Hedging of Variable Annuities under Basis Risk

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Abstract

I study dynamic hedging for variable annuities under basis risk. Basis risk, which arises from the imperfect correlation between the underlying fund and the proxy asset used for hedging, has a highly negative impact on the hedging performance. I analyze whether the choice of a suitable hedging strategy can help to reduce the risk for the insurance company. Comparing several cross-hedging strategies based on the proxy asset, I observe very similar hedging performances. Particularly, I show that well-established but complex strategies from mathematical finance do not outperform very simple and naive approaches. Nonetheless, I find that a naive delta-gamma hedge, where plain vanilla options on the proxy asset are used as additional hedging instruments, performs better than the one-instrument strategies. A more substantial risk reduction could, however, be achieved by diversification, that is, by distributing the policyholders' premium among several underlying funds.

Keywords: Variable annuities; GMAB; Basis risk; Cross-hedging; Diversification

JEL classifications: G13, G22, G32

Introduction

Variable annuities (VAs) are fund-linked life insurance products that are mostly equipped with financial guarantees (Bauer et al., 2008). The VA market is large with total net assets in variable annuity funds accounting for 1,827 billion dollars as of 2017 in the U.S. (Investment Company Institute, 2018). VAs have also gained increasing popularity in Asia and Europe (Gatzert and Schmeiser, 2013). Because of the embedded financial guarantees, VA providers are exposed to market risk which became evident when several VA companies suffered significant losses during the 2008 financial crisis (Chopra et al., 2009). In June 2009, Hartford even received government support through the Troubled Asset Relief Program (Koijen and Yogo, 2016). Thus, risk management for VAs is an important and challenging task.

Insurance companies typically manage the market risk from their VA business by dynamic hedging programs drawing on the insights from modern option pricing (and hedging) theory. An important issue for insurance companies, which has attracted little attention in the VA hedging literature so far, is that they are exposed to *basis risk* which arises because of the imperfect (return) correlation between the underlying investment fund and the proxy asset used for hedging the financial guarantees. There is evidence suggesting that basis risk has significantly contributed to the large losses of insurance companies during the 2008 financial crisis (Chopra et al., 2009; Sun, 2009; Zhang, 2010; Duverne and Hele, 2016). Basis risk is not only relevant for insurance companies but also for investment banks. Davis (2006, pp. 169) reports about his impressions from a bank's trading floor:

> "I found that the traders usually had excellent intuition about the sensitivity of option values to various modelling assumptions and parameter values. Correlation was however one area where their intuition sometimes seemed mis-calibrated. [...] Since the (return) correlation between a representative basket of stocks and the index is very high – perhaps 80% – most traders were perfectly happy to hedge using the index as a "proxy" asset, but had very little idea what the residual risk was in doing so."

In this paper, I study dynamic hedging of VAs under basis risk. Therefore, I consider an insurance company selling a pool of single premium VA contracts with a guaranteed minimum accumulation benefit rider (or GMAB-rider). For these contracts, the single premium is invested in a mutual fund and the fund shares are transferred to the accounts of the policyholders. At the maturity of the GMAB-rider, the maximum of the VA account value and some guaranteed amount is assigned to the policyholders.[1]

The main focus will be on the insurer's "hedged loss" for the pool of GMAB-contracts which is equal to the guarantee payment at maturity to the policyholders (if any) net of accumulated profits from hedging and accumulated fees. Throughout the paper, the "insurer's risk" will be measured by the conditional value-of-risk of this hedged loss. In my baseline analysis, I use a multivariate geometric Brownian motion for modeling the joint dynamics of the underlying funds and the proxy asset.

Overall, I document severe basis risk effects which is demonstrated by a surprisingly strong increase of risk when the correlation between the underlying fund and the proxy asset decreases. I find that these adverse effects are particularly strong for high guarantee levels and for high fund volatilities.

I analyze whether the choice of a suitable hedging strategy can help to reduce the risk for the insurance company. First, I focus on strategies that only use the proxy asset for dynamically hedging the pool of GMAB-contracts. I compare simple and tractable strategies with two well-established but complex "quadratic approaches" from mathematical finance, namely local risk minimization (Föllmer and Sondermann, 1986; Schweizer, 1988; Schweizer, 1991) and mean-variance hedging (Duffie and Richardson, 1991; Schweizer, 1994). Overall, only minor differences between the hedging performances can be observed. This applies for all tested correlations, guarantee levels and fund volatilities. Second, I propose a naive two-instrument hedge where plain vanilla options on the proxy asset are included as additional instruments for delta-gamma hedging. By implementing this strategy instead of a one-instrument hedge, a risk reduction can be achieved.

Furthermore, I analyze to what extent basis risk can be diversified away. Therefore, I assume that the aggregated premium from the policyholders is equally allocated among up to ten funds with a similar investment focus. Hedging is then performed based on the same proxy asset for all funds. If the number of underlying funds gets larger, the hedging risk decreases and the results suggest that basis risk can – at least partly – be diversified away.

I provide further empirical evidence for the strong basis risk impact as well as for the diversification effects within block-bootstrap analyses, where multivariate sample paths are generated based on historical return observations without making any distributional assumption.

[1] Contracts with GMAB-riders are frequently considered in the VA hedging literature (Bernard and Kwak, 2016; Kélani and Quittard-Pinon, 2017; Trottier et al., 2018a). Interestingly, these simple GMAB-contracts seem to have gained popularity recently in the US market (Hopkins, 2018).

Additionally, I conduct out-of-sample analyses where I compare historical hedging losses for different degrees of basis risk. These "backtesting-results" also confirm the surprisingly strong impact of basis risk as well as the diversification benefits.

This paper contributes to the research on variable annuity hedging, where basis risk is mostly ignored.[2] In fact, and to the best of my knowledge, Ankirchner et al. (2014) and Trottier et al. (2018a) are the only papers that address the issue of basis risk in the context of VA hedging.[3] In both papers, a kind of a guaranteed minimum accumulation benefit is considered which is in line with the present paper.

Ankirchner et al. (2014) apply a sophisticated hedging strategy based on the mean-variance approach developed by Schweizer (1994) and conclude that basis risk has a surprisingly strong and adverse impact on the hedging performance. They also derive theoretical bounds for the variance of the insurer's hedging error. Their results are based on Monte Carlo simulations using a multivariate geometric Brownian motion, that is also used in the present paper. The authors do not solely focus on basis risk but also take liquidity risk and parameter risk into account, what is ignored in the current paper.

Trottier et al. (2018a), who worked independently of me on the subject, propose a method for VA hedging under basis risk that is based on a local optimization criterion. Their main approach takes capital requirements according to the Canadian regulatory framework into account. The authors use a regime-switching log-normal model for the joint dynamics of the underlying asset and the hedging instrument. They highlight the importance of considering basis risk when determining the risk for VA writers and for calculating capital requirements.

I extend the work of Ankirchner et al. (2014) and Trottier et al. (2018a) along the following dimensions. Firstly, I compare several cross-hedging strategies based on the proxy asset systematically whereas previous work only focuses on a smaller subset of available hedging approaches. Ankirchner et al. (2014) only apply a quadratic hedging strategy and Trottier et al. (2018a) solely consider local approaches. In the present paper, however, I compare quadratic, local *and* naive strategies. Secondly, I propose a naive delta-gamma strategy that considers plain vanilla put options as additional hedging instruments whereas previous work only considers one-instrument hedges based on the proxy asset. Thirdly, I take a broader perspective on VA hedging under basis risk by analyzing diversification effects in a situation when the total premium is allocated

[2]See, for example, Coleman et al. (2007), Kling et al. (2011), Augustyniak and Boudreault (2012), Kling et al. (2014), Kélani and Quittard-Pinon (2017), Cathcart et al. (2015), and Augustyniak and Boudreault (2017).

[3]There is also a related practical note (Trottier et al., 2018b).

among several funds. Finally, I provide further empirical evidence for the general impact of basis risk as well as for the diversification effects by performing block-bootstrap analyses and by providing out-of-sample hedging results.

