# The Application of Affine Processes in Multi-Cohort Mortality Risk Models


**Abstract**

This paper assesses and extends continuous time affine mortality models based on the Arbitrage-Free Nelson-Siegel (AFNS) framework showing how they have superior forecasting performance and can flexibly incorporate factor dependence into the model structure. As well as the common assumption of Gaussian distributed mortality intensity, we assess the Cox-Ingersoll-Ross (CIR) process based mortality model allowing for Gamma distributed mortality rates. We compare models fitted to age-period data with models fitted to age-cohort data and show how to capture cohort effects more effectively in the continuous time framework. The models have appealing features suited for theoretical and practical application.



Note: This is a draft working paper reporting work in progress prepared for consideration to be presented at the ARIA Annual Meeting in August 2019.

# 1 Introduction

Longevity risk refers to the risk that the overall survival probabilities of the reference population are higher than anticipated (Cairns et al., 2006a). One of the main causes of longevity risk is the mortality improvement over the last few decades, which result in larger payments than expected from pension funds and life insurance companies. The Continuous Mortality Investigation (2018) suggests that mortality rates continue to improve and it has been shown that total longevity risk is significant for longevity risk holders from a financial perspective (The Joint Forum, 2013). Thus, pension funds and life insurance companies require different approaches to manage longevity risk through financial and insurance products.

One way to reduce the exposure to mortality improvement is to transfer the longevity risk to counterparties (Biffis and Blake, 2009), including transferring the longevity risk to reinsurers and capital markets. The capital market solutions can be accomplished through securitization of insurance assets and liabilities, which has become increasingly crucial in longevity risk management in recent years (Blake et al., 2018). With capital markets as the counterparties, market capacity to absorb longevity risk as well as liquidity and pricing of longevity risk can be enhanced (Biffis and Blake, 2009). A variety of instruments that are known as longevity-linked securities have been developed for this purpose, including longevity bonds (Blake and Burrows, 2001), longevity swaps (Dowd et al., 2006), and q-forwards (Coughlan et al., 2007). During the period from 2007 to 2016, the global market of longevity risk transfer has grown significantly (Blake et al., 2018). Therefore, transferring longevity risk to capital market has drawn great attention as an alternative approach for effective longevity risk management.

Despite the importance of longevity-linked securities, the pricing of these instruments is still under discussion and one of the major challenges in pricing is to forecast the mortality improvement for financial applications.

The aim of this article is to provide guidance on applying the mortality model in Blackburn and Sherris, 2013 and the mortality model based on the Arbitrage-Free Nelson-Siegel (AFNS) model (Christensen et al., 2011) and their extensions, which is for efficient financial applications and risk management.

Different types of stochastic mortality models have been proposed to describe the evolution of mortality rates, including discrete-time models and continuous-time models. The discrete-time models employ discrete-time series to capture mortalities rates, among which the Lee-Carter model (Lee and Carter, 1992) is most widely used. Various extensions have been made to the Lee-Carter model for further improvement, including incorporating cohort effects to the model. Cairns et al. (2009) have provided a summary on these extensions, among which the Cairns-Blake-Dowd (Cairns et al., 2006b) and the age-period-cohort model (Renshaw and Haberman, 2006) are more widely applied. Compared with the discrete-time models, the continuous-time models describe mortality intensity with diffusion processes, such as affine processes, which are significant for understanding the evolution of prices of longevity-linked securities.

Affine mortality models are developed based on the Affine Term Structure Model (ATSM) in interest rate modeling developed by Duffie et al. (1996) and Dai and Singleton (2000), due to the similarities between mortality intensity and interst rates (Cairns et al., 2006a). The advantages of the affine mortality include having a similar structure with interest rate models to allow for an integrated pricing framework (Barrieu et al. (2012)) and the producing closed-form solutions to survival probabilities for easier computation (Dahl, 2004). Moreover, the affine framework satisfies the consistency requirement to ensure the same functional form of forecast survival curves for financial purposes (Björk and Christensen, 1999; De Rossi, 2004; Blackburn and Sherris, 2013). Additionally, factor loadings in the affine models can be interpreted. The affine framework also has an advantage in parameter estimation that the model can be expressed as a state space model with the Kalman filter (Kalman, 1960) to generate factor values with up-to-date information and incorporate Poisson measurement errors (Schrager, 2006; Blackburn and Sherris, 2013; Xu et al., 2015; Xu et al., 2018).

A number of stochastic affine mortality models have been proposed to capture the short rate of mortality in the literature. Both Milevsky and Promislow (2001) and Dahl (2004) have introduced a pricing framework for mortality-related products and Dahl (2004) has defined the affine mortality model. Schrager (2006) and Blackburn and Sherris (2013) model and project the whole term structure of mortality but only focus on the age-period effects of mortality rates. A cohort refers to a group of individuals born in a particular period sharing specific experience (Xu et al., 2015). Since cohort effects have been observed in different countries (Willets, 2004;Cairns et al., 2009; Gallop, 2008), possibly attributed to changing lifestyles and medical research advances (Gallop, 2008), some of affine mortality models have incorporated cohort effects. These models contain the ones that capture the evolution of mortality intensity within a single cohort (Dahl, 2004; Biffis, 2005; Dahl and Mller, 2006; Luciano and Vigna, 2008), and the ones that describes the force of mortality across all cohorts simultaneously (Jevtic et al., 2013; Xu et al., 2015; Chang and Sherris, 2018). However, the existing affine mortality models do not produce satisfactory model fit at older ages and predictive performance. Besides, the factors in the existing affine models are unidentifiable, which implies that the factors cannot be interpreted to better understand the evolution of mortality rates.

As affine mortality models are developed based on the ATSMs in Duffie et al. (1996) in interest rate modeling due to the similarities between the mortality intensity and interest rates. Among the ATSMs, the AFNS model (Christensen et al., 2011; Diebold and Rudebusch, 2013) demonstrates outstanding model fit and superior out-of-sample forecasting performance with being parsimonious and tractable at the same time. Furthermore, the imposition of arbitrage-free restriction ensures that the AFNS model is theoretically rigorous in pricing financial products. Similarly, the AFNS model can be applied in mortality modeling as one of the ATSMs.

Gaussian processes have been adopted to describe the dynamics of mortality intensity under the affine framework, such as the models proposed by Blackburn and Sherris (2013) and Xu et al. (2015). A one-factor model Gaussian model used to capture mortality intensity can be represented by the following stochastic differential equation (SDE):

$$\mu_x^i(t) = -\delta\mu_x^i(t)\,dt + \sigma dW_t, \tag{1}$$

where the unobserved state variable $\mu_x^i(t)$ is the instantaneous mortality intensity for individuals aged $x$ at time $t$ of a given cohort $i$, $\delta$ is the mean reversion matrix, $\sigma$ is the volatility matrix, and $W_t$ is a standard Brownian motion.

