Equilibrium recoveries in insurance markets with limited liability

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Abstract

This paper studies optimal insurance in partial equilibrium in case the insurer is protected by limited liability, and the multivariate insured risk is exchangeable. We focus on the optimal allocation of remaining assets in default. We show existence of an equilibrium in the market. In such an equilibrium, we get perfect pooling of the risk in the market, but a protection fund is needed to charge levies to policyholders with low realized losses. If policyholders cannot be forced *ex post* to pay a levy, we show that the constrained equal loss rule is used in equilibrium. This rule gained particular interest in the literature on bankruptcy problems. Moreover, in absence of a regulator, the insurer will always invest all its assets in the risky technology. We illustrate the welfare losses if other recovery rules are used in case of default; a different recovery rule can substantially effect the profit of the insurer.

Keywords: insurance, limited liability, partial equilibrium, recovery rules, incentive compatibility.

1 Introduction

This paper studies optimal recoveries in insurance, and their effects on prices in equilibrium. We use an agency model, where a non-life insurer is protected by limited liability. In case of a default, the remaining assets of the insurer are (at least partially) allocated to the policyholders. In practice, proportional methods are very popular (Araujo and Páscoa, 2002; Sherris, 2006; Ibragimov et al., 2010; Laux and Muermann, 2010). When the multivariate insured risk is exchangeable, which includes the case of *i.i.d.* risk and a common shock model, we show that using a proportional method to allocate the recoveries may yield welfare losses in the economy. Moreover, we characterize the optimal method instead. In the literature on bankruptcy problems, this optimal method is called a constrained equal loss (CEL) rule (see, e.g., Moulin, 2002; Thomson, 2003). We study bankruptcy problems that arise endogenously in insurance contract design.

A bankruptcy problem describes the situation in which we have to allocate a given amount (often referred to as estate) among a group of claimants when the available amount is not sufficient to cover all their claims. A bankruptcy rule calculates shares for claimants such that 1) no agent gets more than its claim, and 2) all get a non-negative share. For an overview of bankruptcy problems in practice and bankruptcy rules, we refer to O'Neill (1982), Aumann and Maschler (1985), Moulin (2000), or the overviews of Moulin (2002), and Thomson (2003). In a natural way, any default situation with limited liability is related to a bankruptcy problem where the realized risks are claims and the realized asset value is the size of the estate. Then, any bankruptcy rule can be taken to define a solution to allocate the remaining assets to the policyholders.

Habis and Herings (2013) and Koster and Boonen (2014) study stochastic bankruptcy problems in risk sharing problems. Kıbrıs and Kıbrıs (2013) and Karagözoğlu (2014) study an investment problem, where bankruptcy rules are applied in case of default. In all these papers, default is however an exogenous event, that is not affected by the aggregate investment decisions. We apply the concept of stochastic bankruptcy rules to a partial equilibrium setting in insurance with limited liability, where default occurs endogenously.

Initially, Doherty and Schlesinger (1990), Cummins and Mahul (2003), and Bernard and Ludkovski (2012) study insurance contract design with limited liability by modeling default as an exogenous event that is correlated with the risk of a policyholder. We follow the approach of Filipović et al. (2015) to study optimal risk taking and premia of an insurer in equilibrium. They study this problem with one insurer and one policyholder. Moreover, Biffis and Millossovich (2011), Asimit et al. (2013) and Cai et al. (2014) study optimal reinsurance contracts with default risk if there is one policyholder. We differ by allowing for multiple policyholders. We assume that the multiple policyholders are *ex ante* identical, by e.g. an exchangeability condition on the multivariate insured risk. Popular examples of exchangeable risk are the case where it is independent and identically distributed (*i.i.d.*), and the case where it is formulated as a common shock model (see, e.g., Marshall and Olkin, 1967; Promislow, 2006).

Pooling risk of multiple policyholders reduces the probability of default, and should therefore be reflected in the insurance price. In case there are multiple policyholders, the issue to allocate the remaining assets in default exists naturally. A recovery rule is used to allocate the remaining assets in case of default. Such a recovery rule naturally affects the premiums for insurance that are paid *ex ante*, and determined by the insurer. Rees et al. (1999) study optimal insurance regulation with a given recovery rule. Moreover, Sherris (2006), Ibragimov et al. (2010), Laux and Muermann (2010) and Bauer and Zanjani (2016) all assume a proportional recovery rule. An exception is Araujo and Páscoa (2002), who focus on existence of general equilibria with a continuum of policyholders. There are frequent real life deviations from the proportional rule, and some are actually contemplated by law (Araujo and Páscoa, 2002).

This paper extends the approach of Mahul and Wright (2004) to the setting of equilibria in case the insurer is protected with limited liability. Mahul and Wright (2004) study optimal risk sharing among insurers via pools in the context of catastrophe insurance. Their perspective is to maximize a weighted utility of all insurers. Then, all insurance risk is pooled *ex post*, and then redistributed among the insurers. The premium is allowed to be decided *ex post* as well. This problem is in line with classical Pareto optimal risk sharing as in Borch (1962), but with constraints. Mahul and Wright (2004) describe the constrained equal loss recovery rule and characterize it via an *ex post* participation constraint. Our focus is different as we study the effect of rules to allocate default losses in equilibrium, and their effects on insurance premia and the risk taking behavior of the insurer.

We show in this paper that the equilibrium exists. Moreover, we find that it is optimal for the insurer to force some policyholders to pay *ex post* levies to cover losses in default. This leads to an equilibrium with perfect pooling of the insurance risk. If the insurer cannot force policyholders to pay *ex post* a levy, we find that the constrained equal loss recovery rule is the optimal recovery rule in equilibrium. Our results also hold in absence of a regulator (monitoring device). Without a regulator, the insurer will invest in such a way that it maximizes its own expected profit - not taking into account the utility of the policyholder. Then, in absence of leverage, we show that the insurer will always invest all its assets in the risky technology. We illustrate in an example that welfare losses may be substantial if other recovery rules are used. Moreover, bankruptcy costs do not affect optimality of the CEL recovery rule, but it may lead to a substantially different insurance premium and risk taking behavior of the insurer. In particular, we show that even providing insurance may not be optimal, which would lead to a break-down of the market in equilibrium.

This paper is set out as follows. Section 2 defines the model set-up. Section 3 characterizes the optimal pooling and recovery rules. Section 4 shows existence of the equilibrium. Section 5 studies incentive compatibility. Section 6 shows in an illustration the welfare losses of suboptimal recovery rules and the effect of the number of policyholders on insurance contracts in equilibrium. Section 7 illustrates the effects of positive dead-weight costs in default. Finally, Section 8 concludes. All proofs are delegated to Appendix A.

2 Preferences

2.1 Preferences insurer

We consider a one-period economy with a given future reference period. The insurer has initial wealth $W \geq 0$. Let $N = \{1, \dots, n\}$ be the finite set of policyholders. There is one class of insurance policies, so that the insurer charges the same premium to everyone. Every policyholder $i \in N$ seeks insurance for a given risk $X_i \in L^1_+$ by paying a single premium $\pi \geq 0$ to the insurer, where L^1_+ is the set of non-negative random variables on a given probability space for which the expectation exists. T Denote the set of risks as $X := (X_i)_{i=1}^n$. The risk-free rate is given by $r \geq 0$. The insurer can invest a fraction $\alpha \in [0, 1]$ of its wealth in a risky technology that generates a stochastic excess return R, for which the support is a subset of $[-(1+r), \infty)$.