The remainder of the paper is structured as follows. In Section 1, the VA contract is introduced and the model framework is described. Section 2 is concerned with pricing and hedging without basis risk. The simple strategies for hedging under basis risk are introduced in section 3, whereas the benchmark approaches from mathematical finance are described in the appendix. The results for the univariate case and the multivariate case are presented in sections 4 and 5, respectively. Section 6 provides further empirical evidence and robustness checks.

1 Product and Model Framework

I consider a homogeneous pool of VA contracts with GMAB-riders.[4] The total initial premium is given by $P = \sum_i P_i$, where P_i is the premium paid by policyholder i. The fraction $P(1-\alpha)$ is invested in a mutual fund and the fund shares are transfered to the investors' private accounts. The remaining fraction of the initial premium $P\alpha$ is assigned to the insurer and covers the cost for the financial guarantee. If not stated otherwise, it is assumed that all policyholders invest in the same fund F with price F_t at time t. The time-t (aggregated) account value is then given by

$$A_t = \frac{P\,(1-\alpha)}{F_0}\,F_t. \tag{1}$$

Without loss of generality, I always assume that $P = F_0$. Consequently,

$$A_t = (1-\alpha)\,F_t. \tag{2}$$

The GMAB-rider provides a downside protection. At the maturity date T – one could think of T as the retirement date of the policyholder – the insurance company guarantees an amount of at least $K := \kappa\,P$ (the "guaranteed amount"). Formally,

$$\max\{A_T, \kappa\,P\} \tag{3}$$

[4] A current example is the VA with the GMAB-rider "GPA 3 Select" offered by the Pacific Life Insurance Company. Notice that I consider a stylized version of this contract since, among others, I assume that guarantee fees are paid upfront whereas VA fees are typically charged periodically as a fixed percentage of the account value.

is assigned to the pool of policyholders at maturity T. The lowest possible amount assigned to the individual policyholder i is given by κP_i. For $\kappa = 100\%$, the payoff to each policyholder is at least equal to the individual premium. The policyholders typically have several options at time T. For example, they can take the amount (3) as a lump sum or annuitize it according to prevalent market conditions at this point in time.

For simplicity, I assume that policyholders cannot surrender the contract and are not allowed to make subsequent premium payments after time $t = 0$. Furthermore, it is assumed that the contract is assigned to an heir in case that a policyholder dies before time T. This allows us to go without modeling mortality and we can have a clear focus on the financial risk.

Next, I describe the model framework that is mostly used in this paper. The financial market consists of a bond with $dB_t = rB_t\,dt$ and $B_0 = 1$. There are two risky assets – S ("*the proxy*") and F ("*the fund*") – whose values are modeled by geometric Brownian motions, that is,

$$S_t = S_0\, e^{(\mu_S - \frac{1}{2}\sigma_S^2)t + \sigma_S W_t} \tag{4}$$

$$F_t = F_0\, e^{(\mu_F - \frac{1}{2}\sigma_F^2)t + \sigma_F \widetilde{W}_t} \tag{5}$$

for $t \geq 0$ and where $(W, \widetilde{W})$ is a two-dimensional Brownian motion with correlation matrix $R = \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$. It follows that the geometric Brownian motions solve the following system of stochastic differential equations:

$$dS_t = \mu_S S_t\, dt + \sigma_S S_t\, dW_t \tag{6}$$

$$dF_t = \mu_F F_t\, dt + \sigma_F F_t\, d\widetilde{W}_t. \tag{7}$$

Since (S, F) is a two-dimensional geometric Brownian motion, vector of corresponding log-returns over some discrete time interval are bivariate normally distributed (Remillard, 2013, p. 35).

2 Pricing and Hedging without Basis Risk

In this section, the proxy asset S is ignored and it is assumed that the fund F is used for hedging. It seems necessary to study this case first before analyzing a situation with basis risk when the proxy asset is used for cross-hedging.

2.1 Pricing

The payoff to the pool of policyholders at maturity is

$$\max\{A_T, \kappa\, P\} = (1-\alpha)F_T + (1-\alpha)\, \max\{\tfrac{\kappa\, P}{1-\alpha} - F_T, 0\}. \tag{8}$$

The fee $\alpha \in (0, 1)$ is called *fair* if the premium P equals the risk-neutral expectation of the discounted payoff, that is, if it holds that

$$P = e^{-rT}\, \mathbb{E}^Q[(1-\alpha)F_T + (1-\alpha)\, \max\{\tfrac{\kappa\, P}{1-\alpha} - F_T, 0\}], \tag{9}$$

where Q is the equivalent martingale measure.[5] Since $\mathbb{E}^Q[e^{-rT}F_T] = F_0 = P$, the fee is fair if and only if the fee payment $P\alpha$ covers the cost of the embedded put option, that is, if and only if

$$(1-\alpha)\, \mathbb{E}^Q[e^{-rT}(\, \tfrac{\kappa\, P}{1-\alpha} - F_T\,)^+] - P\alpha = 0\,. \tag{10}$$

It holds that

$$\mathbb{E}^Q[e^{-rT}(\, \tfrac{\kappa\, P}{1-\alpha} - F_T\,)^+] = \pi_{\text{put}}(0, F_0, \tfrac{\kappa\, P}{1-\alpha})\,, \tag{11}$$

where π_{put} is the Black and Scholes price of a European put option with strike k, time to maturity $T-t$ and time-t price of the underlying x which is given by (Hull, 2017, ch. 14)

$$\pi_{\text{put}}(t, x, k) = k\, e^{-r(T-t)}\, \Phi(-d_2) - x \cdot \Phi(-d_1), \tag{12a}$$

with

$$d_1 = d_1[t, x, k] = \frac{\ln(x/k) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma\sqrt{T-t}. \tag{12b}$$

Thereby, Φ is the cumulative distribution function of a standard normally distributed random variable.

[5] For pricing, I assume "ideal conditions" on the market (Black and Scholes, 1973, p. 640). Among others, there are no transactions cost, no dividends and portfolios can be rebalanced continuously. Under these assumptions there is exactly one equivalent martingale measure and, consequently, there is a unique price for any contingent claim.

To sum up, the contract is fair if and only if $\alpha \in (0,1)$ is such that

$$\bar{\pi}_0(\alpha) := (1-\alpha)\,\pi_{\text{put}}(0, F_0, \tfrac{K}{1-\alpha}) - P\alpha = 0. \tag{13}$$

So far, it is not clear if a fair fee always exists and, if it exists, whether it is unique or not. These questions will be addressed in the following proposition. The proof is given in the appendix.

Propostition 1. *For $K < P\,e^{rT} \Leftrightarrow \kappa < e^{rT}$, a fair upfront fee $\alpha^* \in (0,1)$ exists and it is unique.*

2.2 Delta Hedging

Theoretically, a replication of the payoff (8) – which would eliminate the entire risk for the insurer – would be straightforward. The static replication portfolio at time $t = 0$ would simply be given by

(i) a long position of $(1-\alpha)P/F_0 = (1-\alpha)$ units of the fund and

(ii) a long position of $(1-\alpha)$ put options on the fund with a strike price of $\frac{\kappa P}{1-\alpha}$.