This article has implemented the proposed mortality model using age-cohort data, where the mortality data is listed based on birth year and ages of individuals. Some of the models are based on age-period data, where the data is arranged based on ages of individual and the calendar year which the data was collected in.

With the observed cohort effects, it suggests that a good stochastic mortality model for pricing and forecasting should account for such effects (Cairns et al., 2009). In addition, the pricing of longevity-linked securities depends on the survival rates of certain cohorts (Xu et al., 2015; Jevtic et al., 2013). Directly modeling the mortality rates across cohorts using the age-cohort data enhances the understanding of cohort mortality rates in pricing and allows for projections of survival probabilities across cohorts. On the contrary, implementing the model on the age-period data directly only predicts survival rates across calendar years and thus, the required survival probabilities of cohorts have to be computed by following the cohorts. Therefore, cohort effects should be considered and the age-cohort data is adopted in model calibration.

This article is structured as follows. Section 2 provides a description on the US mortality data. Model specification and estimation of affine mortality models are summarized in Section 3. Section 4 conducts an analysis of the estimation results and compares the mortality models. In Section 5, survival probabilities are predicted with the mortality models out-of-sample forecasting ability is examined. Section 6 concludes the paper with a summary and major findings.

# 2 Data Summary

The mortality data used for estimation is the US mortality data from the Human Mortality Database (2017) (HMD) sponsored by the University of California, Berkeley (USA), and Max Planck Institute for Demographic Research (Germany). To be consistent with the existing affine mortality models (Blackburn and Sherris, 2013; Xu et al., 2015; Chang and Sherris, 2018), the proposed mortality model has been calibrated to the mortality data from age 50 to 100 to study the mortality dynamics at older ages. Thus, the cohorts born from 1883 and 1915 were selected from the available data.

Follow each cohort, the survival probability $S^i(x;t,T)$ and the average force of mortality $\bar{\mu}^i(x;t,T)$ over the period $\tau = T - t$ for cohort $i$ aged $x$ at time $t$ can be calculated from the data set, respectively:

$$S^i(x;t,T) = \prod_{s=1}^{T-t} \left[1 - q^i(x+s-1, t+s-1)\right], \tag{2}$$

$$\bar{\mu}^i(x;t,T) = -\frac{1}{T-t} \log\left[S^i(x;t,T)\right], \tag{3}$$

where $q^i(x,t)$ is the one year death probability for an individual aged $x$ at time $t$ in cohort $i$.

The average force of mortality for the cohorts born between 1883 and 1915, aged 50 to 100, is plotted in Figure 1. There exists mortality improvement across cohorts, with a downward trend of the average force of mortality at each age. Also, each cohort shows an exponential shape of the mortality curve and each age has a different rate of mortality improvement.

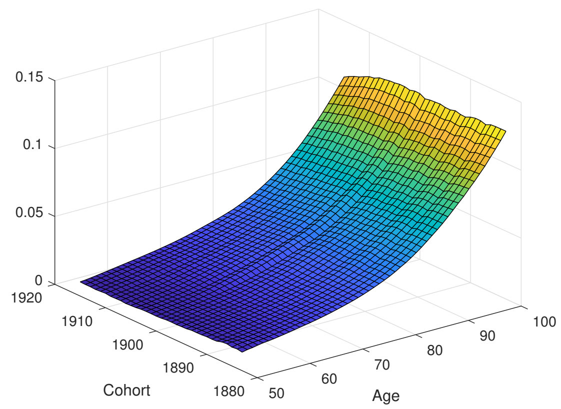



Figure 1: Average Force of Mortality for Males Born from 1883 to 1915

## 3 Affine Mortality Models with Gaussian Processes

Due to the incomplete market of longevity risk, there exists no unique risk-neutral measure $Q$ (Xu et al., 2015). In pricing longevity risk and longevity-linked financial products, the risk-neutral measure $Q$ is defined in terms of the zero-coupon longevity bond (Cairns et al., 2006a; Bauer et al., 2008; Blackburn and Sherris, 2013). The real-world measure $P$ reflects the best estimate of mortality, which is related to the historical mortality data (Bauer et al., 2008).

Similar to what is described in Blackburn and Sherris (2013) and Cairns et al. (2006a), considering a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$, $\Omega$ is the set of possible states of nature and $\mathbb{F} = \{\mathcal{F}_t\}_{0 \leq t \leq T}$ is assumed to satisfy the condition of right continuity.

We define that the instantaneous mortality intensity for individuals aged $x$ at time $t$ of a given cohort $i$:

$$\mu_x^i(t) = \rho_1' X_t, \tag{4}$$

where $\rho_1 \in \mathbb{R}^n$, and $X_t$ is a vector of $n$ latent factors (or state variables) that are driving the mortality intensity. The subscript $x$ indicating ages and $i$ indicating cohorts will be dropped in the rest of the article.

The state variables $X_t$ that are driving the mortality intensity are described by the following system of SDEs under the risk-neutral measure $Q$:

$$dX_t = -K^Q X_t dt + \Sigma dW_t^Q, \tag{5}$$

where $K^Q \in \mathbb{R}^{n \times n}$ is the mean reversion matrix, $\Sigma \in \mathbb{R}^{n \times n}$ is the volatility matrix, and $W_t^Q \in \mathbb{R}^n$ is a standard Brownian motion. The state variables $X_t$ are said to follow Gaussian processes with the above SDEs.

Then, the survival probabilities can be represented as (Blackburn and Sherris, 2013):

$$S(t,T) = E^Q\left[\exp\left(-\int_t^T \mu(s)\, ds\right)\right] = \exp\left(B(t,T)' X_t + A(t,T)\right), \tag{6}$$

where $B(t,T)$ and $A(t,T)$ are the solutions to the following system of ordinary differential equations (ODEs):

$$\frac{dB(t,T)}{dt} = \rho_1 + \left(K^Q\right)' B(t,T), \tag{7}$$

$$\frac{dA(t,T)}{dt} = -\frac{1}{2}\sum_{j=1}^{3}\left(\Sigma' B(t,T) B(t,T)' \Sigma\right)_{j,j}, \tag{8}$$

with boundary conditions $B(T,T) = A(T,T) = 0$.