Before covering the insurance claims, the assets of the insurer at the given future time are given by

$$A(\alpha, \pi) := (W + n\pi)(1 + r + \alpha R),$$

which is stochastic.[1] The insurer remains solvent if the assets are higher than the realized insurance claims, i.e., when the following event occurs:

$$S(\alpha, \pi) := \left\{ A(\alpha, \pi) \geq \sum_{i=1}^{n} X_i \right\}.$$

There are no costs of bankruptcy included for the insurer, but the policyholders are cut in their indemnities to cover the deficits. The objective of the insurer is to maximize

$$U_I(\alpha, \pi) := E\left[(A(\alpha, \pi) - \sum_{i=1}^{n} X_i)^+ \right],$$

under participation constraints of the policyholders which we will specify in Subsection 2.2, where we define $(y)^+ = \max\{y, 0\}$. Hence, we assume that the insurer is risk neutral, and protected by limited liability.

[1] For a return $\hat{R} := R + 1 + r$, this can be written as $A(\alpha, \pi) = (W + n\pi)\left((1-\alpha)(1+r) + \alpha\hat{R}\right)$.

2.2 Preferences policyholders

In this paper, we study the effects of limited liability. In case of default, the remaining assets are allocated to the policyholders. The way this should be done is non-trivial, and the central topic of this paper. It is determined by a function $f : \mathbb{R}_+^{n+1} \to \mathbb{R}^n$ that maps realisations of (A, X) into n-dimensional vectors, where $\mathbb{R}_+$ is the class of non-negative real numbers.

Definition 2.1 *Let $\mathcal{F}$ the collection of the mappings $f : \mathbb{R}_+^{n+1} \to \mathbb{R}^n$ such that*

$$\sum_{i=1}^{n} f_i\left(a, (x_i)_{i=1}^n\right) = \begin{cases} (1-\delta)a, & \text{if } a < \sum_{i=1}^n x_i, \\ \sum_{i=1}^n x_i, & \text{otherwise,} \end{cases} \tag{1}$$

and $f_i\left(a, (x_i)_{i=1}^n\right) \leq x_i$ for all $i \in N$, Moreover, let $\mathcal{R} \subset \mathcal{F}$ the collection of mappings f that are also such that $f(a, (x_i)_{i=1}^n) \geq 0$ for all $(a, (x_i)_{i=1}^n) \in \mathbb{R}^{n+1}$.

The fraction $\delta \in [0, 1]$ reflects the costs of default for the policyholders that are deducted from the remaining assets (see Biffis and Millossovich, 2011).[2] With slight abuse of notation, we denote the mapping $\tilde{f} : (L^1)^{n+1} \to (L^1)^n$, given by $\tilde{f}(A, X)(\omega) = f(A(\omega), X(\omega))$ for all elements ω of the state space, by f as well. Then, $f(A, X)$ is an n-dimensional vector of stochastic variables that represent the payments from the insurer to the n policyholders. In other words, $f(A, X)$ is the vector of insurance indemnities, and we refer to f as a *rule*.

We assume that the rule f is common knowledge before the insurance contract is sold. Therefore, it might influence the insurance premium in equilibrium. We model the preferences of the policyholders by agents with expected utility function u and initial wealth w_0, i.e., the utility of policyholder i is given by

$$U_{PH}^i(f, \alpha, \pi) := E\left[u(w_0 - \pi - X_i + f_i(A(\alpha, \pi), X))\right].$$

[2] In this context of an interbank market, this fixed fraction δ as cost of default is also imposed by Rogers and Veraart (2013).

Ideally, individuals should be differentiated according to their particular utility functions. As argued by Young (1990), this is impossible in practice, and, even if it were possible, would be based on false premises because it requires making fine-tuned interpersonal utility comparisons. Instead, we consider u as a social norm: the utility function of a "representative agent".

Throughout this paper, we impose the following regularity assumptions.

Assumption 2.1: It holds that:

- $0 < E[R] < \infty$.
- the no-default event $S(\alpha, \pi)$ happens with positive probability for all $(\alpha, \pi) \in [0,1] \times \mathbb{R}_+$, and R is non-negatively correlated with the event $S(\alpha, \pi)$: $E[R|S(\alpha, \pi)] \geq E[R]$.
- the utility function $u : \mathbb{R} \to \mathbb{R}$ is such that $u'(\cdot) > 0$, $u''(\cdot) < 0$, and $\lim_{x \to -\infty} u(x) = -\infty$.
- the distribution of (R, X) is such that the utility of the insurer U_I is differentiable in some neighborhood of the domain $[0,1] \times \mathbb{R}_+$ of (α, π).

We assume that the risky technology has a higher expected return than the risk-free rate. Furthermore, we assume that the investment return is non-negatively correlated with the no-default event, so that a high investment return R is not correlated with low insurance risk realisations $\sum_{i=1}^n X_i$, and vice versa. For instance, under Solvency II, investment returns (the additive inverse of market risk) and insurance risk are assumed to have a negative linear correlation coefficient, that is given by -0.25.

The policyholders' individual rationality constraints are given by

$$U_{PH}^i(f, \alpha, \pi) \geq \underline{u}_i, \tag{2}$$

for all $i \in N$, where $\underline{u}_i \leq U_{PH}^i(f^*, \alpha^*, \pi^*)$ for all i for some (f^*, α^*, π^*). For instance, we may set $\underline{u}_i$ at the utility level in the status quo, i.e., $\underline{u}_i = E\left[u(w_0 - X_i)\right]$.

The effect of the rule f is key in the participation constraint (2). As the participation constraint (2) ensures individual rationality, we maximize the expected profit of the insurer $U_I(\alpha, \pi)$ under this constraint. Possible sharing of welfare gains is possible by choosing the values of $\underline{u}_i$ wisely. Note that the utility of the policyholders in (2) is not necessarily decreasing in the premium π.

3 Optimal pooling and recovery rules

3.1 Problem statement

Every policyholder $i \in N$ is seeking to insure its risk given by $X_i \in L^1_+$. We assume that $\{X_1, \ldots, X_n\}|R$ are exchangeable, i.e., the distribution of $\{X_1, \ldots, X_n\}|R$ is invariant under every permutation of the index set $\{1, \ldots, n\}$. In other words, for every natural number $k \leq n$, the joint distribution of any selection of k random variables from $\{X_1, \ldots, X_n\}|R$ is the same (see, e.g., Denuit and Vermandele, 1998; Albrecht and Huggenberger, 2017). This implies that X_i has the same distribution function as X_j for $i, j \in N$. Exchangeability is a generalization of the case where X is *i.i.d.*, and the case where X is generated by a common shock model.

Due to the exchangeability (*ex ante* symmetry) of the risks $X_1, \ldots, X_n$, we let the reservation utilities be equal for all policyholders, i.e., $\underline{u}_i = \underline{u}$ for all $i \in N$. A tuple (f, α, π) is called a partial equilibrium if it yields the highest expected profit for the insurer, provided that the policyholders' individual rationality constraints are satisfied. More precisely, the set of partial equilibria is given by the solutions of the following optimization problem:

$$\begin{array}{ll} \max_{f,\alpha,\pi} & U_I(\alpha, \pi), \\ \text{s. t.} & U^i_{PH}(f, \alpha, \pi) \geq \underline{u}, \text{ for all } i \in N, \\ & (f, \alpha, \pi) \in \hat{F} \times [0,1] \times \mathbb{R}_+, \end{array} \tag{3}$$

where $\hat{F} = \mathcal{F}$ or $\hat{F} = \mathcal{R}$, and $\underline{u} \leq U^i_{PH}(f^*, \alpha^*, \pi^*)$ for all i and some $(f^*, \alpha^*, \pi^*) \in \hat{F} \times [0,1] \times \mathbb{R}_+$. In this section, we assume that the problem in (3) has a solution. In

Section 4, we will show existence of this solution formally in Theorem 4.2.