However, these put options are typically not traded since the maturities are often very long and the underlying is not a usual index but a specific mutual fund. Even if an investment bank would offer such a contract in the over-the-counter market, it is at least questionable whether the insurer would take on the default risk of the bank and the probably high charges on this specific contract. For these reasons, I assume that the insurer establishes a long position of $(1-\alpha)$ units of the fund and implements a dynamic replication (or hedging) strategy for the put payoff

$$(1-\alpha^*)\,\max\{\tfrac{\kappa\,P}{1-\alpha} - F_T, 0\}, \tag{14}$$

whose time-t value is given by

$$V_t = (1-\alpha^*)\,\pi_{\text{put}}(t, F_t, \tfrac{\kappa\,P}{1-\alpha^*}) =: v(t, F_t). \tag{15}$$

In practice, hedging portfolios cannot be rebalanced continuously (Boyle and Emanuel, 1980) and I assume that the insurer manages the risk by implementing a hedging strategy that is adjusted every h years. For example, if $h = 1/12$, the insurer rebalances the hedge every month. A delta hedger tries to offset the value change

$$\delta V_{t+h} := V_{t+h} - V_t \tag{16}$$

by a position of x_t units in the fund which generates a profit (or loss) equal to

$$x_t \, (F_{t+h} - F_t). \tag{17}$$

The value change (16) can be approximated by the first-order Taylor approximation

$$\begin{aligned} \delta V_{t+h} &\approx \Delta_t \, (F_{t+h} - F_t) && (18) \\ &=: \delta V^{[1]}_{t+h}, && (19) \end{aligned}$$

where (Hull, 2017, ch. 19)

$$\Delta_t := \tfrac{\partial v}{\partial F}(t, F_t) = -(1-\alpha^*) \, \Phi(-d_1[t, F_t, \tfrac{\kappa P}{1-\alpha^*}]). \tag{20}$$

Based on the approximation, the hedger should hold

$$x_t = \Delta_t \tag{21}$$

units of F at time t since the (approximated) periodic hedging loss

$$\Delta_t \, (F_{t+h} - F_t) - x_t \, (F_{t+h} - F_t) \tag{22}$$

would be zero in that case.

I assume that the insurer implements this hedging strategy dynamically in a self-financing manner (using the risk-free asset) and with zero initial capital. Then, the terminal value of the hedging portfolio with rebalancing at times $0, h, \ldots T/h-1$ is given by (Augustyniak, 2013, p. 136)

$$\Pi_T = \sum_{i=0}^{T/h-1} \Delta_{ih} \left(F_{(i+1)h} - F_{ih} e^{rh} \right) e^{r(T-(i+1)h)}. \tag{23}$$

The insurer's total *hedged loss* (at time T) is then equal to[6]

$$HL_T = (K - A_T)^+ - \Pi_T - e^{rT}\alpha^* P\,. \tag{24}$$

If h is "sufficiently small", mean and standard deviation of the hedged loss are close to zero and the uncertainty could even be eliminated entirely if the hedging portfolio would be rebalanced continuously (Björk, 2009, Prop. 9.7).

In the following, I will also consider the *unhedged loss* which is simply given by

$$L_T = (K - A_T)^+ - e^{rT}\alpha^* P\,. \tag{25}$$

2.3 Delta-Gamma Hedging

For a fixed rebalancing frequency, the performance of discretely-adjusted delta hedges can often be improved by incorporating options on the underlying in order to make the hedger's position not only delta-neutral but also gamma-neutral. This hedging strategy is often referred to as "delta-gamma hedging".[7] Intuitively, gamma-neutrality allows "to keep the delta hedge for a longer period" (Björk, 2009, p. 132). A delta-gamma hedger tries to offset the approximated value change (second-order Taylor approximation)

$$\begin{aligned} \delta V_{t+h} &\approx \Delta_t\,(F_{t+h} - F_t) + \frac{1}{2}\,\Gamma_t\,(F_{t+h} - F_t)^2 && (26)\\ &=: \delta V^{[2]}_{t+h} && (27) \end{aligned}$$

by a position of x_t units of F and x^p_t units of short-term at-the-money put options on F.[8] This hedging position generates a profit (or loss) equal to

$$x_t\,(F_{t+h} - F_t) + x^p_t\,(P^F_{t+h} - P^F_t). \tag{28}$$

[6] Notice that the first part of the payoff (8), i.e. $(1-\alpha)F_T$, is perfectly replicated by the long position in the fund that is established at time $t = 0$.

[7] See Alexander (2008, pp. 164) or Björk (2009, pp. 132).

[8] Short-term at-the money options are typicallly liquidly traded but, clearly, other options could be considered here.

Thereby, P_t^F is the Black and Scholes price of the European put option with strike price F_t which is used for hedging. This price is given by

$$P_t^F = \pi_{\text{put}}(t, F_t, F_t). \tag{29}$$

Notice that (Hull, 2017, ch. 19)

$$\Gamma_t = \frac{\partial^2 v}{\partial F^2}(t, F_t) = (1-\alpha^*)\,\frac{\Phi'\left(d_1[t, F_t, \frac{\kappa P}{1-\alpha^*}]\right)}{F_t \sigma \sqrt{T-t}}. \tag{30}$$

For the value change of the put option in (28), the second-order Taylor approximation

$$P_{t+h}^F - P_t^F \approx \Delta_t^p (F_{t+h} - F_t) + \frac{1}{2}\,\Gamma_t^p\,(F_{t+h} - F_t)^2 \tag{31}$$

is used. Δ_t^p is the delta and Γ_t^p the gamma of the European put option. Formal expressions for these “Greeks” are, for example, given by Hull (2017, ch. 19). Consequently, the (approximated) periodic loss is given by

$$\begin{aligned} &\Delta_t\,(F_{t+h} - F_t) + \tfrac{1}{2}\,\Gamma_t\,(F_{t+h} - F_t)^2 - x_t\,(F_{t+h} - F_t) \\ &\quad - x_t^p\,\left[\Delta_t^p (F_{t+h} - F_t) + \tfrac{1}{2}\,\Gamma_t^p\,(F_{t+h} - F_t)^2\right]. \end{aligned} \tag{32}$$

By rearranging the terms and comparing the coefficients in equation (32), we observe that the delta-gamma hedger should hold

$$x_t^p = \frac{\Gamma_t}{\Gamma_t^p} \tag{33}$$

put options and

$$x_t = \Delta_t - \frac{\Gamma_t}{\Gamma_t^p}\,\Delta_t^p \tag{34}$$

units of the proxy. The hedge is implemented in a self-financing manner with zero initial capital and the insurer's total loss is then given by (24) with

$$\Pi_T = \sum_{i=0}^{T/h-1} x_{ih} \left(F_{(i+1)h} - F_{ih}\, e^{rh} \right) e^{r(T-(i+1)h)} \\ + \sum_{i=0}^{T/h-1} x^p_{ih} \left(P^F_{(i+1)h} - P^F_{ih}\, e^{rh} \right) e^{r(T-(i+1)h)} \,. \tag{35}$$

The strategy proposed here is consistent with the delta-gamma approach described by Björk (2009, p. 134). Kolkiewicz and Liu (2012) propose a similar delta-gamma hedging approach in the context of variable annuity hedging (without basis risk) for a VA contract with a "Guaranteed Minimum Withdrawal Benefit" (GMWB).

3 Simple Approaches for Hedging with Basis Risk

For deriving the simple hedging approaches under basis risk, I assume that the insurer still tries to offset the value changes δV_{t+h} for each period by implementing a discretely adjusted hedge *based on the proxy* S. If the insurer could roughly offset the value changes δV_{t+h} in each period, our discussion in section 2.2 implies that the insurer's terminal loss tends to be small then. It is worth noting that the simple hedging strategies proposed in this section are flexible in the sense that they can directly be applied for variable annuity contracts other than the VA/GMAB that is considered here. The simplicity, however, comes at a price. First, simplifying assumptions and approximations will be necessary. Second, some of the approaches presented in this section are *local* which means that one does only control for the total hedging risk indirectly by considering each period separately.