The average force of mortality over the duration $(T-t)$ is affine in the latent factors and is defined as (Blackburn and Sherris, 2013; Xu et al., 2015):

$$\bar{\mu}(t,T) = -\frac{1}{T-t}\log\left[S(t,T)\right] = -\frac{B(t,T)'}{T-t}X_t - \frac{A(t,T)}{T-t}. \tag{9}$$

### 3.1 Change of Measure

Since the affine mortality model with Gaussian processes is defined under the risk-neutral measure $Q$, the measure needs to be changed to the real-world measure $P$ to fit the observed mortality data to the model, with the market price of longevity risk. The relationship between the dynamics under the measure $P$ and the ones under the measure $Q$ is written as:

$$dW_t^Q = dW_t^P + \Lambda_t dt, \tag{10}$$

where $\Lambda_t$ is the risk premium.

To specify the structure of the risk premium of longevity risk, we have adopted the essentially affine model proposed by Duffee (2002). The market price of risk has the following form:

$$\Lambda_t = \lambda^0 + \lambda^1 X_t, \tag{11}$$

where $\Lambda_t \in \mathbb{R}^{n\times 1}$, $\lambda^0 \in \mathbb{R}^{n\times 1}$ and $\lambda^1 \in \mathbb{R}^{n\times n}$.

With the above specification, the SDEs of the state variables $X_t$ under the real-world measure $P$ can be written as following:

$$dX_t = K^P \left[\theta^P - X_t\right] dt + \Sigma dW_t^P, \tag{12}$$

where

$$K^P = K^Q - \Sigma\lambda^1, \quad \theta^P = \frac{\Sigma\lambda^0}{K^Q} \tag{13}$$

and $K^P$ is the mean-reversion matrix and $\Sigma$ is the volatility under measure $P$. Due to the flexible structure of $\Lambda_t$, we are free to choose the mean reversion matrix $K^P$ and the mean vector $\theta^P$.

### 3.2 Model Specification

This section presents the affine mortality model with three-factor proposed by Blackburn and Sherris (2013) (hereafter Blackburn-Sherris model) and the three-factor affine mortality model developed based on the AFNS model, with independent and dependent factors.

#### 3.2.1 The Blackburn-Sherris Model

The instantaneous mortality intensity is defined as:

$$\mu(t) = X_t^1 + X_t^2 + X_t^3, \tag{14}$$

with $\rho_1 = (1,1,1)^T$ in Equation (4).

**The independent-factor model:** the dynamics of the state variables $X_t = \left(X_t^1, X_t^2, X_t^3\right)$ have the following form under the risk-neutral measure $Q$:

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = -\begin{pmatrix} \delta_{11} & 0 & 0 \\ 0 & \delta_{22} & 0 \\ 0 & 0 & \delta_{33} \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_t^{1,Q} \\ dW_t^{2,Q} \\ dW_t^{3,Q} \end{pmatrix}, \tag{15}$$

and can be represented by this system of SDEs under the real-world measure $P$:

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = -\begin{pmatrix} k_{11}^P & 0 & 0 \\ 0 & k_{22}^P & 0 \\ 0 & 0 & k_{33}^P \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix} \begin{pmatrix} dW_t^{1,P} \\ dW_t^{2,P} \\ dW_t^{3,P} \end{pmatrix}. \tag{16}$$

$B(t,T)$ and $A(t,T)$ are derived as solutions to the ODEs in Equations (7) and (8):

$$B^j(t,T) = -\frac{1 - e^{-\delta_{jj}(T-t)}}{\delta_{jj}}, \quad j = 1,2,3, \tag{17}$$

$$A(t,T)=\frac{1}{2}\sum_{j=1}^{3}\frac{\sigma_{jj}^2}{\delta_{jj}^3}\left[\frac{1}{2}\left(1-e^{-2\delta_{jj}(T-t)}\right)-2\left(1-e^{-\delta_{jj}(T-t)}\right)+\delta_{jj}(T-t)\right]. \tag{18}$$

**The dependent-factor model:** both of the mean reversion matrix $K^Q$ and the volatility matrix $\Sigma$ are lower-triangular matrices. Thus, the dynamics of state variables $X_t$ are driven by the following SDEs under the risk-neutral measure $Q$:

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = -\begin{pmatrix} \delta_{11} & 0 & 0 \\ \delta_{21} & \delta_{22} & 0 \\ \delta_{31} & \delta_{32} & \delta_{33} \end{pmatrix}\begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}\begin{pmatrix} dW_t^{1,Q} \\ dW_t^{2,Q} \\ dW_t^{3,Q} \end{pmatrix}. \tag{19}$$

The real-world dynamics can be represented as:

$$\begin{pmatrix} dX_t^1 \\ dX_t^2 \\ dX_t^3 \end{pmatrix} = -\begin{pmatrix} k_{11}^P & 0 & 0 \\ 0 & k_{22}^P & 0 \\ 0 & 0 & k_{33}^P \end{pmatrix}\begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}\begin{pmatrix} dW_t^{1,P} \\ dW_t^{2,P} \\ dW_t^{3,P} \end{pmatrix}. \tag{20}$$

### 3.2.2 The AFNS model

The latent factors are $X_t = (L_t, S_t, C_t)$ for the 3-factor AFNS model. The instantaneous mortality intensity is defined as:

$$\mu(t) = L_t + S_t, \tag{21}$$

with $\rho_1 = (1,1,0)^T$ in Equation (4).

**The independent-factor model:** the dynamics of the factors under the $Q$-measure are:

$$\begin{pmatrix} dL_t \\ dS_t \\ dC_t \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta & -\delta \\ 0 & 0 & \delta \end{pmatrix}\begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}\begin{pmatrix} dW_t^{1,Q} \\ dW_t^{2,Q} \\ dW_t^{3,Q} \end{pmatrix}. \tag{22}$$

and the SDEs of the latent factors under the real-world measure $P$ are:

$$\begin{pmatrix} dL_t \\ dS_t \\ dC_t \end{pmatrix} = -\begin{pmatrix} k_{11}^P & 0 & 0 \\ 0 & k_{22}^P & 0 \\ 0 & 0 & k_{33}^P \end{pmatrix}\begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{33} \end{pmatrix}\begin{pmatrix} dW_t^{1,P} \\ dW_t^{2,P} \\ dW_t^{3,P} \end{pmatrix}. \tag{23}$$