In the following lemma, we show the qualitative behavior of the preferences of the insurer.

Lemma 3.1 *Let Assumption 2.1 hold. For all* $(\alpha, \pi) \in (0,1) \times \mathbb{R}_{++}$*, we have*

$$\frac{\partial}{\partial \alpha} U_I(\alpha, \pi) > 0,$$

and for all $(\alpha, \pi) \in [0,1] \times \mathbb{R}_{++}$*, we have*

$$\frac{\partial}{\partial \pi} U_I(\alpha, \pi) > 0,$$

where $\mathbb{R}_{++}$ *is the class of strictly positive real numbers.*

From Lemma 3.1, we get that for a fixed $\alpha \in [0,1]$ the utility of the insurer is strictly increasing in premium π. For a given premium π, we get from $E[R] > 0$ and the risk-loving preferences of the insurer that the utility of the insurer is strictly increasing in the exposure α.

3.2 Optimal pooling

In this subsection, we consider the case that $\hat{F} = \mathcal{F}$, i.e., the case where we allow that $f_i(A(\alpha, \pi), X) < 0$. Then, an insurer in default can force policyholders with small realized losses to sponsor the policyholders with large losses. This mechanism is for instance enforced by a protection fund, that charges levies in case of default. Charging *ex post* levies is common practice in banking such as for deposit insurance (Schich and Kim, 2011). Moreover, insurance guarantee funds exist, but the market is still limited (European Commission, 2010).

In an optimal insurance contract, the total claims at default are pooled and, then, the losses are pro rata shared among the policyholders. In other words, the assets are allocated such that the risk $X_i - f_i(A(\alpha, \pi), X)$ is the same for every policyholder i. We call this solution perfect pooling (PP), and we show this result in the following theorem.

Theorem 3.2 *Let Assumption 2.1 hold, and let* (f, α, π) *be a solution of* (3) *with* $\hat{F} = \mathcal{F}$*. Then* $f = PP$*, where*

$$PP_i(A(\alpha,\pi), X) = \begin{cases} X_i + ((1-\delta)A(\alpha,\pi) - \sum_{j=1}^{n} X_j)/n & \text{if } A(\alpha,\pi) < \sum_{j=1}^{n} X_j, \\ X_i & \text{otherwise,} \end{cases} \tag{4}$$

for all $i \in N$*.*

The solution PP in (4) can be seen as perfect risk pooling as all insurance risk is pooled and then shared equally among all policyholders. Because all policyholders are equal *ex ante*, but not *ex post*, the solution PP resembles the concept of Harsanyi's "veil of ignorance" (Harsanyi, 1953). Because *ex ante* policyholders do not know whether the realizations of their risks are good or bad, they strive for egalitarianism *ex post*.

To enforce this egalitarian mechanism, some policyholders might need to pay after the risk occurs (*ex post*). This happens when $f_i(A, X) < 0$. This may be difficult to enforce, as it requires policyholders to pay a compensation on top of their risk after the risks are realized. This is why we focus in the next subsection on the case where we impose the condition $f(A, X) \geq 0$. So, such a rule is *ex ante* not necessarily optimal. However, we do not need to enforce cross-payments among policyholders at the future time period.

3.3 Optimal recovery rules

There is substantial literature on bankruptcy problems, which are also called rationing problems. In a standard bankruptcy problem, there is one deterministic estate $E > 0$ and a deterministic claim vector $d \in \mathbb{R}_+^n$ such that $\sum_{i=1}^{n} d_i > E$. A bankruptcy rule $\varphi : \mathbb{R}_{++} \times \mathbb{R}_+^n \to \mathbb{R}_+^n$ is such that $0 \leq \varphi(E, d) \leq d$ and $\sum_{i=1}^{n} \varphi_i(E, d) = E$ (see, e.g., O'Neill, 1982, or the overviews of Moulin, 2002, and Thomson, 2003).

In this paper, we apply the concept of bankruptcy rules to the case where the estate and claims are stochastic. For a realization of the assets A and the claims X such that $A < \sum_{i=1}^{n} X_i$ (default), we use the bankruptcy rule. Moreover, we extend the bankruptcy rules to allow also for the case where $A \geq \sum_{i=1}^{n} X_i$ (no default); then all claims are covered. We call such a rule a *recovery rule*, and it is given by a mapping in $\mathcal{R}$ (see Definition 2.1).

We focus on the following recovery rules that are inspired by well-known bankruptcy rules (Moulin, 2000, 2002; Thomson, 2003):

- Proportional rule: for each (A, X),

$$f_i\,(A, X) = PROP_i(A, X) = \begin{cases} (1-\delta)\frac{A}{\sum_{j=1}^{n} X_j} \cdot X_i & \text{if } \sum_{j=1}^{n} X_i > A, \\ X_i & \text{otherwise,} \end{cases}$$

  for all $i \in N$.

- Constraint Equal Award (CEA): for each (A, X), $f_i(A, X) = CEA_i(A, X) = \min\{X_i, \gamma\}$, where γ is such that $\sum_{j=1}^{n} \min\{X_j, \gamma\} = (1-\delta)A$ if $A < \sum_{j=1}^{n} X_j$, and $\gamma = \infty$ otherwise.

- Constraint Equal Loss (CEL): for each (A, X), $f_i(A, X) = CEL_i(A, X) = \max\{0, X_i - \gamma\}$, where γ is such that $\sum_{j=1}^{n} \max\{0, X_j - \gamma\} = (1-\delta)A$ if $A < \sum_{j=1}^{n} X_j$, and $\gamma = 0$ otherwise.

- Talmud rule:

$$\begin{aligned} f(A, X) &= TR(A, X) \\ &= \begin{cases} X & \text{if } \sum_{i=1}^{n} X_i \leq A, \\ CEA(A, \frac{1}{2}X) & \text{if } \sum_{i=1}^{n} X_i > \max\{2(1-\delta)A, A\}, \\ X - CEA(\frac{\sum_{i=1}^{n} X_i}{1-\delta} - A, \frac{1}{2}X), & \text{otherwise,} \end{cases} \end{aligned}$$

  for each (A, X).

The intuition of the first three recovery rules is straightforward. Proportional recovery rules seem the most natural way to allocate assets in default, and is popular in the insurance literature (Sherris, 2006; Ibragimov et al., 2010; Laux and Muermann, 2010). It is easy to communicate to the policyholders. The constrained equal award rule strives to obtain egalitarianism in bankruptcy problems (see, e.g., Koster and Boonen, 2014). The constrained equal loss rule strives to obtain egalitarianism for the dual problem, i.e., for the risks $X_i - f_i(A, X), i \in N$. The Talmud rule is more advanced, and is characterized for bankruptcy problems by a consistency axiom (Aumann and Maschler, 1985).

For all these four recovery rules above, it holds that

$$\begin{aligned} f_i(A(\alpha,\pi), X)|R &\stackrel{d}{=} f_j(A(\alpha,\pi), X)|R, \\ f_i(A(\alpha,\pi), X) &\stackrel{d}{=} f_j(A(\alpha,\pi), X), \\ X_i - f_i(A(\alpha,\pi), X) &\stackrel{d}{=} X_j - f_j(A(\alpha,\pi), X) \end{aligned}$$

for all $i, j \in N$. So, there is an *ex ante* equal treatment of the policyholders. Note that some recovery rules might yield the same posterior joint risk $f(A, X)$. For instance, if $X_i = Z$ for all $i \in N$, we have that all recovery rules defined above yield the same solution, which is $f_i(A, X) = \frac{(1-\delta)A}{n}$ when $A < nZ$, and $f_i(A, X) = Z$ otherwise, for all $i \in N$.