3.1 Naive Approaches

3.1.1 Naive Delta Hedging

Just as in the section before, the delta hedger tries to offset the (approximated) value change

$$\delta V^{[1]}_{t+h} := \Delta_t \,(F_{t+h} - F_t), \tag{36}$$

where Δ_t is given by (20). Under basis risk, however, the proxy asset S has to be used for hedging. The profit from the offsetting position is then equal to

$$\hat{x}_t\,(S_{t+h} - S_t), \tag{37}$$

where $\hat{x}_t$ is the hedge ratio. Thus, the (approximated) hedging loss is given by

$$\Delta_t\,(F_{t+h} - F_t) - \hat{x}_t\,(S_{t+h} - S_t). \tag{38}$$

A naive hedge ratio can be obtained by proceeding on the assumption that (log-)returns are equal. Then, the insurer should hold

$$\hat{x}_t = \frac{\Delta_t\,F_t}{S_t} \tag{39}$$

units of the proxy asset S from time t to $t+h$. This strategy is implemented in a self-financing manner and the insurer's total loss is then given by (24) with

$$\Pi_T = \sum_{i=0}^{T/h-1} \hat{x}_{ih}\left(S_{(i+1)h} - S_{ih}\,e^{rh}\right) e^{r(T-(i+1)h)}. \tag{40}$$

Finally, notice that the hedging strategy based on the hedge ratio (39) is very similar to the naive strategy proposed by Davis (2006, p. 172).[9] The only difference is the multiplicative factor σ_F/σ_S and, therefore, the strategies coincide for equal volatilities.

3.1.2 Naive Delta-Gamma Hedging

The hedger tries to offset the (approximated) value change

$$\delta V_{t+h}^{[2]} = \Delta_t\,(F_{t+h} - F_t) + \frac{1}{2}\,\Gamma_t\,(F_{t+h} - F_t)^2\,, \tag{41}$$

where

$$\Gamma_t = \frac{\partial^2 v}{\partial F^2}(t, F_t) = (1-\alpha^*)\,\frac{\Phi'\left(d_1[t, F_t, \frac{\kappa P}{1-\alpha^*}]\right)}{F_t \sigma \sqrt{T-t}}. \tag{42}$$

[9] See also Davis (2000).

Two hedging instruments are considered: The proxy S and at-the-money put options on the proxy with time to maturity of $\hat{T} < T$ years. The price of an at-the-money option on S at time t is given by

$$P_t^S = \pi_{\text{put}}(t, S_t, S_t). \tag{43}$$

The profit from the offsetting position is equal to

$$\hat{x}_t \,(S_{t+h} - S_t) + \hat{x}_t^p (P_{t+h}^S - P_t^S)\,, \tag{44}$$

where $\hat{x}_t$ is the number of proxy assets and $\hat{x}_t^p$ is the number of put options. For the value change of the put option, the second-order Taylor approximation

$$P_{t+h}^S - P_t^S \approx \Delta_t^p (S_{t+h} - S_t) + \frac{1}{2}\,\Gamma_t^p\,(S_{t+h} - S_t)^2 \tag{45}$$

is used. Δ_t^p is the delta and Γ_t^p the gamma of a European put option on S in the Black and Scholes model. Consequently, the (approximated) periodic hedging loss is given by

$$\begin{aligned} &\Delta_t\,(F_{t+h} - F_t) + \tfrac{1}{2}\,\Gamma_t\,(F_{t+h} - F_t)^2 - \hat{x}_t\,(S_{t+h} - S_t) \\ &\quad - \hat{x}_t^p \cdot \Delta_t^p (S_{t+h} - S_t) + \tfrac{1}{2}\,\Gamma_t^p\,(S_{t+h} - S_t)^2. \end{aligned} \tag{46}$$

A naive hedger who proceeds on the assumption that (log-)returns are equal, which implies that squared (log-)returns are also identical, should hold

$$\hat{x}_t^p = \frac{\Gamma_t\,F_t^2}{\Gamma_t^p\,S_t^2} \tag{47}$$

put options and

$$\hat{x}_t = \frac{\Delta_t F_t}{S_t} - \hat{x}_t^p \Delta_t^p \tag{48}$$

units of the proxy. This can be seen by rearranging the terms and comparing the coefficients in equation (46). Notice that the naive delta gamma hedging strategy is a generalization of the strategy proposed in section 2.3 since, for $F = S$, the hedge ratios would be equal. The hedge is implemented in a self-financing manner with zero initial capital and the insurer's total loss is then given by (24) with

$$\Pi_T = \sum_{i=0}^{T/h-1} \hat{x}_{ih} \left(S_{(i+1)h} - S_{ih}\, e^{rh} \right) e^{r(T-(i+1)h)} + \sum_{i=0}^{T/h-1} \hat{x}^p_{ih} \left(P^S_{(i+1)h} - P^S_{ih}\, e^{rh} \right) e^{r(T-(i+1)h)} . \tag{49}$$

3.2 Local Approaches

In this section, the periodic delta hedging loss approximated by

$$\Lambda^{[1]}_{t+h} := \Delta_t \left(F_{t+h} - F_t \right) - \hat{x}_t \left(S_{t+h} - S_t \right) \tag{50}$$

is considered. Similar to Trottier et al. (2018a), a mean-variance criterion – where the mean error is traded off against the variance of the error in a linear way – is used. I propose the local mean-variance hedge ratio

$$\hat{x}^*_t = \underset{\hat{x}_t \in \mathbb{R}}{\arg\min} \quad \mathrm{var}_t[\Lambda^{[1]}_{t+h}] + 2\,\lambda\,\mathbb{E}_t[\Lambda^{[1]}_{t+h}]. \tag{51}$$

Thereby, $\mathrm{var}_t[\ ]$ ($\mathbb{E}_t[\ ]$) denotes the conditional variance (expected value) under the physical probability measure P.

Propostition 2. *The local mean-variance hedge ratio is given by*

$$\hat{x}^*_t = \Delta_t \, \frac{\mathrm{cov}_t[S_{t+h}, F_{t+h}]}{\mathrm{var}_t[S_{t+h}]} + \lambda \, \frac{\mathbb{E}_t[S_{t+h}] - S_{t+h}}{\mathrm{var}_t[S_{t+h}]} \, . \tag{52}$$

Under the model framework described in section 1, it holds that

$$\hat{x}^*_t = \frac{\Delta_t\, F_t}{S_t} \, \frac{e^{\rho\sigma_S\sigma_F h} - 1}{e^{\sigma_S^2\, h} - 1} \, e^{h(\mu_F - \mu_S)} + \lambda \, \frac{e^{\mu_S h} - 1}{S_t e^{2\mu_S h}(e^{\sigma_S^2 h} - 1)} \, . \tag{53}$$

For $\lambda = 0$, we get the minimum-variance hedge ratio*.*

Remark 1. The general structure (52) of the hedge ratio is equal to the structure of the hedge ratio proposed by Trottier et al. (2018a, Prop. 3.2). Notice, however, that the authors use a regime-switching log-normal model. Thus, the hedge ratio (53) is in general not equal to the hedge ratio used by *Trottier et al.* However, for the special case of one regime, the hedge ratios are identical. ◊

Remark 2. Interpreting $\hat{x}_t^*$ as a function of h, it follows that

$$\hat{x}_t^* \to \rho \, \frac{\sigma_F}{\sigma_S} \, \frac{\Delta_t \, F_t}{S_t} + \lambda \, \frac{\mu_S}{S_t \, \sigma_S^2} \tag{54}$$

for $h \to 0$ by l'Hôpital's rule. Clearly, this applies to all parameter choices of our model. Thus, for sufficiently small h, the approximation

$$\hat{x}_t^* \approx \rho \, \frac{\sigma_F}{\sigma_S} \, \frac{\Delta_t \, F_t}{S_t} + \lambda \, \frac{\mu_S}{S_t \, \sigma_S^2} \tag{55}$$

could be used. This approximation turns out to be very accurate. ♢

Remark 3. The approximated minimum variance hedge ratio ($\lambda = 0$)

$$\rho \, \frac{\sigma_F}{\sigma_S} \, \frac{\Delta_t \, F_t}{S_t} \tag{56}$$

is equal to naive hedge ratio (39) up to the multiplication by $\rho \, \frac{\sigma_2}{\sigma_1}$. Interestingly, the drift terms, which are difficult to estimate accurately (Rogers, 2001; Monoyios, 2007), do no appear in equation (56). The hedge ratio (56) is equal to the naive hedge ratio proposed by Davis (2006, p. 172) multiplied by the correlation ρ. Furthermore, the strategy based on the hedge ratio (56) coincides with the naive strategy proposed by Hulley and McWalter (2015, sec. 8) which is derived using simplifying assumptions inspired by the Capital Asset Pricing Model. Hulley and McWalter (2015, p. 95) call it an "improved benchmark strategy" as compared to the naive approach proposed by *Mark Davis*. ♢

4 Results for the Standard Case

4.1 General Remarks and Data

In this section, I compare the performance of several strategies. Furthermore, I analyze the general impact of basis risk for the standard case, where it is assumed that the premium of the policyholders is invested in the same fund. The first step of the analysis is to get reasonable parameters for the two-dimensional geometric Brownian motion that will be used for the Monte Carlo simulations. Therefore, I consider the following situation. The hedging instrument is the S&P 500 total return index.[10] The underlying VA fund is given by a portfolio consisting of the

[10] In practice, index futures are used for hedging purposes. For simplicity, I use the index directly.

