(K^*, \overline{K}^*) \in \{(K, \overline{K}) | K \geq 0, \overline{K} \geq 0, K + \overline{K} \leq nl_0\}, \tag{28}$$

$$M^* = E\left\{\min\{\overline{K}^*, \max\{0, l-K^*\}\}\right\}. \tag{29}$$

If $\frac{\lambda}{1-\lambda} \leq (U^{-1})'[pU(-l_0)]$,

$$(U^{-1})'[pU(-l_0 - T_0^*) + (1-p)U(-T_0^*)] = \frac{\lambda}{1-\lambda}, \tag{30}$$

$$(K^*, \overline{K}^*) \in \{(K, \overline{K}) | K \geq 0, \overline{K} \geq 0, K + \overline{K} \leq nl_0, E\left\{\min\{\overline{K}, \max\{0, l-K\}\}\right\} \geq nT_0^*\}, \tag{31}$$

$$M^* = E\left\{\min\{\overline{K}^*, \max\{0, l-K^*\}\}\right\} - nT_0^*. \tag{32}$$

Then, we prove Remark 3. It is easy to find that the optimal reinsurance contract $(K^*, \overline{K}^*)$ is not unique. The optimal tax T_0^* is irrelevant with $(K^*, \overline{K}^*)$. Since

$$\begin{aligned} M^* &= E\left\{\min\{\overline{K}^*, \max\{0, l-K^*\}\}\right\} - nT_0^* \\ &= \int_{\frac{K}{nl_0}}^{\frac{K+\overline{K}}{nl_0}} (nxl_0 - K) f(x) dx + \int_{\frac{K+\overline{K}}{nl_0}}^{1} \overline{K} f(x) dx - nT_0^*, \end{aligned} \tag{33}$$

with some algebra, we have

$$\frac{\partial M^*}{\partial K^*} = F\left(\frac{K}{nl_0} | p, \delta\right) - F\left(\frac{K+\overline{K}}{nl_0} | p, \delta\right) \leq 0, \tag{34}$$

$$\frac{\partial M^*}{\partial \overline{K}^*} = 1 - F\left(\frac{K+\overline{K}}{nl_0} | p, \delta\right) \geq 0. \tag{35}$$

Proof of Proposition 1

To prove Proposition 1, we need to find a reinsurance contract $(K^*, \overline{K}^*)$ and corresponding (T_0^*, M^*) such that

$$Prob\left(\max\{l - \overline{K}^*, \min\{l, K^*\}\} > n\alpha_0^*(T_0^*) + C - M^*\right) = 0.$$

To achieve a zero default risk, the insurer should be capable to cover the most severe situation. That is

$$nl_0 - \overline{K}^* \leq n\alpha_0^*(T_0^*) + C - M^*. \tag{36}$$

Carrying out some algebra, we have

$$nl_0 + E\left\{\min\{\overline{K}^*, \max\{0, l - K^*\}\}\right\} - \overline{K}^* \leq -nU^{-1}[pU(-T_0^* - l_0) + (1-p)U(-T_0^*)] + C, \quad (37)$$

where the right hand side is decreasing in $\overline{K}^*$ and the left hand side is positive. If we fix $K^* = 0$ and denote $g(\overline{K}) := nl_0 + E\left\{\min\{\overline{K}^*, \max\{0, l - K^*\}\}\right\} - \overline{K}^*$ then $g(0) = nl_0$ and $g(nl_0) = pnl_0$. Thus, to prove Remark 3, we only need to prove

$$g(nl_0) \leq -nU^{-1}[pU(-T_0^* - l_0) + (1-p)U(-T_0^*)] + C.$$

$$\begin{aligned} -nU^{-1}[pU(-T_0^* - l_0) + (1-p)U(-T_0^*)] + C &\geq -nU^{-1}[U(-T_0^* - pl_0)] + C \\ &= npl_0 + nT_0^* + C \\ &> npl_0. \end{aligned} \quad (38)$$

In conclusion, $\exists(K^*, \overline{K}^*)$ and corresponding (T_0^*, M^*) such that

$$Prob\left(\max\{l - \overline{K}^*, \min\{l, K^*\}\} > n\alpha_0^*(T_0^*) + C - M^*\right) = 0.$$