Theorem 3.3 *Let Assumption 2.1 hold, $X|R$ is exchangeable, and let (f, α, π) be a solution of* (3) *with $\hat{F} = \mathcal{R}$. Then*

$$f(A(\alpha,\pi), X) = CEL(A(\alpha,\pi), X).$$

Note that if $f = CEL$ and $\delta = 0$, the participation constraint in (2) writes as

$$E\left[u(w_0 - \pi - \min\{X_i, \gamma\})\right] \geq \underline{u}, \tag{5}$$

where γ is a random variable such that

$$\sum_{i=1}^{n} \min\{X_i, \gamma\} = \left(\sum_{i=1}^{n} X_i - (W + n\pi)(1 + r + \alpha R)\right)^{+}.$$

Hence, if $\delta = 0$, the Constrained Equal Loss recovery rule resembles deductible insurance, but where the deductible is random as well. The CEL rule strives to *ex post* egalitarianism under non-negativity constraints.

Remark The findings in this section are based on optimizing the expected profit of the insurer under participation constraints of the policyholders. Optimality of $f = PP$ when $\hat{F} = \mathcal{F}$ and optimality of $f = CEL$ when $\hat{F} = \mathcal{R}$ also holds when a social planner maximizes a weighted sum all utilities in the market. We formalize this setting and findings in Appendix B.

4 Existence of the equilibrium

In this section, we show existence of the equilibrium. For the case where $\delta = 0$, we illustrate the qualitative behavior of utility functions of the insurer and policyholder. We show convexity of the utility of the insurer, and concavity of the utility of the policyholder, both with respect to α and π. This result holds straightforward in case $n = 1$ (see Filipović et al., 2015), but concavity of the utility function of the policyholder is more complicated to show in case $n > 1$. We assert this result in the following lemma.

Lemma 4.1 *Let Assumption 2.1 hold, $\delta = 0$, $X|R$ is exchangeable, and $f \in \{PP, CEL\}$. Then $U_I(f, \alpha, \pi)$ is convex in α and π, and $U_{PH}^i(f, \alpha, \pi)$ is concave in α and strictly concave in π, for all $i \in N$.*

Assumption 4.1: The distribution of (R, X) is such that the utility of the policyholder is real-valued and differentiable in some neighborhood of the domain $[0, 1] \times \mathbb{R}_+$ of (α, π) whenever $f \in \{PP, CEL\}$.

Theorem 4.2 *Let Assumptions 2.1 and 4.1 hold, $X|R$ is exchangeable, and $\hat{F} = \mathcal{F}$ or $\hat{F} = \mathcal{R}$. For any $\underline{u}$, there exists at least one (f^*, α^*, π^*) that solves (3). It is such that the participation constraints are binding. Moreover, for any given $\alpha \in [0, 1]$, there exists at most one (f, α, π) solving (3). If $\delta = 0$, then (f, α, π) is such that $\frac{\partial}{\partial \pi} U^i_{PH}(f, \alpha, \pi) \leq 0$.*

It is important to remark that if $\underline{u}$ is high, it is not rational for the insurer to offer the insurance contracts. Therefore, we need to verify *ex post* whether the equilibrium solving (3) is rational for the insurer; if rationality is violated there is no insurance in equilibrium.

Remark Background risk is an important topic in the literature on insurance contract design (see Dana and Scarsini, 2007). We would like to point out that adding a bounded background risk Y to the income of the insurer does not affect our results as long as Assumption 2.1 still holds.

5 Incentive compatibility

In this section, we study incentive compatibility in insurance. For instance, suppose that initial wealth W and the premium π are such that $(W + n\pi)(1 + r) > \sum_{i=1}^n X_i$ for any X, the policyholder would prefer the insurer to invest completely risk-free. The policyholder is also willing to pay a higher premium to achieve this. In absence of a regulator, there is however no guarantee that the insurer will invest everything risk-free. This is called counterparty risk, risk shifting (see, e.g., Filipović et al., 2015) or incentive compatibility in insurance.

Suppose the investment decision is not observed by the policyholder. After the policyholders pay the insurance premium, the insurer will invest its assets in order to maximize its own utility. We impose the incentive compatibility constraint: $\alpha \in \text{argmax}_{\alpha' \in [0,1]} U_I(\alpha', \pi)$. Then, the set of partial equilibria with incentive compatibility is given by the solutions of the following optimization problem:

$$
\begin{array}{ll}
\max_{f,\alpha,\pi} & U_I(\alpha,\pi), \\
\text{s. t.} & U_{PH}^i(f,\alpha,\pi) \geq \underline{u}, \text{ for all } i \in N, \\
& (f,\pi) \in \hat{F} \times \mathbb{R}_{++}, \\
& \alpha \in \text{argmax}_{\alpha' \in [0,1]} U_I(\alpha',\pi),
\end{array}
\tag{6}
$$

where $\hat{F} = \mathcal{F}$ or $\hat{F} = \mathcal{R}$, and where $\underline{u} = U_{PH}^i(f^*, 1, \pi)$ for some $\pi > 0$, and where $f^* = PP$ if $\hat{F} = \mathcal{F}$ and $f^* = CEL$ if $\hat{F} = \mathcal{R}$. Note that we explicitly require $\pi > 0$ in (6), which we assume to prove the following result.

Theorem 5.1 *Let Assumptions 2.1 and 4.1 hold, $X|R$ is exchangeable, and $\hat{F} \in \{\mathcal{F}, \mathcal{R}\}$. Then, there exists a unique solution (f,α,π) to (6). This is such that $(f,\alpha,\pi) = (f^*, 1, \pi)$, where $f^* = PP$ if $\hat{F} = \mathcal{F}$ and $f^* = CEL$ if $\hat{F} = \mathcal{R}$.*

Theorem 5.1 states that if the insurer decides to maximize profit after it received the premiums, then it will invest all assets in the risky technology. This is not in the interest of the policyholders. Since the participation constraints for the policyholders are binding, the utilities of the policyholders remain the same as in Sections 2-4. Hence, regulation could be welfare-improving for insurer. In absence of regulation, the insurer invests risky. In line with, e.g., Caillaud et al. (2000), a regulated market makes the risk-neutral insurer more risk-averse.

6 Numerical example

In this section, we show the effect of recovery rules on equilibrium prices, and risk taking behavior of the insurer. We provide an extensive example of an insurer whose financial position is relatively poor. In this case, we show that the effect of the type of recovery rules is important.

Let $r = 0\%$, $\delta = 0$, $X_1, \ldots, X_n \overset{i.i.d.}{\sim} \exp(1)$, and $R = e^G - 1, G \sim N(\mu, \sigma^2)$, with $\mu = 0\%$ and $\sigma = 16\%$, and independent of X_i. Moreover, policyholders use the exponential

(Constant Absolute Risk Aversion) utility function $u(x) = -\exp(-\lambda x)$ with $\lambda = 0.2$. It is well-known that initial wealth w_0 is irrelevant for exponential utilities. Clearly, the assumptions $X_i \in L^1_+, E[R] > 0, u'(\cdot) > 0, u''(\cdot) < 0$, and $\lim_{x\to-\infty} u(x) = -\infty$ are satisfied. The initial assets of the insurer are set at $W = 0$, and moreover we have $n = 10$. We set $\underline{u} = E[u(w_0 - X_i)]$. In absence of default, we get from straightforward calculations that the indifference price for insurance is approximately 1.108, i.e., the risk premium is given by 10.8%. We simulate the risks in the economy 100,000 times for every case.