Table 1
Parameter

	x	$\hat{\mu}$	$\hat{\sigma}$	$\hat{\rho}$
S&P 500 (TR)	1.0	10.2	14.3	100.0
International Equity Fund 80	0.8	9.6	14.3	99.2
International Equity Fund 50	0.5	8.6	14.6	95.2
International Equity Fund 30	0.3	8.0	15.0	90.9
MSCI World ex US (TR)	0.0	7.1	16.1	83.1

Note: The table reports annualized maximum likelihood estimates for the parameters of the bivariate geometric Brownian motions (S&P 500, fund) based on monthly data from 06:1993 to 06:2018. The S&P 500 fraction is denoted by x. $\hat{\rho}$ is the empirical correlation coefficient between the monthly log-returns of the fund and the proxy. See Remillard (2013, section 2.2) for details on maximum likelihood estimation for multivariate geometric Brownian motions. All numbers are in per cent.

S&P 500 index and the MSCI World ex USA (total return) index where the fraction $x \in [0, 1]$ is invested in the S&P 500 and the fraction $1 - x$ is invested in the MSCI World ex USA index. For $x < 1$, the underlying can be considered as a global equity fund while the VA guarantee is hedged using the S&P 500 index. I assume that the portfolio is rebalanced on a monthly basis to restore the targeted weightings. The monthly fund return at time t is given by

$$R_t^{\text{Fund}} = x \cdot R_t^{\text{US}} + (1 - x) \cdot R_t^{\text{World}}, \tag{57}$$

where the monthly returns of the S&P 500 and the MSCI World ex USA index are denoted by R_t^{US} and R_t^{World}, respectively. Table 1 presents maximum likelihood estimates for the model parameters based on return data from 06:1993 to 06:2018. Based on these estimates, I decided to use – if not stated otherwise – the following model parameters by default:

$$\rho = 0.95$$

$$\sigma_F = \sigma_S = 0.15$$

$$\mu_F = \mu_S = 0.08$$

$$r = 3\%.$$

I always assume that the fair guarantee fee is charged and I consider an insurer selling a pool of homogeneous contracts. Unless otherwise stated, the following product characteristics are used by default:

$$P = 100$$
$$T = 10$$
$$\kappa = 100\%.$$

I evaluate the hedge effectiveness by analyzing the distribution of the insurer's hedged loss HL_T. Particularly, I look at relevant characteristics of the distribution namely the mean, standard deviation (std) and conditional value-at-risk (CVaR). The $\text{CVaR}_{1\%}$, which corresponds to the average loss for the 1% worst scenarios, will be the main evaluation criterion.[11] If not stated otherwise, all characteristics are calculated using a Monte-Carlo simulation based on the two-dimensional geometric Brownian motion with $m = 100\,000$ sample paths.

4.2 Basis Risk Effects and Comparison of Hedging Strategies

Beside analyzing the general impact of basis risk, I compare the simple hedging strategies developed in section 3 with more sophisticated approaches from the mathematical finance literature in this section. More precisely, I compare naive delta hedging (*n*, section 3.1.1), local minimum variance hedging (*minv*, section 3.2), local mean variance hedging (*mv*, section 3.2), local risk minimization hedging (*lrm*, appendix B) and mean-variance hedging (*mvh*, appendix B). The latter two are so-called "quadratic strategies" that are well-established in mathematical finance (Heath et al., 2001). In a nutshell, both quadratic approaches (*lrm*, *mvh*) minimize expected squared hedging costs over one global time step (*mvh*) or "on each infinitesimal time interval" (*lrm*) (Heath et al., 2001, p. 393). I refer to appendix B for more details. Hulley and McWalter (2015, Proposition 3) use the basis risk model introduced in section 1 and derive the *lrm*- and *mvh*-hedging strategy for a plain vanilla call and put option explicitly (see appendix B for details). Their results can directly be applied here. The continuous-time versions of these quadratic strategies are used and in order to do justice on it, the hedging portfolios should be

[11] The standard sample-based estimator definition of the CVaR for the hedged loss is

$$\text{CVaR}_\alpha[HL_T] = \frac{1}{m\,\alpha} \sum_{i=1}^{m\,\alpha} HL_T^{\{i\}}, \tag{58}$$

where $HL_T^{\{i\}}$ is the ith largest loss among the $m = 100\,000$ simulated scenarios (Augustyniak and Boudreault, 2017, p. 512). The CVaR for the unhedged loss L_T is defined equivalently. The CVaR is sometimes also referred to as the conditional tail expectation (CTE). Notice that $m\,\alpha$ is always a positive integer here; otherwise, the definition (58) would have to be modified slightly.

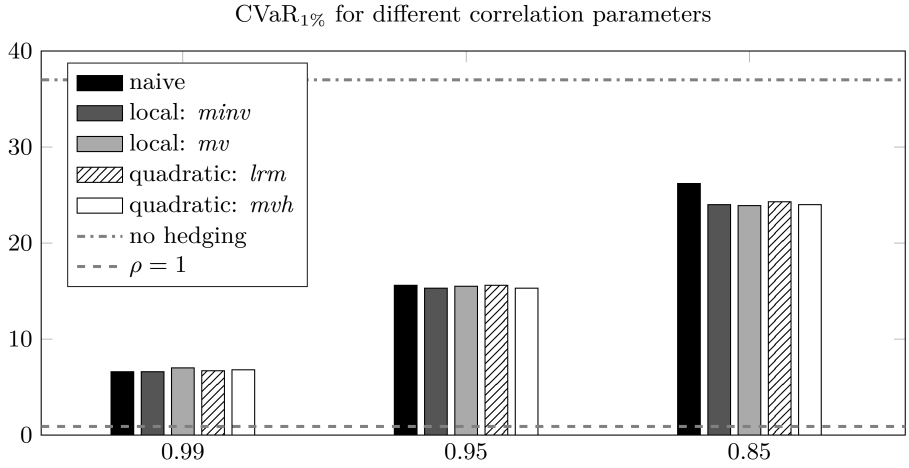

Figure 1
The figure illustrates the hedging performances for different magnitudes of basis risk.

rebalanced frequently. I decided to use a daily rebalancing frequency in this section. For a fair comparison, I focus on one-instrument hedges solely based on the proxy asset, whereas naive delta gamma hedging, where plain vanilla options serve as a additional hedging instruments, will be analyzed separately in section 4.3.

At first, I compare the hedging performances of the different strategies for several correlation parameters. The main results are presented in figure 1 and details can be found in table 2. Comparing the different hedging strategies, we observe very similar CVaRs across all levels of correlations. Interestingly, we observe that the sophisticated quadratic hedging strategies do not outperform the much simpler naive and local strategies. For $\rho = 0.99$ the CVaR for the different strategies ranges between 6.6 and 7.0 and, also for lower correlations, the CVaR-range is rather narrow. Notice, however, that the naive strategy does slightly worse in comparison with the local and quadratic approaches for very low correlations. Furthermore, it is worth noting that the standard deviations for the *mvh*-strategy are typically lowest, but, in terms of the CVaR, the strategy does not outperform the others.