**The dependent-factor model:** the $Q$-dynamics of state variables $X_t = (L_t, S_t, C_t)$ can be represented as:

$$\begin{pmatrix} dL_t \\ dS_t \\ dC_t \end{pmatrix} = -\begin{pmatrix} 0 & 0 & 0 \\ 0 & \delta & -\delta \\ 0 & 0 & \delta \end{pmatrix}\begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}\begin{pmatrix} dW_t^{1,Q} \\ dW_t^{2,Q} \\ dW_t^{3,Q} \end{pmatrix}. \tag{24}$$

With a diagonal $K^P$, the factor dependence is added through correlated shocks only. Thus, the dynamics of $X_t$ can be written as the following SDEs under measure $P$:

$$\begin{pmatrix} dL_t \\ dS_t \\ dC_t \end{pmatrix} = \begin{pmatrix} k_{11}^P & 0 & 0 \\ 0 & k_{22}^P & 0 \\ 0 & 0 & k_{33}^P \end{pmatrix}\begin{pmatrix} L_t \\ S_t \\ C_t \end{pmatrix} dt + \begin{pmatrix} \sigma_{11} & 0 & 0 \\ \sigma_{21} & \sigma_{22} & 0 \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}\begin{pmatrix} dW_t^{1,P} \\ dW_t^{2,P} \\ dW_t^{3,P} \end{pmatrix}. \tag{25}$$

The solutions of $B(t,T)$ and $A(t,T)$ in the independent-factor and dependent-factor AFNS models are given in the *Thesis*.

### 3.3 Parameter Estimation

The Kalman filter (Kalman, 1960) maximum likelihood estimation framework is adopted for estimating the parameters in the affine mortality models, as suggested by Christensen et al. (2011) and Blackburn and Sherris (2013).

The estimation process proceeds as follows:

1. Represent the affine mortality models in the state space form which consists of two components, a measurement equation and a state transition equation (Xu et al., 2015; Shumway and Stoffer, 2017).

   A measurement equation describes the affine relationship between the average force of mortality and the state variables (Xu et al., 2015; Durbin and Koopman, 2012). The measurement equation can be written with the average force of mortality (Blackburn and Sherris, 2013; Xu et al., 2015; Christensen et al., 2011):

   According to Blackburn and Sherris (2013) and Xu et al. (2015), the measurement equation is based on the average force of mortality:

$$\bar{\mu}(t,T) = -\frac{B(t,T)'}{T-t} X_t - \frac{A(t,T)}{T-t} + \varepsilon_t, \quad \varepsilon_t \sim N(0,H), \tag{26}$$

   where the measurement error $\varepsilon_t$ is independently and identically distributed noise with $H$ being diagonal and the covariance matrix of the measurement error.

   The parameteric form used of the diagonal for the covariance matrix $H$ is

$$H(t,T) = \frac{1}{T-t} \sum_{i=1}^{T-t} \left[ r_c + r_1 e^{r_2 i} \right], \tag{27}$$

   where the values of $r_c$, $r_1$ and $r_2$ are estimated as part of the optimal parameters set.

   For example, for a 3-factor affine mortality model with $X_t = \left(X_t^1, X_t^2, X_t^3\right)$, the measurement equation with $N$ observed average forces of mortality is written as:

$$\begin{pmatrix} \bar{\mu}(t,t+1) \\ \bar{\mu}(t,t+2) \\ \vdots \\ \bar{\mu}(t,t+N) \end{pmatrix} = \begin{pmatrix} -B^1(t,t+1) & -B^2(t,t+1) & -B^3(t,t+1) \\ -\frac{B^1(t,t+2)}{2} & -\frac{B^2(t,t+2)}{2} & -\frac{B^3(t,t+2)}{2} \\ \vdots & \vdots & \vdots \\ -\frac{B^1(t,t+N)}{N} & -\frac{B^2(t,t+N)}{N} & -\frac{B^3(t,t+N)}{N} \end{pmatrix} \begin{pmatrix} X_t^1 \\ X_t^2 \\ X_t^3 \end{pmatrix} + \begin{pmatrix} -A(t,t+1) \\ -\frac{A(t,t+2)}{2} \\ \vdots \\ -\frac{A(t,t+N)}{N} \end{pmatrix} + \begin{pmatrix} \varepsilon_t(1) \\ \varepsilon_t(2) \\ \vdots \\ \varepsilon_t(N) \end{pmatrix}. \tag{28}$$

   A state transition equation represents the unobserved dynamics of the state variables (Xu et al., 2015; Durbin and Koopman, 2012). Working on the conditional moments, the state transition equation is obtained as:

$$X_t = \exp\left(-K^P\right) X_{t-1} + \eta_t, \quad \eta_t \sim N(0,R), \tag{29}$$

where $\eta_t$ is the transition error vector with a diagonal matrix $R$ being the covariance matrix of the transition error.

Matrix $R$ has the following structure:

$$R = \int_{t-1}^{t} e^{-K^P(t-s)} \Sigma\Sigma' e^{-\left(K^P\right)'(t-s)} ds. \tag{30}$$

2. Use the Kalman filter to evaluate the likelihood function of affine mortality models and to extract the values of state variables. Let the information available at time $t$ denoted by $Y_t = (y_1, \ldots, y_t)$ and the model parameters represented by $\psi$.

   In the forecasting step, assuming the state update $X_{t-1}$ and its mean square error $\Sigma_{t-1}$ are obtained at $t-1$,

$$X_{t|t-1} = E\left[X_t|Y_{t-1}\right] = \Phi\left(\psi\right) X_{t-1}, \tag{31}$$

$$\Sigma_{t|t-1} = \Phi\left(\psi\right) \Sigma_{t-1} \Phi\left(\psi\right)' + R\left(\psi\right), \tag{32}$$

   where $\Phi = \exp\left(-K^P\right)$.

   In the update step, the information at time $t$, $Y_t$, is used to update the forecasts $X_{t|t-1}$ and we obtain:

$$X_t = E\left[X_t|Y_t\right] = X_{t|t-1} + \Sigma_{t|t-1} B\left(\psi\right)' F_t^{-1} \nu_t, \tag{33}$$

$$\Sigma_t = \Sigma_{t|t-1} - \Sigma_{t|t-1} B\left(\psi\right)' F_t^{-1} B\left(\psi\right) \Sigma_{t|t-1}, \tag{34}$$

   where

$$\nu_t = y_t - E\left[y_t|Y_{t-1}\right] = y_t - A\left(\psi\right) - B\left(\psi\right) X_{t|t-1}, \tag{35}$$

$$F_t = cov\left(\nu_t\right) = B\left(\psi\right) \Sigma_{t|t-1} B\left(\psi\right)' + H\left(\psi\right). \tag{36}$$

3. Evaluate the following log-likelihood function with the values obtained in the previous step:

$$\log L\left(y_1, \ldots, y_t; \psi\right) = \sum_{t=1}^{T} \left(-\frac{N}{2}\log\left(2\pi\right) - \frac{1}{2}\log\left|F_t\right| - \frac{1}{2}\nu_t' F_t \nu_t\right), \tag{37}$$

   where $N$ is tthe number of observed average force of mortality. The log-likelihood function is maximized with respect to $\psi$ to obtain the optimal parameter set.