In the baseline model, we let $f = CEL$, but we first study different recovery rules as well. For instance, and in line with Araujo and Páscoa (2002) and Ibragimov et al. (2010), bankruptcy losses may be *ex post* pro rata shared among policyholders. In this example, it turns out to be the case that the equilibrium is unique. We show the outcome on prices and risk taking in Table 1. We find that the effects of the choice of the recovery rule are substantial. For instance, when the insurer uses CEA instead of CEL, then the insurance premium will drop from 0.96 to 0.88. As a result, the probability of default increases and the expected profit for the insurer is smaller. For recovery rules, the results in Table 1 confirm Theorem 3.3 in that CEL is optimal to use for the insurer. It leads to a higher premium, and the utility for the insurer is highest. If it is possible to have perfect pooling as in Section 3.2, we find that there exist additional expected profits for the insurer, but the difference is rather small. For the optimal recovery rule CEL, the insurer will invest less in the risky asset, which leads to the highest solvency probability. Because insurer receives a higher premium π if it uses CEL than for any other recovery rule, it does not need to invest very risky to guarantee solvency. On the other hand, if the premium is much lower than the expected loss, the policyholders may want the insurer to invest more risky in order to benefit from the risk premium.

Next, we show the effect of the number of policyholders, which is given by n. We display these effects in Table 2. If $n = 1$, then it is likely that default occurs when the insured risk X_1 is high. In order to prevent this, the policyholder is willing to pay a higher premium. This is beneficial for the profit of the insurer. If $n = 10$, we get

f	CEL	CEA	$PROP$	TR	PP
π	0.96	0.88	0.94	0.95	0.96
α	86%	94%	100%	90%	93%
$\mathbb{P}(S(\alpha, \pi))$	49.8%	40.0%	47.5%	48.7%	49.8%
$U_I(\alpha, \pi)$	1.21	0.83	1.15	1.17	1.24

Table 1: Overview of numerical result corresponding to Section 6. This table displays the effect of the recovery rule f. The definition of PP is provided in (4), and the other alternatives of f are shown in Subsection 3.3. This table shows the equilibrium solution (α, π) of (3) with given f, and the no-default probability and the utility of the insurer in this equilibrium.

that the premium in equilibrium is rather low. The default event is less correlated with X_i, but the risk aversion of the policyholders is such that they are not willing to pay more than 1, which is the expected loss in absence of limited liability. The profit per contract for the insurer is therefore low. When n gets larger, the total insurance losses get approximately normally distributed due to the central limit theorem. Then, default particularly occurs when investment returns are low, which is assumed to be independent of the insured risk. Since the insurer is risk-loving due to limited liability, it is not true that diversification of risk is good for the insurer. However, more policyholders lead to more aggregately received premia that can be invested in the risky technology.

n	1	100	$1,000$	$10,000$
π	1.11	1.08	1.09	1.09
α	100%	89%	95%	96%
$\mathbb{P}(S(\alpha, \pi))$	66.9%	68.2%	71.8%	71.6%
$U_I(\alpha, \pi)$	0.45	12.8	129.4	1,292
$\frac{1}{n}U_I(\alpha, \pi)$	0.45	0.13	0.13	0.13

Table 2: Overview of numerical results corresponding to Section 6, where we vary the number of policyholders n. This table shows the equilibrium (α, π) as defined in (3) with given $f = CEL$, and the no-default probability and the utility of the insurer in this equilibrium.

Finally, we conclude this section with analyzing the effect of a common shock. Let X be given by

$$X_i = Y_i + Z, i \in N, \tag{7}$$

where $X \in (L^1_+)^n$. Here, $Y_i, i \in N$, are independent and identically distributed (*i.i.d.*), and independent of Z. This is a common shock model, where the risk Z is a common shock that effects all insurance claims (see, e.g., Marshall and Olkin, 1967; Promislow, 2006). Let $n = 10$, and the common shock be given by $Z = \gamma e^{\bar{G}}$, with $\bar{G} \sim N(\mu_\gamma, 1)$ and $\gamma \in [0, 1]$. Moreover, we assume $X_i = Z + (1 - \gamma)Y_i$, $i = 1, \ldots, 10$, with $Y_1, \ldots, Y_{10} \overset{i.i.d.}{\sim} \exp(1)$. For every γ, we let μ_γ be such that expectation of X_i is the same. The marginal distribution of R is the same as above, but $(\bar{G}, G)$ are bivariate normally distributed where the correlation coefficient is assumed to be -0.25.[3] We adjust the reservation utility $\underline{u}$ to be the utility of the policyholder in case it does not insure its risk.

γ	0	0.25	0.5	0.75	1
π	0.96	0.80	0.71	0.85	1.87
α	86%	99%	95%	94%	100%
$\mathbb{P}(S(\alpha, \pi))$	49.8%	34.9%	36.4%	59.2%	85.3%
$U_I(\alpha, \pi)$	1.21	0.67	0.69	2.12	11.07

Table 3: Overview of numerical results corresponding to Section 6 with the common shock, where we vary the parameter γ. This table shows the equilibrium (α, π) as defined in (3) with given $f = CEL$, and the no-default probability and the utility of the insurer in this equilibrium.

From Table 3, we get that the common shock has substantial impact on the profits in equilibrium. We get that the equilibrium premiums are U-shaped in the severity of the common shock. In case of the common shock is high, the losses in default may be substantial. These losses are borne by the policyholder. As a result, the policyholder is willing to pay a higher premium in equilibrium if the common shock is severe. This high premium prevents that the insurer is likely to become bankrupt. In particular, it prevents bankruptcy in cases where the policyholder's risk is high as well. On the other hand, due to the risk-loving preferences of the insurer, the insurer benefits from the systematic risk as the common shock increases the aggregate risk in the economy. When the common shock is smaller ($\gamma = 0.25$ or 0.5), the policyholder is not willing to pay a high premium anymore. The reason is that the risk of default is too high to justify the

[3] This yields an approximate linear correlation of 0.25 between $-R$ and X_i. Under Solvency II, the market and insurance risk are assumed to have a linear correlation coefficient of 0.25.

premium, where the default event is less strongly correlated with the policyholder's risk.

7 Effects bankruptcy costs

In this section, we discuss the effect of deadweight bankruptcy costs δ. Recall from (1) how δ affects the insurance indemnities. Of course, when $\delta = 1$, the choice for recovery rules is irrelevant.

We study the effect of δ numerically in cases where the equilibrium exists. We assume that the $F = \mathcal{R}$, i.e., the recoveries need to be non-negative. The recovery rule in equilibrium is due to Theorem 3.3 given by $f = CEL$. We use the same setting as in Section 6, but vary δ. We find that the equilibrium exists and is unique. For δ larger than approximately 30%, we obtain that the equilibrium is such that $\pi = 0$, i.e., there is no trade. If the deadweight welfare losses are small, we find that there is an insurance trade. We display the equilibrium contracts in Table 4. We find that the insurer will ask a relatively low premium if δ is large, so that it is unlikely to be solvent. As a result, it will gamble by investing all it assets in the risky technology. This effect diminishes when δ gets closer to 0.

δ	5%	10%	15%	20%	25%	30%
π	0.90	0.81	0.64	0.43	0.30	0
α	88%	100%	100%	100%	100%	-
$\mathbb{P}(S(\alpha, \pi))$	42.7%	32.7%	13.7%	1.6%	0.3%	0
$U_I(\alpha, \pi)$	0.95	0.62	0.17	$1.2 \cdot 10^{-3}$	$6.4 \cdot 10^{-4}$	0

Table 4: Overview of the equilibrium (α, π) as defined in (3) corresponding to Section 7, where the initial wealth is given by $W = 0$ and where we vary the value of bankruptcy cost δ.