Next, analyzing the general impact of basis risk, we observe very strong effects. Comparing the cases $\rho = 1$ and $\rho = 0.99$, we see that even small deviations between the underlying and the hedging instrument make a surprisingly significant difference. The CVaR of the hedged loss HL_T increases from 0.9 to 6.6 for the naive strategy. For $\rho = 0.85$, the CVaR even corresponds to 24% of the premium ($P = 100$) for the minv-strategy and this CVaR is thus not too far away from the CVaR without hedging (37%). Thus, I confirm the strong basis risk effects that have already been documented in the literature by Ankirchner et al. (2014) and Trottier et al.

Table 2
Correlation

		Hedged Loss HL_T					
			correlation = 0.99				
				local		quadratic	
	L_T	$\rho = 1$	n	minv	mv	lrm	mvh
mean	-9.1	0.0	0.0	-0.1	-1.0	-0.2	-0.2
std	7.4	0.3	1.7	1.7	1.9	1.7	1.3
$\mathrm{CVaR}_{1\%}$	37.0	0.9	6.6	6.6	7.0	6.7	6.8
			0.95				
				local		quadratic	
	L_T	$\rho = 1$	n	minv	mv	lrm	mvh
mean	-9.1	0.0	0.0	-0.5	-1.4	-1.0	-0.9
std	7.4	0.3	3.9	3.7	3.8	3.7	2.8
$\mathrm{CVaR}_{1\%}$	37.0	0.9	15.6	15.3	15.5	15.6	15.3
			0.90				
				local		quadratic	
	L_T	$\rho = 1$	n	minv	mv	lrm	mvh
mean	-9.1	0.0	-0.1	-1.0	-2.0	-2.0	-1.7
std	7.4	0.3	5.3	4.9	4.9	4.8	3.7
$\mathrm{CVaR}_{1\%}$	37.0	0.9	20.8	20.0	20.1	20.5	20.7
			0.85				
				local		quadratic	
	L_T	$\rho = 1$	n	minv	mv	lrm	mvh
mean	-9.1	0.0	-0.1	-1.5	-2.4	-2.8	-2.5
std	7.4	0.3	6.4	5.7	5.7	5.5	4.5
$\mathrm{CVaR}_{1\%}$	37.0	0.9	26.2	24.0	23.9	24.3	24.0

Note: The table reports characteristics of the insurer's unhedged loss L_T and hedged loss HL_T for different correlation parameters.

(2018a). Similar to Trottier et al. (2018a), I conclude that basis risk must not be ignored for risk evaluation or for determining capital requirements. Otherwise, the risk would be significantly underestimated.

Now, the hedging performances are compared for different guarantee levels

$$\kappa \in \{80\%, 100\%, 120\%\}. \tag{59}$$

The results are illustrated in figure 2 and further details can be found in table A1. Again, we observe very similar hedging performances and there seems to be no superior hedging strategy. For $\kappa = 120\%$, the CVaR ranges between 20.7 for the local minimum variance strategy and 22.5 for the quadratic *mvh*-approach. The differences are even smaller for $\kappa = 80\%$. I chose a correlation parameter of $\rho = 0.95$, but the picture is very similar for higher or lower correlations between the fund and the proxy (results are not reported).

When comparing the general impact of the guarantee level, we see that the risk can be eliminated almost entirely for any guarantee level κ if there is no basis risk, i.e., for $\rho = 1$. This no longer holds true when basis risk is an issue, that is, for $\rho < 1$. In this case and for all strategies, the CVaR increases significantly when the guarantee level κ gets larger. Whereas for $\kappa = 80\%$ the CVaR of HL_T corresponds to (approximately) 8% of the premium P, it corresponds to more than 20% of the premium when $\kappa = 120\%$, which represents a significant increase by around 150%. Thus, from the perspective of the insurance company, lower guarantee levels seem to be beneficial when basis risk is an issue.

Finally, the hedging results are analyzed for different fund volatilities

$$\sigma_F \in \{10\%, 15\%, 20\%\}. \tag{60}$$

For simplicity, I assume that $\sigma_F = \sigma_S$ and that the drift parameters remain unchanged. Figure 3 shows that the performances of the different strategies in terms of the CVaR are again very similar across all fund volatilities. More detailed results can be found in table A2. Furthermore, we observe that the CVaR of the hedged loss HL_T without basis risk is very low for all fund volatilities. The picture with basis risk is different. In this case, the risk in terms of the standard deviation and CVaR increases significantly when the fund volatility σ_F gets larger. Whereas for $\sigma_F = 10\%$ the CVaR of HL_T corresponds to approximately 5.5% of the policyholder's premium P, it corresponds to more than 20% of the premium when the fund volatility is equal to 20%,

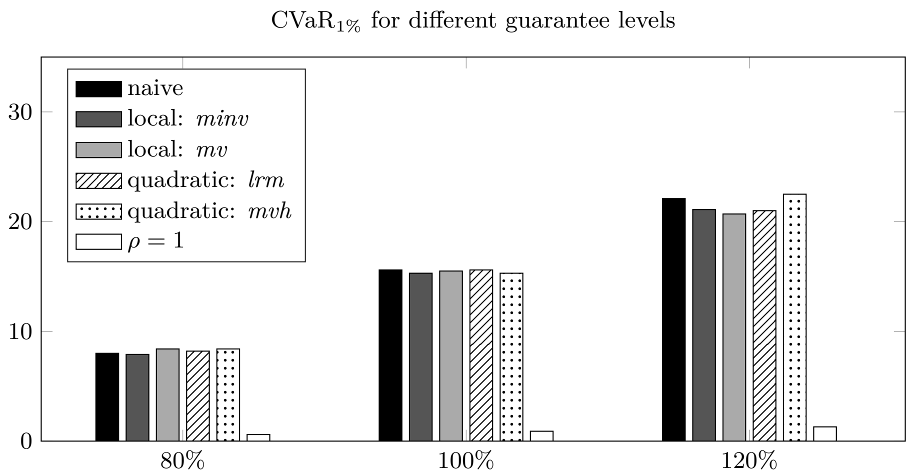

Figure 2
The figure illustrates the hedging performances for different levels of financial protection.

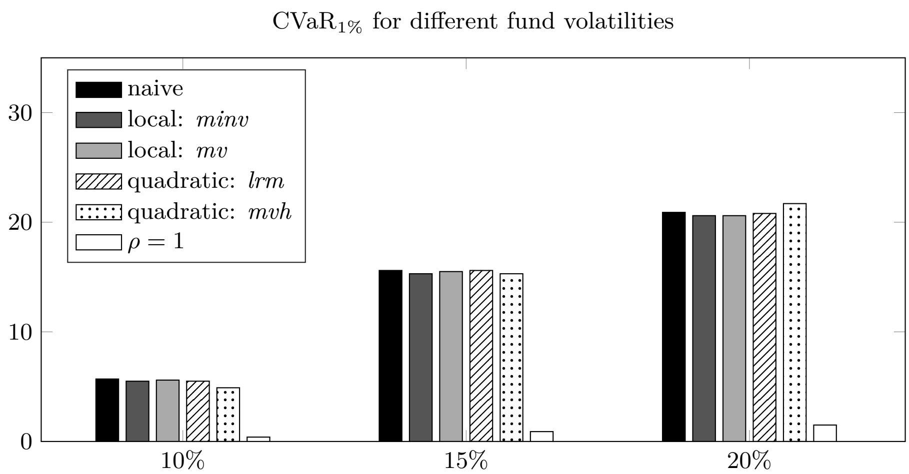

Figure 3
The figure illustrates the hedging performances for different fund volatilities.

which is a 264% increase. Thus, lower fund volatilities seem to be beneficial for an insurer which is exposed to basis risk.