# 4 Calibration Results and Model Comparison

## 4.1 Parameter Estimates

Table 1 summarizes the parameter estimates for each model, along with standard errors of $\delta$s. The standard errors of the $\delta$s help to determine the significance of factor loadings, which further indicates whether the corresponding factor has significant influence on mortality rates.

The mean reversion $K^P$ indicates the speed to revert to the long-term mean 0. The parameters in $K^Q$ ($\delta'$s) represents the sensitivity of mortality rates to each of the factors at different ages. The $\delta$s of all models are significant, suggesting that the factors in each model have significant impacts on mortality rates. Comparing the two AFNS models, $\delta$'s are both negative to account for the sensitivity of older ages to the slope factor. The dependent-factor AFNS model $\delta$'s are smaller in absolute value than the one in the independent-factor AFNS model. The possible reason is that $\delta$ in the independent model is the only parameter to capture the factor dependence.

The volatility indicates the variability of the factors across time. Because of the structure of matrix $H$ in Equation (27), the measurement errors are age-dependent and exponentially increasing with age. By comparing the $r_2$, which is the scalar in the exponential function in matrix $H$, the independent-factor Blackburn-Sherris model has the largest $r_2$, resulting in larger measurement error volatility than other models.

Table 1: Estimated Parameters

| | The Blackburn-Sherris Model | | The AFNS Model | |
|---|---|---|---|---|
| | Independent-Factor | Dependent-Factor | Independent-Factor | Dependent-Factor |
| $\delta_{11}$ (AFNS: $\delta$) | -0.01106 (0.00123) | -0.20183 (2.944e-05) | -0.08348 (1.580e-04) | -0.04725 (8.096e-05) |
| $\delta_{21}$ | - | 0.56206 (4.091e-05) | - | - |
| $\delta_{22}$ | 0.07484 (0.00432) | -0.07092 (1.766e-05) | - | - |
| $\delta_{31}$ | - | 0.24075 (1.555e-05) | - | - |
| $\delta_{32}$ | - | 0.80809 (4.102e-05) | - | - |
| $\delta_{33}$ | -0.06883 (2.452e-04) | 0.77825 (1.461e-05) | - | - |
| $k_{11}^P$ | 0.38753 | -0.04248 | 0.18793 | 0.01810 |
| $k_{22}^P$ | 0.13910 | 0.01869 | 0.01361 | 0.02002 |
| $k_{33}^P$ | 0.00718 | 0.01827 | 0.02701 | 0.04972 |
| $\sigma_{11}$ | 0.00782 | 7.557e-11 | 9.593e-04 | 0.00400 |
| $\sigma_{21}$ | - | 0.01110 | - | -0.00387 |
| $\sigma_{22}$ | 0.00125 | 3.370e-11 | 1.120e-04 | 0.00091 |
| $\sigma_{31}$ | - | -0.01190 | - | -0.00183 |
| $\sigma_{32}$ | - | 0.00047 | - | 0.00123 |
| $\sigma_{33}$ | 5.409e-04 | 0.00029 | 3.549e-05 | 0.00023 |
| $r_1$ | 1.071e-11 | 4.337e-08 | 1.422e-10 | 6.272e-08 |
| $r_2$ | 0.37797 | 0.11375 | 0.17784 | 0.10742 |
| $r_c$ | 4.360e-08 | 5.705e-08 | 4.963e-07 | 4.636e-13 |

### 4.2 Model Goodness-of-Fit

For each model, we have calculatedthe Root Mean Square Error (RMSE), the Akaike information criterion (AIC), the Bayesian information criterion (BIC).

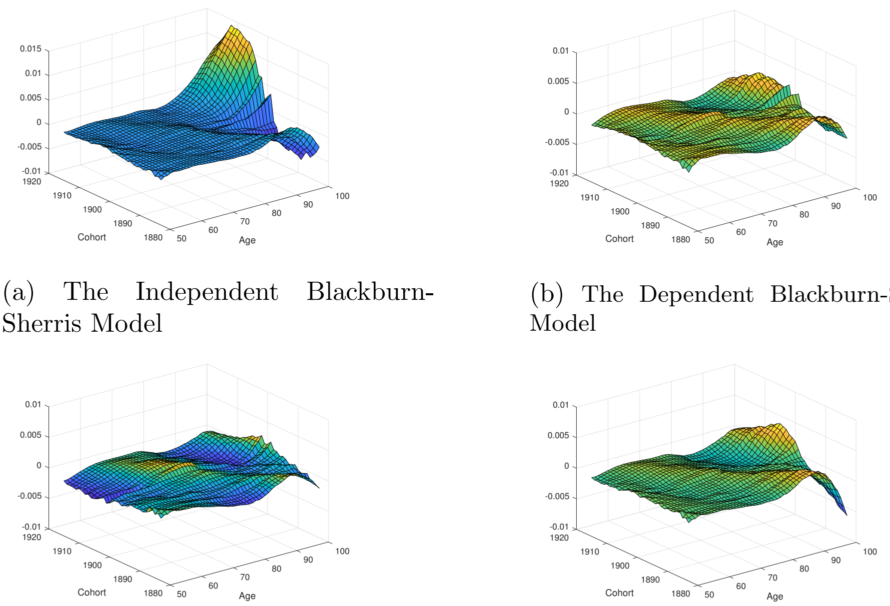
Table 2: Comparison of Affine Mortality Models

| | The Blackburn-Sherris Model | | The AFNS Model | |
|---|---|---|---|---|
| | Independent-Factor | Dependent-Factor | Independent-Factor | Dependent-Factor |
| Log Likelihood | 9896.419 | 9938.696 | 9665.801 | 9887.878 |
| RMSE | 0.00250 | 7.601e-04 | 6.856e-04 | 9.160e-04 |
| No. of Parameteres | 12 | 18 | 10 | 13 |
| AIC | -19570.837 | -19643.392 | -19113.602 | -19551.757 |
| BIC | -18968.292 | -19008.277 | -18521.914 | -18943.783 |
| Probability of Negative Mortality | 0.02700 | 1.011e-32 | 1.722e-31 | 4.34e-14 |

The results of the above numeric measures are summarized in Table 2. Due to the more complicated model structure in the dependent-factor Blackburn-Sherris model, this model has the largest log-likelihood and better AIC and BIC than the other models. The possible reason is that more parameters allow this model to better capture the dependent structure in the mortality. Among the Blackburn-Sherris models, dependence between factors could improve the model fit of the independent model in terms of log- likelihood, RMSE, AIC and BIC. However, in terms of the RMSE, the independent-factor AFNS model outperforms the others. Additionally, since Gaussian processes are adopted to describe mortality intensity, the probabilities of negative mortality in these models are positive. Both the dependent-factor Blackburn-Sherris model and the independent-factor AFNS model produce relatively negligible probabilities of negative mortality.