Proof of Proposition 2

The weighted social utility without the government reinsurance is:

$$V_1 := \lambda n\left[pU(-l_0 - T_1) + (1-p)U(-T_1)\right] + (1-\lambda)[n\alpha_0^*(T_1) - pnl_0]. \quad (39)$$

The weighted social utility with the government reinsurance is:

$$V_2 := \lambda n\left[pU(-l_0 - T_0^*) + (1-p)U(-T_0^*)\right] + (1-\lambda)[n\alpha_0^*(T_0) - M^* - E\left\{\max\{l - \overline{K}, \min\{l, K\}\}\right\}]. \quad (40)$$

Inserting $M^* = E\left\{\min\{\overline{K}, \max\{0, l - K\}\}\right\} - nT_0^*$ into (40) yields

$$V_2 = \lambda n\left[pU(-l_0 - T_0^*) + (1-p)U(-T_0^*)\right] + (1-\lambda)[n\alpha_0^*(T_0^*) + nT_0^* - pnl_0]. \quad (41)$$

For $\forall\ 0 \leq T_1 \leq T_0^*$, we have

$$\begin{aligned} V_1 &= \lambda n\left[pU(-l_0 - T_1) + (1-p)U(-T_1)\right] + (1-\lambda)[n\alpha_0^*(T_1) - pnl_0] \\ &\leq \lambda n\left[pU(-l_0 - T_1) + (1-p)U(-T_1)\right] + (1-\lambda)[n\alpha_0^*(T_1) + nT_1 - pnl_0] \\ &\leq \max_{T_1}\ n\left[pU(-l_0 - T_1) + (1-p)U(-T_1)\right] + (1-\lambda)[n\alpha_0^*(T_1) + nT_1 - pnl_0] \\ &= \lambda n\left[pU(-l_0 - T_0^*) + (1-p)U(-T_0^*)\right] + (1-\lambda)[n\alpha_0^*(T_0^*) + nT_0^* - pnl_0] \\ &= V_2. \end{aligned} \quad (42)$$

Proof of Proposition 3

To ensure that inhabitants are better off with government reinsurance, the primary insurance premium, α_1, should satisfy that

$$U(-\alpha_1 - T_0) \geq U(-\alpha^*(T_1) - T_1), \quad (43)$$

i.e.

$$\alpha_1 \leq -T_0 - U^{-1}[pU(-l_0 - T_1) + (1-p)U(-T_1)]. \quad (44)$$

Suppose $\alpha_1 := -\frac{1}{2}T_1 - T_0 - U^{-1}[pU(-l_0 - T_1) + (1-p)U(-T_1)]$. Obviously inhabitants are better off with this primary insurance premium compared to U_{c1}. Then, we need to prove that compared to Π_1, the insurer also earns a higher profit with α_1. The insurer would enter the market if it can earn a non-negative profit, or leave the market if it cannot. And the profit of the insurer is

$$\Pi(\alpha_1, M, K, \overline{K}, p, \delta) = n\alpha_1 - M - E\left\{\max\{l - \overline{K}, \min\{l, K\}\}\right\}. \tag{45}$$

Following the similar simplifying ways to the maximization problem (Equ.(23)), we have

$$\begin{aligned}\Pi(\alpha_1, M, K, \overline{K}, p, \delta) &= n\alpha_1 - M - E\left\{\max\{l - \overline{K}, \min\{l, K\}\}\right\} \\ &= -\frac{1}{2}nT_1 - nU^{-1}[pU(-l_0 - T_1) + (1-p)U(-T_1)] - npl_0 \\ &\geq -nT_1 - nU^{-1}[pU(-l_0 - T_1) + (1-p)U(-T_1)] - npl_0 \\ &= \Pi_1.\end{aligned} \tag{46}$$

That is, there exists a primary insurance premium, α_1, to ensure that both inhabitants and the insurer are better off with government reinsurance.