In this section, the available (one-instrument) hedging strategies were compared systematically and, thereby, we observed that the performances were very similar and there seems to be no superior hedging strategy.[12] *Due to these small differences, I conclude that – from a practical point of view – using the simpler strategies seems to be sufficient and I decided to focus on the simplest one-instrument approach in what follows, namely the naive delta hedging strategy.* In the following section, I analyze whether the performance of this naive approach can be improved by considering plain vanilla options as additional hedging instruments (two-instrument hedge).

4.3 Rebalancing Frequency and Gamma Hedging

Without basis risk, one should expect that the hedging performance can be improved if the hedge is rebalanced more often. With basis risk, however, the impact of the rebalancing frequency is not clear a priori. Thus, it is necessary to evaluate possible benefits from increasing the rebalancing frequency. This is relevant because a higher frequency typically comes along with higher monitoring and trading costs (Augustyniak and Boudreault, 2017, section 5.6).

I compare the situation without ($\rho = 1$) and with basis risk ($\rho = 0.95$), and consider half-yearly (hy), monthly (m) and daily (d) rebalancing frequencies. Results are presented in figure 4 and table A3 (appendix). I do only consider the naive strategy but the results for the local strategies ($minv$, mv) are very similar.[13]

First, the case without transactions cost is discussed. For $\rho = 1$, it is not surprising that the CVaR decreases significantly if the hedge is adjusted more often. The CVaR corresponds to 11.1% of the initial premium P for half-yearly rebalancing and only to 0.9% of the initial premium P for daily rebalancing. The picture is different for $\rho < 1$. Whereas a reduction of the CVaR can be observed when switching from half-yearly to monthly rebalancing, a further increase in the frequency has only a minor impact on the CVaR. When comparing the CVaR for monthly rebalancing (15.4% of premium P) with the CVaR for daily rebalancing (15.3% of premium P), it seems at least questionable whether it is worth to switch from monthly to daily

[12] Trottier et al. (2018a, table 3) find larger differences between the local hedging strategies that are proposed in their paper. However, the authors do use another evaluation criterion. Instead of the CVaR of the hedged loss, that is used here, they mainly focus on the CVaR of the "discounted sum of injections". Furthermore, their strategies are to some extent optimized for the Canadian regulatory framework and they use a regime-switching log-normal model for their simulation study.

[13] Notice that it is obviously not meaningful to compare the *continuous-time* quadratic approaches for lower rebalancing frequencies.

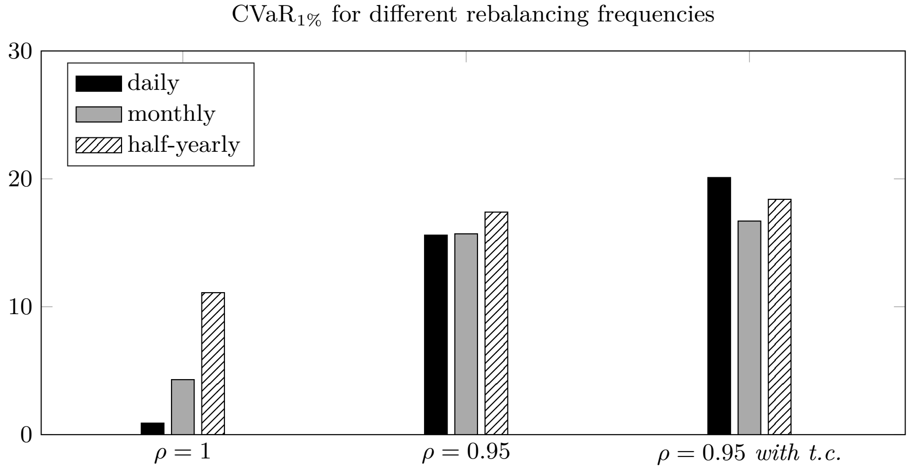

Figure 4
The figure illustrates the hedging performances for different rebalancing frequencies. For $\rho = 0.95$, the naive delta hedging strategy is used.

rebalancing. If we consider transactions cost of 0.2% of the change in the market value of the hedging instrument (Augustyniak and Boudreault, 2012), the CVaR for monthly rebalancing is lower as compared to the CVaR for daily rebalancing and adjusting the hedge on a monthly basis is superior.

In this paper, a time-based delta-hedging strategy is used where trading takes place at fixed points in time. Besides time-based hedging, move-based approaches where trades are triggered by a "significant" movement in the underlying or the hedge ratio are also popular.[14] In fact, there is an ongoing discussion in the finance literature (see, for example, Toft (1996) and the references therein) and the insurance literature (Boyle and Hardy, 1997; Coleman et al., 2007; Lin et al., 2016) on the question which approach should be favored. According to Toft (1996), this question cannot be answered in general and the answer depends on the market environment (market volatility, size of transactions cost). Due to the small range of the CVaRs (and standard deviations) for different rebalancing frequencies in a setting with basis risk, I would not expect that switching to move-based hedging makes a great difference when basis risk is an issue.

Furthermore, I test whether incorporating short-term plain vanilla put options on the proxy as additional hedging instruments adds value in a situation with basis risk. More precisely, I compare the performance of the naive delta hedge (section 3.1.1) and the naive delta-gamma hedge (section 3.1.2). I use at-the-money put options with a maturity of $T = 0.5$ years for hedging. Nevertheless, results are similar for other maturities or other degrees of moneyness. Black and Scholes prices are used for these options.

[14]Delta move-based approaches are dicussed in Martellini and Priaulet (2002) for example.

Table 3
Gamma hedging

	without t.c.				with t.c. (0.2%)			
	monthly		half-yearly		monthly		half-yearly	
	n	n-g	n	n-g	n	n-g	n	n-g
mean	0.1	-0.1	0.3	-0.1	0.8	0.5	0.7	0.2
std	4.0	3.2	4.7	3.4	4.0	3.2	4.8	3.5
$\text{CVaR}_{1\%}$	15.7	13.4	17.4	14.6	16.7	14.4	18.4	15.6

Note: Naive delta (n) and delta-gamma (n-g) hedging are compared. Hedging portfolios are either rebalanced monthly or half-yearly. The results are also reported for the case with transactions costs.

Table 3 shows that the hedging performance can be improved when options are used for delta-gamma hedging. For monthly rebalancing, the CVaR can be reduced from 15.7 to 13.4, which corresponds to a reduction of 14.6%. One could argue that delta-gamma hedging could come along with higher transactions cost since two hedging instruments are involved. Nevertheless, even if transactions cost of 0.2% are used for the proxy asset as well as the proxy-options, delta-gamma hedging is still superior. For monthly (half-yearly) rebalancing and with transactions cost, the CVaR can be reduced by 13.8% (15.2%) if put options are used for naive delta-gamma hedging.

5 Results for the Multivariate Case

5.1 Extended Model Framework

In this section, I investigate the extent to which basis risk can be diversified away by distributing the premium to several funds, and not just one single fund as for the standard case. In order to do so, the model framework needs to be generalized. The financial market still consists of a bond with $dB_t = rB_t\,dt$ and $B_0 = 1$ but – in contrast to section 1 – there are $n+1$ risky assets, namely the proxy S and n funds $F^{(1)}, \ldots, F^{(n)}$ with

$$S_t = S_0\, e^{(\mu_S - \frac{1}{2}\sigma_S^2)t + \sigma_S W_t} \tag{61}$$

$$F_t^{(1)} = F_0^{(1)}\, e^{(\mu_1 - \frac{1}{2}\sigma_1^2)t + \sigma_1 W_t^{(1)}} \tag{62}$$

$$\vdots \tag{63}$$

$$F_t^{(n)} = F_0^{(n)}\, e^{(\mu_n - \frac{1}{2}\sigma_n^2)t + \sigma_n W_t^{(n)}}, \tag{64}$$

where $(W, W^{(1)}, \ldots, W^{(n)})$ is a $(n+1)$-dimensional Brownian motion with correlation matrix $R = (\rho_{jk})_{j,k \in \{1,\ldots,n+1\}}$. Thereby, $\rho_{jk} = \rho_{kj}$ and $\rho_{kk} = 1$ for all j, k. It follows that the geometric Brownian motions solve the following system of stochastic differential equations ($j = 1, \ldots, n$):

$$dS_t = \mu_S S_t \, dt + \sigma_S S_t \, dW_t \tag{65}$$

$$dF_t^{(j)} = \mu_j F_t^{(j)} \, dt + \sigma_j F_t^{(j)} \, dW_t^{(j)}. \tag{66}$$

I assume that the total premium P is *equally* allocated among the n funds. Thus, the amount P/n is invested in each fund. For each policyholder, the insurance company guarantees $\kappa \cdot 100\%$ of the initial individual premium. In this section, funds with similar characteristics are used and it is assumed that the fee α^* is the same for all funds for simplicity.