### 4.3 Residual Analysis

Figure 2 shows the residuals of mortality models at the same scale on the z-axis (except for the independent Blackburn-Sherris model). The residuals are the differences between the average force of mortality calculated from the actual mortality data and the estimated one from mortality models.




(a) The Independent Blackburn-Sherris Model

(b) The Dependent Blackburn-Sherris Model

(c) The Independent AFNS Model

(d) The Dependent AFNS Model

Figure 2: Residuals of Affine Mortality Models

The independent Blackburn-Sherris model has relatively larger residuals, especially at older ages and younger cohorts. By looking at the independent-factor Blackburn-Sherris (Figure 2a) model and the independent-factor AFNS model (Figure 2c), both have three factors, but the identifiable factors in the AFNS model help to reduce the magnitude of residuals at older ages. Therefore, the identifiable factors are able to account for the variation in mortality curves, especially at older ages.

Comparing the dependent Blackburn-Sherris model (Figure 2b) with the independent model (Figure 2a), factor dependence helps to reduce the size of residuals and to account for the variation at older ages. However, comparing to residuals of the independent AFNS model in Figure 2c, the size of residuals in the dependent AFNS model (Figure 2d) is larger, particularly at older ages. Thus, the independent AFNS model is able to capture the mortality variability better than the dependent AFNS model. Given more pa- rameter to be estimated in the dependent AFNS model, it does not improve the ability to capture the variation in mortality, compared with the independent AFNS model. Between the two dependent-factor models, the dependent AFNS model produces larger residuals of the older cohorts at ages above 90.

### 4.4 In-Sample Analysis

The in-sample model performance analysis is conducted by comparing the estimated cohort survival probabilities from our mortality models and the cohort survival probabili- ties derived from the actual data. Figure 3 summarizes the in-sample model fit results by adopting the Mean Absolute Percentage Error (MAPE) for each age, across all cohorts. The scale of the percentage error is different above and below age 85.

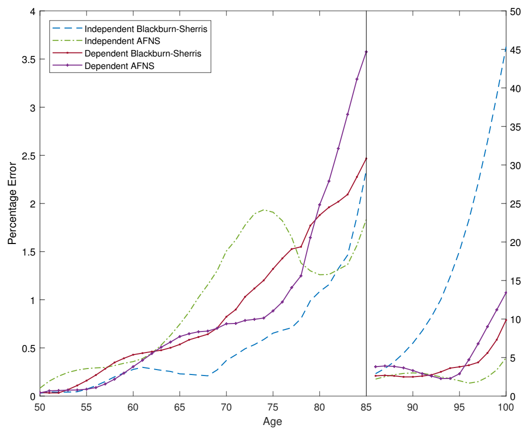



Figure 3: MAPE of Affine Mortality Models

Below age 85, all models have similar performance and the differences between the percentage errors of different models are relatively small. However, above the age 85, the independent Blackburn-Sherris model produces significantly larger percentage errors, while the independent AFNS model provides a better fit. Given that the longevity-linked products have greater concern at older ages where mortality rates are more volatile, the in-sample model fit at older ages is significant. Thus, the independent AFNS model is preferable.

### 4.5 Estimated Factors and Factor Loadings

Figures 4 and 5 show the factors and factor loadings of the dependent Blackburn-Sherris model. $X^1$ is stable around zero across time, whereas $X^2$ and $X^3$ demonstrate general an upward or downward trend. The convex $B_2$ indicates that younger ages around 50 and older ages are more sensitive to $X^2$, and thus, the decline in $X^2$ has improved the mortality at these ages. Since the values of $X^3$ are all negative and increasing, it suggests the mortality improvement at younger ages has slowed down across time, given the exponentially decreasing $B_3$.

The values of factors and factor loadings of the 3-factor AFNS model are displayed in Figures 6 and 7. Prior to 1950, all factors are relatively stable but have shown obvious upward or downward trends since 1950. Given the exponentially increasing factor loading $B_2$, mortality rates at older ages are more sensitive to $S$. After 1950, the rise in $L$ corresponds to the decline in $S$. The curvature factor $C$ means that a change in this factor affects the mortality rates at middle age (around 65 to 85) more than the younger age and older ages. After 1950, the decline in $C$ indicates that the mortality curves are less convex across time. The factor loading $B_3$ is negative and decreasing across all ages and thus, the convexity at older ages decreases faster than younger ages. This suggests that the mortality improvement at older ages

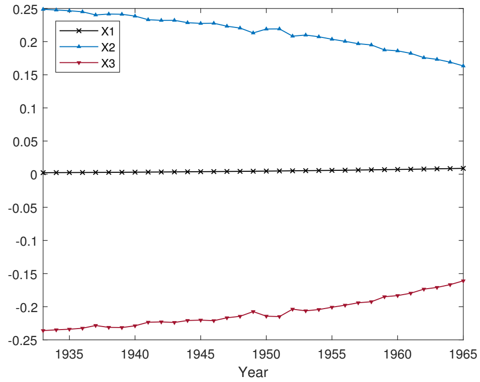
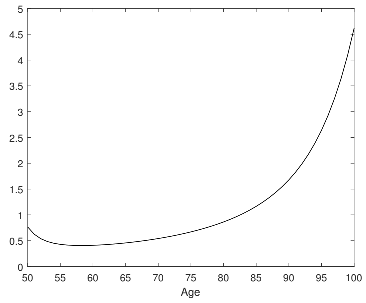
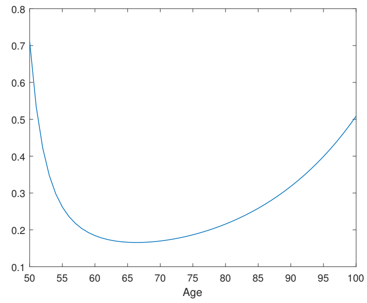