The results in Table 4 partially follow from the fact that we set $W = 0$, i.e., the insurer has no initial wealth. Next, we assume that $W = 5$. We show the results in Table 5. Note that we should compare the utility of the insurer with the utility in case the insurer does not provide insurance, and only invests its initial assets. We find that the reservation utility of the insurer in this case is given by approximately 5.06. Hence, if δ is 90% or 100%, the insurer will not offer insurance to the policyholders. Moreover, we find that if δ gets larger, the insurer will invest less in the risky technology. This follows from the fact that bankruptcy gets more harmful for the policyholders. As a result, the insurer has to charge a lower insurance premium, which leads to a lower profit.

δ	0%	10%	25%	50%	75%	90%	100%
π	1.10	1.09	1.07	1.04	1.00	0.96	0.90
α	100%	100%	100%	64%	39%	40%	32%
$\mathbb{P}(S(\alpha,\pi))$	93.8%	93.1%	92.4%	93.1%	92.9%	91.7%	89.5%
$U_I(\alpha,\pi)$	6.39	6.29	6.04	5.71	5.19	4.88	4.36

Table 5: Overview of the equilibrium (α, π) as defined in (3) corresponding to Section 7, where the initial wealth is given by $W = 5$ and where we vary the value of δ.

8 Conclusion

This paper studies the effect of recovery rules on insurance policies in equilibrium. We study the well-known case where the risks of the policyholders are exchangeable. We find that the *constrained equal loss* rule is optimal. This rule is popular in the literature on bankruptcy problems (e.g., Moulin, 2002; Thomson, 2003), but it is not commonly studied in the literature about limited liability in insurance. In the literature, proportionality is typically assumed exogenously.

If a protection fund can charge levies to policyholders with low losses, it is optimal to perfectly pool the risk. This yields the highest utility in the market. We show existence of an equilibrium in the market, and study the welfare losses if other recovery rules are used in default. Moreover, we show that in absence of a regulator, the insurer will always invest all its assets in the risky technology. Therefore, the insurance price should include this risk-taking as it affects likelihood and magnitude of a default event.

A very interesting extension of our proposed model would be to consider more general distributions of the insured risks of the policyholders. In this case, asymmetric information will be important to consider, which may lead to separating or pooling equilibria. As a result, the insurer needs to consider selection effects as well (see, e.g., Finkelstein and Poterba, 2004). This may lead to optimal recovery rules that are *ex ante* discriminating across policyholders. The characterization of such recovery rules is a question we leave open for further research.

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A Proofs

Proof of Lemma 3.1 Let $(\alpha, \pi) \in [0,1] \times \mathbb{R}_{++}$. We get

$$\begin{aligned}
\frac{\partial}{\partial \pi} U_I(\alpha, \pi) &= \frac{\partial}{\partial \pi} E\left[((W + n\pi)(1 + r + \alpha R) - \sum_{i=1}^{n} X_i)^+\right] \\
&= E[n(1 + r + \alpha R)1_{S(\alpha,\pi)}] \\
&= n(1 + r + \alpha E[R|S(\alpha, \pi)])\mathbb{P}(S(\alpha, \pi)) > 0,
\end{aligned}$$

where the inequality follows the assumption that $\mathbb{P}(S(\alpha, \pi)) > 0$, and $E[R|S(\alpha, \pi)] \geq E[R] > 0$.

Moreover, we get for any $(\alpha, \pi) \in (0,1) \times \mathbb{R}_{++}$ that:

$$\begin{aligned}
\frac{\partial}{\partial \alpha} U_I(\alpha, \pi) &= \frac{\partial}{\partial \alpha} E\left[((W + n\pi)(1 + r + \alpha R) - \sum_{i=1}^{n} X_i)^+\right] \\
&= E[(W + n\pi)R1_{S(\alpha,\pi)}] \\
&= (W + n\pi)E[R1_{S(\alpha,\pi)}] \\
&= (W + n\pi)E[R|S(\alpha, \pi)]\mathbb{P}(S(\alpha, \pi)) > 0,
\end{aligned}$$

which is again due to the fact that $\mathbb{P}(S(\alpha, \pi)) > 0$, and $E[R|S(\alpha, \pi)] \geq E[R] > 0$. This concludes the proof.

Proof of Theorem 3.2 It holds by construction that $\sum_{i=1}^{n} PP_i(A(\alpha, \pi), X) = (1 - \delta)A(\alpha, \pi)$ if $A(\alpha, \pi) < \sum_{i=1}^{n} X_i$, and $PP_i(A(\alpha, \pi), X) = X_i$ otherwise. Moreover, it holds that $PP_i(A(\alpha, \pi), X) \leq X_i$ for all i, and so we have $PP \in \mathcal{F}$. Fix (α, π). Let $f \in \mathcal{F}$, and define

$$\hat{W} := \begin{cases} w_0 - \pi - (\sum_{j=1}^{n} X_j - (1 - \delta)A(\alpha, \pi))/n & \text{if } \sum_{j=1}^{n} X_j > A(\alpha, \pi), \\ w - \pi & \text{otherwise.} \end{cases}$$

Clearly, $\hat{W}$ does not depend on the policyholder i. We take a Taylor expansion of u

around $\hat{W}$ to the second order:

$$
\begin{aligned}
u(w_0 - \pi - X_i + f_i(A(\alpha,\pi),X)) = u(\hat{W}) + u'(\hat{W})(w_0 - \pi - X_i + f_i(A(\alpha,\pi),X) - \hat{W}) \\
+ \frac{1}{2}u''(\zeta_i)(w_0 - \pi - X_i + f_i(A(\alpha,\pi),X) - \hat{W})^2,
\end{aligned}
$$

where ζ_i is in between $w_0 - \pi - X_i + f_i(A(\alpha,\pi),X)$ and $\hat{W}$. Clearly, it holds that $\sum_{i=1}^n (w_0 - \pi - X_i + f_i(A(\alpha,\pi),X) - \hat{W}) = 0$, and so the second term vanishes when we sum it over i. Therefore, we get by summing over all policyholders i and taking the expectation that

$$
\begin{aligned}
&\sum_{i=1}^n E[u(w_0 - \pi - X_i + f_i(A(\alpha,\pi),X))] \\
&= nE[u(\hat{W})] + \frac{1}{2}\sum_{i=1}^n E[u''(\zeta_i)(w_0 - \pi - X_i + f_i(A(\alpha,\pi),X) - \hat{W})^2] \\
&\leq nE[u(\hat{W})],
\end{aligned}
$$

which is due to $u''(\cdot) < 0$. If $f_i \neq PP_i$ for some $i \in N$, we get a strict inequality. Hence, PP, which is defined in (4), uniquely solves the following the system

$$
\begin{array}{ll}
\max_f & \sum_{i=1}^n U_{PH}^i(f,\alpha,\pi), \\
\text{s. t.} & f \in \mathcal{F}.
\end{array} \tag{8}
$$

Suppose that $f^*(A(\alpha,\pi),X)$ is an optimal rule such that $f^* \neq PP$. Since PP solves (8) uniquely, we get that there exists a policyholder $i \in N$ such that

$$U_{PH}^i(PP,\alpha,\pi) > U_{PH}^i(f^*,\alpha,\pi).$$

Then, we have for this policyholder i that

$$U_{PH}^i(PP,\alpha,\pi) > U_{PH}^i(f^*,\alpha,\pi) \geq \underline{u}.$$

By construction, we have that the utility level $U_{PH}^i(PP,\alpha,\pi)$ is the same for every policyholder i. So, if $f^*(A(\alpha,\pi),X)$ is optimal, then the participation constraint in (3) is slack. Since PP is continuous in the first argument, the utility of the policyholder is continuous in π. So, there exists a premium $\hat{\pi} > \pi$ such that the participation constraint in (3) is still satisfied. Since the utility of the insurer is strictly increasing in the price π, we get a higher utility for the insurer. This is a contradiction with the assumption that f^* is optimal. Hence, $f^* = PP$ is the unique rule for all solutions (f^*,α^*,π^*) to the problem (3). This concludes the proof.