Appendix B

Distribution function and density function of event size x

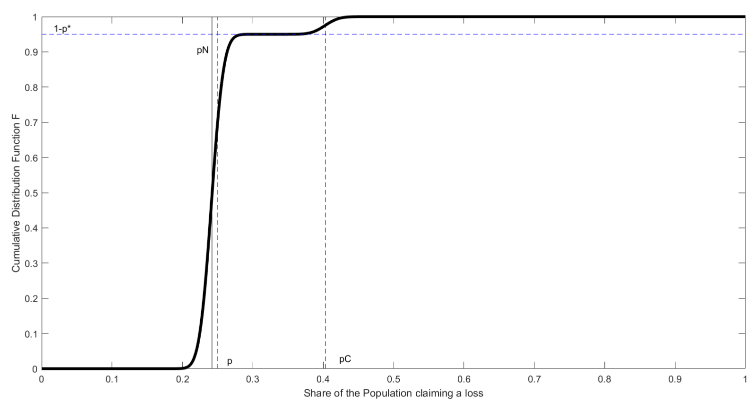


Figure 5: Distribution function of event size($n = 1000, p^* = 0.05, p = 0.25, \delta = 0.4$)

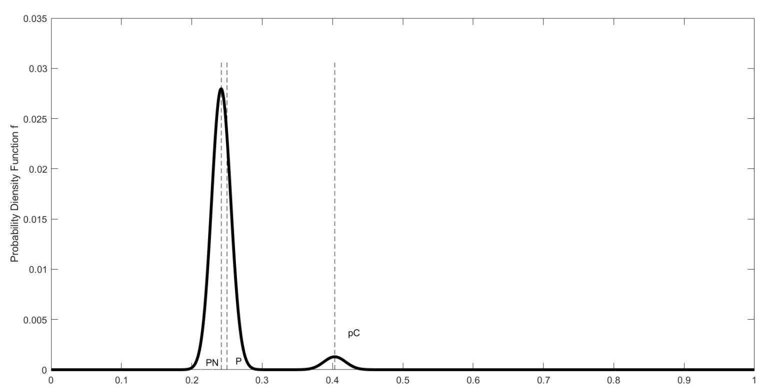


Figure 6: Density function of event size($n = 1000, p^* = 0.05, p = 0.25, \delta = 0.4$)

Following Charpentier and Le Maux (2014), we denote the density and cumulative distribution function of x, the share of inhabitants that claim a loss, as below. Both distribution and density function depend on n, p^* , p and δ.

$$\begin{aligned} F(x) &= P(N \leq k = P(N \leq k|No\ Cat) \times P(No\ Cat) + P(N \leq k|Cat) \times P(Cat) \\ &= \sum_{j=0}^{k} \binom{n}{j} \left[(p_N)^j(1-p_N)^{n-j}(1-p^*) + (p_C)^j(1-p_C)^{n-j}p^*\right], \end{aligned} \tag{47}$$

where $k = \lfloor nx \rfloor$, $p* = P(Cat)$is the the probability of a natural disaster. $P(No\ Cat) = 1 - p^*$. $p_N =$ and p_C are probability that an individual claims loss of no natural disaster and of that of natural disaster respectively. Thus, the individual probability of loss is

$$p = p_N(1-p^*) + p_C p^*. \tag{48}$$

And $p_N =$ and p_C are set as followed.

$$p_N = \frac{(1-\delta)p}{1-\delta+\delta p^*}, \tag{49}$$

$$p_C = \frac{p}{1-\delta+\delta p^*}, \tag{50}$$

where coefficient $\delta \in \left[0, \min\left\{1, \frac{1-p}{1-p^*}\right\}\right]$.

Reference

[1] Bernard, C., & Tian, W. (2009). Optimal reinsurance arrangements under tail risk measures. Journal of Risk and Insurance, 76(3), 709-725.

[2] Bernard, C., & Vanduffel, S. (2014). Financial bounds for insurance claims. Journal of Risk and Insurance, 81(1), 27-56.

[3] Bound, J., Cullen, J. B., Nichols, A., & Schmidt, L. (2004). The welfare implications of increasing disability insurance benefit generosity. Journal of Public Economics, 88(12), 2487-2514.

[4] Bruggeman, V., Faure, M. G., & Fiore, K. (2010). The government as reinsurer of catastrophe risks?. The Geneva Papers on Risk and Insurance-Issues and Practice, 35(3), 369-390.

[5] Bühlmann, H. (1980). An economic premium principle. ASTIN Bulletin: The Journal of the IAA, 11(1), 52-60.

[6] Charpentier, A., & Le Maux, B. (2014). Natural catastrophe insurance: How should the government intervene?. Journal of Public Economics, 115, 1-17.