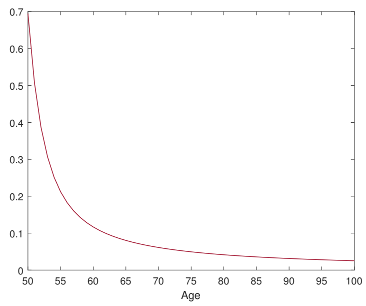

Figure 4: Factors in the Dependent Blackburn-Sherris Model



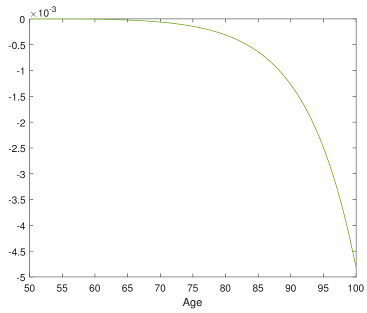

(a) $B_1 = -\frac{B^1(t,T)}{T-t}$




(b) $B_2 = -\frac{B^2(t,T)}{T-t}$




(c) $B_3 = -\frac{B^3(t,T)}{T-t}$




(d) $A = -\frac{A(t,T)}{T-t}$

Figure 5: Factors Loadings in the Dependent Blackburn-Sherris Model

is more significant. The adjustment term $A$ has negative and decreasing values, suggesting more downward adjustment is made at older ages.

# 5 Forecasts of Survival Probabilities

In this section, the out-of-sample forecasting is conducted to investigate the predictive performance of the mortality models, since the mortality models are developed to project survival probabilities for pricing longevity-linked securities.

This out-of-sample forecasting is to predict the mortality improvement across cohorts by using the behaviors of other cohorts, which allows for the cohort effects, which is adopted by Xu et al. (2015). We have used the parameter values estimated from the cohorts born 1883 to 1915 to predict the survival probabilities of the cohort born in 1916.

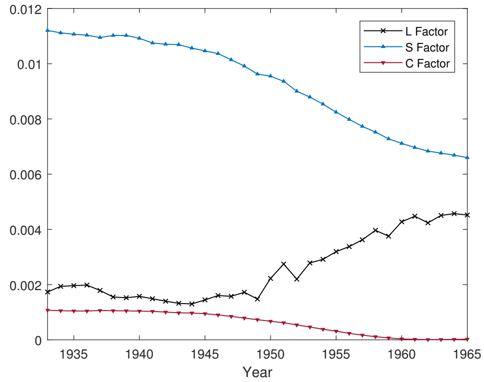
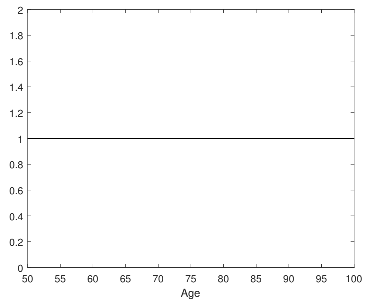
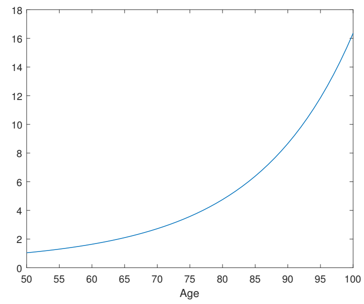


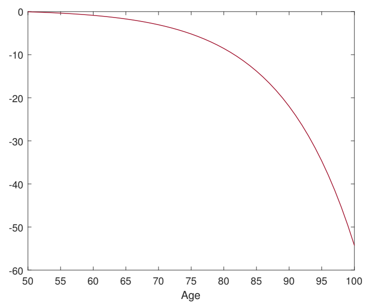

Figure 6: Factors in the Independent AFNS Model



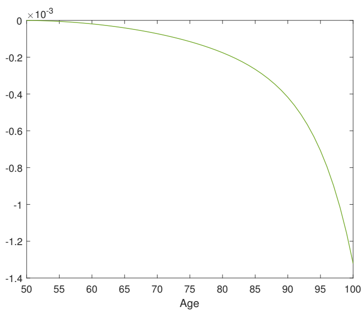

(a) $B_1 = -\frac{B^1(t,T)}{T-t}$




(b) $B_2 = -\frac{B^2(t,T)}{T-t}$




(c) $B_3 = -\frac{B^3(t,T)}{T-t}$




(d) $A = -\frac{A(t,T)}{T-t}$

Figure 7: Factors Loadings in the Independent AFNS Model

### 5.0.1 Simulation Algorithm

To obtain the projected survival probabilities for each model under consideration, the values of Xt needs to be simulated first (Xu et al., 2015). Both the independent Blackburn-Sherris model and the 3-factor independent AFNS model have the same structure of factor dynamics under the real-world measure P in Equations (16) and (23). Thus, the simulation algorithm of state variables in these two models is the same. The simulation algorithm for state variables $X_t = \left(X_t^1, X_t^2, X_t^3\right)$ in the independent Blackburn-Sherris model is described by the following equations:

$$
\begin{aligned}
X_{t+1}^1 &= e^{-k_{11}^P} X_t^1 + \sqrt{\frac{\sigma_{11}^2}{2k_{11}^P}\left(1 - e^{-2k_{11}^P}\right)}\mathcal{Z}_{t+1}^1, \\
X_{t+1}^2 &= e^{-k_{22}^P} X_t^2 + \sqrt{\frac{\sigma_{22}^2}{2k_{22}^P}\left(1 - e^{-2k_{22}^P}\right)}\mathcal{Z}_{t+1}^2, \\
X_{t+1}^3 &= e^{-k_{33}^P} X_t^3 + \sqrt{\frac{\sigma_{33}^2}{2k_{33}^P}\left(1 - e^{-2k_{33}^P}\right)}\mathcal{Z}_{t+1}^3,
\end{aligned}
\tag{38}
$$

where $\mathcal{Z}_{t+1}^1$, $\mathcal{Z}_{t+1}^2$ and $\mathcal{Z}_{t+1}^3$ are independent standard normal random variables. For the independent AFNS model, the algorithm can be simply achieved by replacing $X_t = \left(X_t^1, X_t^2, X_t^3\right)$ in Equation (38) with $X_t = (L_t, S_t, C_t)$.