Proof of Theorem 3.3 Fix (π,α), and moreover fix a realization $R = \hat{r}$ and $X = (x_i)_{i=1}^n$. Then, if $(W+n\pi)(1+r+\alpha\hat{r}) \geq \sum_{i=1}^n x_i$, then the recovery rule $f \in \mathcal{R}$ is fixed. So, let $\hat{A} := (W+n\pi)(1+r+\alpha\hat{r}) < \sum_{i=1}^n x_i$. Define the following problem:

$$\begin{array}{lll} \max_{b_1,\ldots,b_n} & \sum_{i=1}^n u(w_0-\pi-x_i+b_i), & \\ \text{s. t.} & b_i \geq 0, & \\ & \sum_{i=1}^n b_i = (1-\delta)\hat{A}. & \end{array} \tag{9}$$

The objective function in (9) is concave and the constraints are affine. Hence, we get all solutions from the Karush-Kuhn-Tucker (KKT) conditions:

$$\begin{array}{ll} u'(w_0-\pi-x_i+b_i)+\gamma_i & = u'(w_0-\pi-x_1+b_1)+\gamma_1, \text{ for all } i \in N, \\ \sum_{i=1}^n b_i & = (1-\delta)\hat{A}, \end{array} \tag{10}$$

where $\gamma_i b_i = 0$ and $\gamma_i \geq 0$. If policyholder i is such that $b_i > 0$, then $\gamma_i = 0$. So, due to $u''(\cdot) < 0$, we have that all $x_i + b_i$ is the same for all policyholders i such that $b_i > 0$. If $b_i = 0$, then $\gamma_i \geq 0$ and, so, $u'(w_0-\pi-x_i) \leq u'(w_0-\pi-x_1+b_1)+\gamma_1$. So, due to $u''(\cdot) < 0$, we get that if $b_i = 0$, the utility of policyholder i is higher than the utility of policyholder j with $b_j > 0$: $x_i \leq x_j - b_j$. Moreover, $(1-\delta)\hat{A} < \sum_{i=1}^n x_i$ and (10) guarantee that $-x_i + b_i \leq 0$. Therefore, we directly get that $b = CEL(\hat{A},(x_i)_{i=1}^n)$ is the unique solution of (9). Hence, when we solve (9) for any realization of (R,X), we get that $f(A(\alpha,\pi),X) = CEL(A(\alpha,\pi),X)$ solves uniquely the problem

$$\begin{array}{ll} \max_f & \sum_{i=1}^n U_{PH}^i(f,\alpha,\pi), \\ \text{s. t.} & f \in \mathcal{R}. \end{array} \tag{11}$$

Suppose that $f^*(A(\alpha,\pi),X)$ is an optimal recovery rule. Since $f = CEL$ solves (9) uniquely, we get that there exists a policyholder i such that

$$U_{PH}^i(CEL,\alpha,\pi) > U_{PH}^i(f^*,\alpha,\pi).$$

Then, we have for this policyholder i that

$$U_{PH}^i(CEL,\alpha,\pi) > U_{PH}^i(f^*,\alpha,\pi) \geq \underline{u},$$

where CEL is a recovery rule as well. Since $X|R$ is exchangeable, we have $X_i - CEL_i(A(\alpha,\pi),X) \stackrel{d}{=} X_j - CEL_j(A(\alpha,\pi),X)$ for all $i,j \in N$. So, we have that the *ex ante* expected utility level $U_{PH}^i(CEL,\alpha,\pi)$ is the same for every policyholder i. So, we get that if $f^*(A(\alpha,\pi),X)$ is an optimal recovery rule, then the participation constraint in (3) is slack. Since the rule CEL is continuous in the first argument, the utility of the policyholder is continuous in π. So, there exists a premium $\hat{\pi} > \pi$ such that the participation constraint in (3) is still satisfied. Since the utility of the insurer is strictly increasing in the premium π, we get a higher utility for the insurer. This is a contradiction with the assumption that the recovery rule f^* is optimal. Hence, $f = CEL$ is the unique solution to (8). This concludes the proof.

Proof of Lemma 4.1 Let $f = CEL$. For fixed $R = \hat{r}$ and $X = (x_i)_{i=1}^n$, the function $((W+n\pi)(1+r+\alpha\hat{r}) - \sum_{i=1}^n x_i)^+$ is convex in π and in α. Taking expectation preserves these properties. Hence, the utility of the insurer is convex in α and π

Next, we show strict concavity of the utility of the policyholder with respect to premium π. Let $0 \leq \pi_1 < \pi_2$, $\alpha \in [0,1]$, and $\lambda \in (0,1)$. Then, we get

$$\begin{aligned}
&\sum_{i=1}^{n} [\lambda CEL_i(A(\alpha,\pi_1),X) + (1-\lambda)CEL_i(A(\alpha,\pi_2),X)] \\
&= \lambda \min\{A(\alpha,\pi_1), \sum_{i=1}^{n} X_i\} + (1-\lambda)\min\{A(\alpha,\pi_2), \sum_{i=1}^{n} X_i\} \\
&\leq \min\{A(\alpha, \lambda\pi_1 + (1-\lambda)\pi_2), \sum_{i=1}^{n} X_i\}, \qquad (12)
\end{aligned}$$

which holds due to $\lambda A(\alpha,\pi_1) + (1-\lambda)A(\alpha,\pi_2) = A(\alpha, \lambda\pi_1 + (1-\lambda)\pi_2)$. Moreover, we get

$$0 \leq \lambda CEL_i(A(\alpha,\pi_1),X) + (1-\lambda)CEL_i(A(\alpha,\pi_2),X) \leq X_i \text{ for all } i. \qquad (13)$$

Moreover, we get from Theorem 3.3 that for all $f \in \mathcal{R}$ there exists a policyholder i such that

$$\begin{aligned}
&E[u(w_0 - (\lambda\pi_1 + (1-\lambda)\pi_2) - X_i + CEL_i(A(\alpha, \lambda\pi_1 + (1-\lambda)\pi_2), X))] \\
&\geq E[u(w_0 - (\lambda\pi_1 + (1-\lambda)\pi_2) - X_i + f_i(A(\alpha, \lambda\pi_1 + (1-\lambda)\pi_2), X))].
\end{aligned} \qquad (14)$$

From (12)-(13) and the assumption that u is increasing, we get that this also holds for $f_i = \hat{f}_i$, where $\hat{f}_i = \lambda CEL_i(A(\alpha,\pi_1),X) + (1-\lambda)CEL_i(A(\alpha,\pi_2),X), i \in N$. Since $\hat{f}$ yields the same *ex ante* expected utility $U_{PH}^i(\hat{f},\alpha,\pi)$ for all policyholders i, we get

$$\begin{aligned}
&E[u(w_0 - (\lambda\pi_1 + (1-\lambda)\pi_2) - X_i + CEL_i(A(\alpha, \lambda\pi_1 + (1-\lambda)\pi_2), X))] \\
&\geq E[u(w_0 - (\lambda\pi_1 + (1-\lambda)\pi_2) - X_i + \lambda CEL_i(A(\alpha,\pi_1),X) + (1-\lambda)CEL_i(A(\alpha,\pi_2),X))] \\
&> \lambda E[u(w_0 - \pi_1 - X_i + CEL_i(A(\alpha,\pi_1),X))] \\
&\qquad + (1-\lambda)E[u(w_0 - \pi_2 - X_i + CEL_i(A(\alpha,\pi_2),X))].
\end{aligned}$$

Here, the last inequality follows from strict concavity of u, and the fact that from $\pi_1 < \pi_2$, $S(\alpha,\pi_2) \geq S(\alpha,\pi_1)$, and $\mathbb{P}(S(\alpha,\pi_1)) > 0$ it follows that $-\pi_1 - X_i + CEL_i(A(\alpha,\pi_1),X) \overset{d}{\neq} -\pi_2 - X_i + CEL_i(A(\alpha,\pi_2),X)$. Hence, the utility of the policyholder is strictly concave in π.