[7] Chen, C. R., Steiner, T. L., & White, A. M. (2001). Risk taking behaviour and managerial ownership in the United States life insurance industry. Applied Financial Economics, 11(2), 165-171.

[8] Chetty, R. (2009). Sufficient statistics for welfare analysis: A bridge between structural and reduced-form methods. Annual Review of Economics, 1(1), 451-488.

[9] Chetty, R. & Looney, A. (2006). Consumption smoothing and the welfare consequences of social insurance in developing economies. Journal of Public Economics, 90(12), 2351-2356.

[10] Chiappori, P. A., Jullien, B., Salanié, B., & Salanié, F. (2006). Asymmetric information in insurance: General testable implications. The RAND Journal of Economics, 37(4), 783-798.

[11] Cummins, J. D., & Zi, H. (1998). Comparison of frontier efficiency methods: An application to the US life insurance industry. Journal of Productivity Analysis, 10(2), 131-152.

[12] Cutler, D. M., & Zeckhauser, R. J. (1999). Reinsurance for catastrophes and cataclysms. In The Financing of Catastrophe Risk (pp. 233-274). University of Chicago Press.

[13] Doherty, N. A., & Schlesinger, H. (1990). Rational insurance purchasing: Consideration of contract non-performance. Quarterly Journal of Economics. 105, 245-253.

[14] Eckstein, Z., Eichenbaum, M., & Peled, D. (1985). Uncertain lifetimes and the welfare enhancing properties of annuity markets and social security. Journal of Public Economics, 26(3), 303-326.

[15] Einav, L., Finkelstein, A., & Cullen, M. R. (2010). Estimating welfare in insurance markets using variation in prices. The Quarterly Journal of Economics, 125(3), 877-921.

[16] Emons, W. (2001). Imperfect tests and natural insurance monopolies. The Journal of Industrial Economics, 49(3), 247-268.

[17] Filipović, D.,Kremslehner, R., & Muermann, A. (2015). Optimal investment and premium policies under risk shifting and solvency regulation. Journal of Risk and Insurance, 82(2), 261-288.

[18] Froot, K. A. (2001). The market for catastrophe risk: A clinical examination. Journal of Financial Economics, 60(2-3), 529-571.

[19] Goodspeed, T. J., & Haughwout, A. F. (2012). On the optimal design of disaster insurance in a federation. Economics of Governance, 13(1), 1-27.

[20] Ho, C. L., Lai, G. C., & Lee, J. P. (2013). Organizational structure, board composition, and risk taking in the US property casualty insurance industry. Journal of Risk and Insurance, 80(1), 169-203.

[21] Jaffee, D. M., & Russell, T. (1997). Catastrophe insurance, capital markets, and uninsurable risks. Journal of Risk and Insurance, 205-230.

[22] Johnson, H., & Stulz, R. (1987). The pricing of options with default risk. The Journal of Finance, 42(2), 267-280.

[23] Kunreuther, H. (1984). Causes of underinsurance against natural disasters. Geneva Papers on Risk and Insurance, 206-220.

[24] Kousky, C., & Cooke, R. (2012). Explaining the failure to insure catastrophic risks. The Geneva Papers on Risk and Insurance. Issues and Practice, 37(2), 206-227.

[25] Kunreuther, H. (2006). Reflections on US disaster insurance policy for the 21st century (No. w12449). National Bureau of Economic Research.

[26] Kunreuther, H. (2008). Reducing losses from catastrophic risks through long-term insurance and mitigation. Social Research, 905-930.

[27] Litan, R. E. (2006). Sharing and reducing the financial risks of future "mega-catastrophes". Brookings Institution.

[28] Sonnenholzner, M. , & Wambach, A. (2006). Oligopoly in insurance markets. Encyclopedia of Actuarial Science. John Wiley & Sons, Ltd.

[29] Sugarman, S. D. (2007). Roles of government in compensating disaster victims. Issues in Legal Scholarship, 6(3).

[30] Tapiero, C. S., Kahane, Y., & Jacque, L. (1986). Insurance premiums and default risk in mutual insurance. Scandinavian Actuarial Journal, 1986(2), 82-97.

[31] Wildasin, D. (2008). Disaster avoidance, disaster relief, and policy coordination in a federation. Working Paper. University of Kentucky.