The algorithm for the dependent models is more complicated, due to the lower-triangular $\Sigma$ matrix (Equations (20) and (25)). The simulation algorithm for the state variables $X_t = \left(X_t^1, X_t^2, X_t^3\right)$ in the dependent Blackburn-Sherris model is:

$$
X_{t+1} = e^{-K^P} X_t + \Delta^{\frac{1}{2}} \mathcal{Z}_{t+1}, \tag{39}
$$

where

$$
e^{-K^P} = \begin{pmatrix} e^{-k_{11}^P} & 0 & 0 \\ 0 & e^{-k_{22}^P} & 0 \\ 0 & 0 & e^{-k_{33}^P} \end{pmatrix}, \tag{40}
$$

and

$$
\Delta = \int_{t-1}^{t} e^{-K^P(t-s)} \Sigma \Sigma' e^{-\left(K^P\right)'(t-s)}, \tag{41}
$$

with $K^P$ and $\Sigma$ specified in Equation (20), and $\Delta^{\frac{1}{2}}$ denoting the Cholesky decompostion of the matrix $\Delta$.

The algorithm of the dependent AFNS model can be obtained by replacing $X_t = \left(X_t^1, X_t^2, X_t^3\right)$ in Equation (39) with $X_t = (L_t, S_t, C_t)$.

#### 5.0.2 Simulation Results

### 5.1 Simulation Results

We have performed 100,000 simulations for 50-year-old males that are born in 1916. Figure 8 shows the average of simulated survival curves of the selected mortality models and the actual survival probabilities estimated from the historical data. The independent Blackburn-Sherris model has a simulated survival curve that underestimates the survival probabilities around age 60 to 80.

To further investigate the projected survival probabilities, we have computed the RMSE by comparing the actual survival probabilities with the 100,000 simulations for each model, which is shown in Table 3. Between the two Blackburn-Sherris models, the dependent model has a smaller RMSE. However, the independent AFNS model has a lower RMSE, compared with the dependent AFNS model. Thus, the dependence structure in the Blackburn-Sherris model has improved the forecasting performance, but the factor dependence in the AFNS model results in poorer projections, in terms of RMSE.

To have a thorough investigation of the projected survival curves in Figure 8, we have computed

Table 3: RMSE by Comparing the Actual and Simulated Survival Probabilities of the 1916 Cohort

| | The Blackburn-Sherris Model | | The AFNS Model | |
|---|---|---|---|---|
| | Independent | Dependent | Independent | Dependent |
| RMSE | 0.04683 | 0.01404 | 0.01103 | 0.01179 |

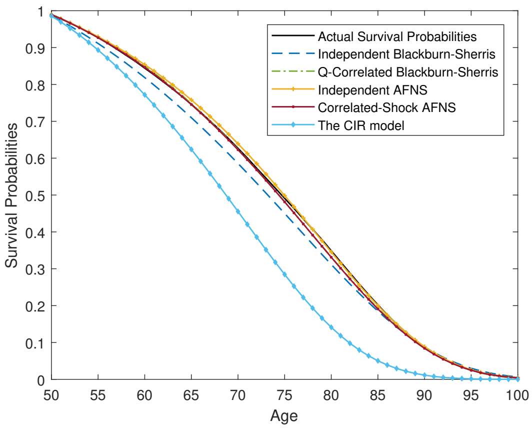



Figure 8: Actual and Simulated Survival Probabilities of the 1916 Cohort

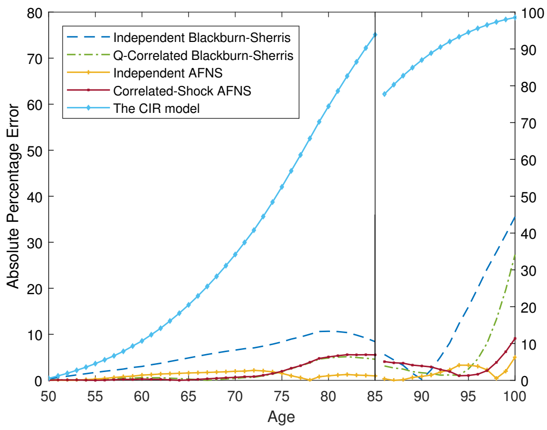



Figure 9: Absolute Percentage Errors between Actual and Mean of Simulated Survival Probabilities

Figure 10: *The CIR graph will be removed later and the legend will be fixed*

the absolute percentage errors of each model at each age, shown in Figure 9. The independent Blackburn-Sherris model has the largest errors across almost all ages. Additionally, the dependent Blackburn-Sherris model and the dependent AFNS model have similar trends and stay close below age 95, but forecasting percentage errors of the former increases fast up to 30% above age 95. Moreover, the independent AFNS model produces the lowest forecasting percentage errors from age 75 to 90, and above 95. The percentage errors of the independent AFNS model are slightly higher than some of the other models, but stay relatively low across all ages.

Therefore, the dependence between factors in the Blackburn-Sherris model is able to improve the forecasting performance, but it cannot address the forecasting errors at older ages. The forecasting percentage errors of the independent and dependent AFNS model have similar performance, but the RMSE of the independent model is lower. Hence, the independent AFNS model is preferred to the dependent AFNS model.

# 6 Conclusion

This paper aims to provide guidance on implementing an affine mortality model with factors following Gaussian processes and predicting survival rates of future cohorts. The mortality models under consideration include the Blacburn-Sherris model (Blackburn and Sherris, 2013) and the mortality model based on the AFNS model (Christensen et al., 2011), with and without factor dependence structure. The mortality models are fit to cohort data to address the cohort effects, which is crucial for pricing and longevity risk management. The in-sample performance and the predictive ability of the mortality models are compared.

Affine mortality models are able to provide consistent survival rates with a structure similar to interest rate models, which implies that these models are suitable for financial application and longevity risk management. Besides, the factors in the AFNS model can be identified and interpreted as level, slope and curvature factors, which allows us to interpret the factors and factor loadings to better understand the evolution of mortality.

By comparing the independent Blackburn-Sherris model and the independent AFNS mortality model, although the single free parameter in the drift term ($\delta$) in of the drift term of the AFNS model results in a poorer model goodness-of-fit than the Blackburn-Sherris model, the specific structure of factor loadings and $K^Q$ in the AFNS model enables the model to better capture the variation in mortality rates and produce smaller residuals at older ages. In terms of forecasting, the independent AFNS model has outperformed the both Blackburn-Sherris models. Thus, the identifiable factors in the AFNS not only allow for factor interpretation, but also improve the model fit at older ages and predictive performance. In the Blackburn-Sherris model family, the factor dependence has improved the in-sample model performance and out-of-sample forecasting. However, between the two AFNS models, the independent model is preferred, given its ability to capture mortality variability at older ages. This implies that different dependence structures does not necessarily result in improved model performance and the particular dependence structure should be carefully selected, considering the factor interpretation.

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