Showing concavity of the utility of the policyholder with respect to parameter α is analogous to the proof of concavity with respect to the premium π.

Next, we prove concavity of the utility of the policyholder when $f = PP$. This follows directly from the fact that for any fixed $R = \hat{r}$ and $X = (x_i)_{i=1}^n$, it holds that $u(w_0 - \pi - (\sum_{i=1}^n x_i - (W + n\pi)(1 + r + \alpha\hat{r}))^+/n)$ is concave in α and π.

Proof of Theorem 4.2 Let $\hat{F} = \mathcal{F}$ or $\hat{F} = \mathcal{R}$. If a solution to (3) exists, we get $f^* = CEL$ if $\hat{F} = \mathcal{R}$ (Theorem 3.3), or $f^* = PP$ if $\hat{F} = \mathcal{F}$ (Theorem 3.2). Since the objective $U_I(\alpha, \pi)$ is strictly increasing in π, we aim for every $\alpha \in [0, 1]$ to find the largest π such that the participation constraints in (3) are satisfied. Moreover, $U_{PH}^i(f^*, \alpha, \pi)$ is the same for all i, so that we fix i in the remainder of the proof. If $\pi \to \infty$, we get $U_{PH}^i(f^*, \alpha, \pi) < E[u(w_0 - \pi)] \to -\infty$ due to $\lim_{x\to-\infty} u(x) = -\infty$. Then, the participation constraint in (3) is violated. By Assumption 4.1, the policyholder's expected utility U_{PH}^i is continuous in the premium π. Since the utility of the insurer is strictly increasing in π, we get that for any fixed $\alpha \in [0, 1]$ there can be at most one optimal premium π solving (3). If it exists, (α, π) is such that the participation constraint is binding. By strict concavity of the utility of the policyholder for given α when $\delta = 0$ (see Lemma 4.1), it is characterized by the fact that it must also satisfy $\frac{\partial}{\partial\pi} U_{PH}^i(f^*, \alpha, \pi) \leq 0$.

By assumption on $\underline{u}$, we have that there exist $(f, \alpha, \pi) \in \hat{F} \times [0, 1] \times \mathbb{R}_+$ with $U_{PH}^i(f, \alpha, \pi) \geq \underline{u}$. From Theorem 3.2 and Theorem 3.3, it follows that this also hold for $f = PP$ when $\hat{F} = \mathcal{F}$ and for $f = CEL$ when $\hat{F} = \mathcal{R}$.

By Assumption 4.1, we have that $U_{PH}^i(f^*, \cdot, \cdot)$ is continuous on $[0, 1] \times \mathbb{R}_+$. From this and $\lim_{\pi\to\infty} \max_{\alpha\in[0,1]} U_{PH}^i(f^*, \alpha, \pi) = -\infty$, we get that the level set $\{(\alpha, \pi) \in [0, 1] \times \mathbb{R}_+ : U_{PH}^i(f^*, \alpha, \pi) \geq \underline{u}\}$ is a compact subset of $[0, 1] \times \mathbb{R}_+$. Moreover, this set is non-empty by assumption. Since U_I is continuous on $(\alpha, \pi) \in [0, 1] \times \mathbb{R}_+$ as well, we conclude that the maximum in (3) for the respective reservation utility level $\underline{u}$, is attained in $[0, 1] \times \mathbb{R}_+$ due to Weierstrass' extreme value theorem.

Proof of Theorem 5.1 Let $\hat{F} \in \{\mathcal{F}, \mathcal{R}\}$ and $\pi > 0$. From Lemma 3.1, we get for any $\alpha \in (0,1)$ that

$$\frac{\partial}{\partial \alpha} U_I(\alpha, \pi) > 0.$$

So, since the utility of the insurer is continuous, we get that the incentive compatibility constraint in (6) yields $\alpha = 1$. Hence, all optimal solutions to (6) are such that $\alpha = 1$.

Then, (6) boils down to maximize for a fixed $\alpha = 1$ the objective function over all $\pi \geq 0$ and f such that the participation constraints are satisfied. In line with Theorem 3.2 and Theorem 3.3, it holds that the optimal rule f is unique, and given by $f^* = PP$ if $\hat{F} = \mathcal{F}$, and $f^* = CEL$ if $\hat{F} = \mathcal{R}$. The objective function in (6) is continuous and strictly increasing in the premium $\pi \geq 0$. Moreover, by definition, there exists a $\pi^* > 0$ such that $U^i_{PH}(f^*, 1, \pi^*) \geq \underline{u}$, and moreover we have $\lim_{\pi \to \infty} U^i_{PH}(f^*, 1, \pi) = -\infty$. Hence, there is a unique solution, and it is such that the participation constraints are binding. This concludes the proof.

B Total social welfare

In this appendix, we briefly discuss social welfare. Suppose there is a social planner that optimizes a weighted sum of the utilities of all agents (Harsanyi, 1955). Then, the problem is given by

$$\begin{array}{ll} \max_{f,\alpha,\pi} & U_I(\alpha, \pi) + k \cdot \sum_{i=1}^n U^i_{PH}(f, \alpha, \pi), \\ \text{s. t.} & (f, \alpha, \pi) \in \hat{F} \times [0,1] \times \mathbb{R}_+, \end{array} \tag{15}$$

where $\hat{F} = \mathcal{F}$ or $\hat{F} = \mathcal{R}$, $k > 0$, and $X|R$ is exchangeable. Here, the preferences of the policyholders are weighted with factor k to compare the utilities with the expected profit of the insurer.

Theorem B.1 *Let $X|R$ be exchangeable, and let (f, α, π) be a solution of* (15) *with $\hat{F} = \mathcal{F}$, then $f = PP$. Let $X|R$ be exchangeable, and let (f, α, π) be a solution of* (15) *with* $\hat{F} = \mathcal{R}$, *then* $f = CEL$.

Proof For given $(\alpha, \pi) \in [0,1] \times \mathrm{I\!R}_+$, we get that the utility of the insurer is unaffected by $f \in \hat{F}$. If $\hat{F} = \mathcal{F}$ (resp. $\hat{F} = \mathcal{R}$),

$$\begin{aligned} &\max_f \sum_{i=1}^{n} U^i_{PH}(f, \alpha, \pi), \\ &\text{s. t. } f \in \hat{F}, \end{aligned}$$

is solved uniquely for the rule $f = PP$ (resp. $f = CEL$) due to (8) (resp. (11)). This concludes the proof.

Theorem B.1 shows that the results of Section 3 also hold if we focus on a social planner. In other words, the total social welfare is optimal when the rule $f = PP$ ($f = CEL$) is used when $\hat{F} = \mathcal{F}$ ($\hat{F} = \mathcal{R}$).