The hedge is based on the proxy S and the insurer's hedged loss is then given by

$$HL_T = \sum_{j=1}^{n} (\kappa \frac{P}{n} - A_T^{(j)})^+ - \sum_{j=1}^{n} \Pi_T^{(j)} - e^{rT} \alpha^* P \tag{67}$$

$$= \sum_{j=1}^{n} \left((\kappa \frac{P}{n} - A_T^{(j)})^+ - \Pi_T^{(j)} - e^{rT} \alpha^* \frac{P}{n} \right), \tag{68}$$

where $\Pi_T^{(j)}$ is the terminal hedging profit. I assume that a standard hedging strategy is applied for each fund (or "sub-account") separately without further adjustments. Finally, notice that $A_t^{(j)} = [F_t^{(j)} \frac{P}{n} (1 - \alpha^*)] / F_t^{(j)}$.

5.2 Data

First, I try to identify reasonable parameters for the multivariate geometric Brownian motion. Therefore, I consider a universe of ten US equity funds that mainly invest in stocks with high market capitalizations. I chose actively managed funds with high assets under management that have been launched in 1993 at the latest such that at least 25 years of monthly return data is available. VA provider often offer multi asset funds that invest into several asset classes. Nevertheless, I decided to focus on pure equity funds which has the main advantage that there is an obvious hedging instrument, namely the S&P 500 index.[15]

The monthly total net returns for the funds for 25 years from 06:1993 to 06:2018 are downloaded from CRSP. Maximum likelihood estimates for the correlations with the S&P 500 index,

[15] Trottier et al. (2018a) also consider a pure equity fund as the underlying for the VA contract with GMAB rider that is considered in their paper.

Table 4
Funds

	CRSP Identifier	Fund Name	$\hat{\rho}_{jj}$	$\hat{\mu}$	$\hat{\sigma}$
S	-	S&P 500 = **Hedging intstrument**	100.0	10.2	14.3
$F^{(1)}$	26985	T Rowe Price Blue Chip Growth Fund, Inc	95.1	11.8	15.7
$F^{(2)}$	5060	American Century Mutual Funds, Inc: Select Fund	94.7	8.9	14.9
$F^{(3)}$	11980	Fidelity Securities Fund: Fidelity Blue Chip Growth Fund	94.4	10.9	16.0
$F^{(4)}$	20126	Matrix Advisors Value Fund, Inc	90.1	9.2	17.1
$F^{(5)}$	9195	Dodge & Cox Funds: Dodge & Cox Stock Fund	89.9	11.8	15.2
$F^{(6)}$	12082	Fidelity Trend Fund	89.8	9.8	17.3
$F^{(7)}$	11809	Fidelity Contrafund	88.4	11.8	13.3
$F^{(8)}$	22631	Oak Associates Funds: White Oak Select Growth Fund	87.1	11.3	22.1
$F^{(9)}$	14612	Harris Associates Investment Trust: Oakmark Fund	86.0	11.1	15.0
$F^{(10)}$	7317	Clipper Fund, Inc	82.0	10.4	14.1

Note: The table reports annualized maximum likelihood estimates for the drift parameters, the volatilities and the diagonal of the correlation matrix R for the multivariate geometric Brownian motion. See Remillard (2013, section 2.2) for details on maximum likelihood estimation for multivariate geometric Brownian motions. The estimation is based on monthly total net (log-)returns from 06:1993 to 06:2018 downloaded from CRSP. The estimate of the full correlation matrix R is reported in the appendix. Numbers are in %.

drift terms and volatilities are reported in table 4. The entire empirical correlation matrix $\hat{R}$ is given in the appendix (table A4). Notice that the funds are sorted by the (estimated) correlation with the S&P 500. $F^{(1)}$ has the highest correlation (0.951) and $F^{(10)}$ has the lowest correlation with the proxy (0.82). Estimates for the drift (volatility) parameters for the funds range between 8.9% and 11.8% (13.3% and 22.1%).

For reasons of consistency and to ensure comparability with the standard case, I use the following model parameters by default, unless otherwise stated:

$$\sigma_S = \sigma_1 = \cdots = \sigma_{10} = 0.15$$
$$\mu_S = \mu_1 = \cdots = \mu_{10} = 0.08\,.$$

Furthermore, the estimated correlations are used. Notice that the estimated drift parameters are on average slightly higher than 0.08. Nevertheless, I come to the same conclusions when the individual drift and volatility parameters of table 4 are used (results are not reported).

5.3 Diversification Benefits

In this section, I present and discuss the results of the diversification analysis. I use the default product characteristics ($T = 10$ and $\kappa = 100\%$; unless otherwise stated) and assume that the total premium $P = 100$ is equally allocated among $n = 1$ (the univariate case), $n = 5$ and $n = 10$ funds. Due to the minor differences between the hedging strategies in the standard case,

Table 5
Diversification

	Panel A: Diversification Benefits			Panel B: Product Characteristics					
	$n = 1$	$n = 5$	$n = 10$	Guarantee Level κ			Fund Volatilities σ		
	$F^{(1)}$	$F^{(1)} - F^{(5)}$	$F^{(1)} - F^{(10)}$	80%	100%	120%	10%	15%	20%
mean	-0.1	0.1	0.2	0.0	0.1	-0.2	0.1	0.1	-0.1
std	4.0	3.0	2.8	1.3	3.0	5.3	1.0	3.0	4.5
$\text{CVaR}_{1\%}$	15.7	10.4	9.1	5.0	10.4	14.3	3.9	10.4	13.7

Note: Panel A reports characteristics of the hedged loss for different degrees of diversification. Panel B reports characteristics of the hedged loss for different guarantee levels and fund volatilities, whereby the premium is allocated among five funds..

Table 6
Diversification: Gamma hedging

	$n = 1$		$n = 5$		$n = 10$	
	$F^{(1)}$		$F^{(1)} - F^{(5)}$		$F^{(1)} - F^{(10)}$	
	n	gamma	n	gamma	n	gamma
mean	-0.1	-0.1	0.1	0.0	0.2	-0.2
std	4.0	3.1	3.0	2.1	2.8	2.2
$\text{CVaR}_{1\%}$	15.7	12.8	10.4	7.8	9.1	7.0

Note: Naive delta and delta-gamma hedging are compared. The total premium is allocated among five funds.

I will solely use the naive hedging strategy in this section. Furthermore, the results of section 4.3 suggest to use a monthly rebalancing frequency which I also follow.

The results reported in table 5:A suggest that basis risk can – to some extent – be diversified away. For the univariate case ($n = 1$), the CVaR corresponds to 15.7% of the total premium P whereas it only corresponds to 9.1% if the premium is equally allocated among all funds ($n = 10$). This corresponds to a CVaR reduction of 39%.

Next, table 5:B shows that – just as in the univariate case – the insurer's risk increases for higher levels of financial protection and for higher volatilities. For $\kappa = 120\%$ the CVaR corresponds to 14.3% of the total initial premium whereas for $\kappa = 80\%$, the CVaR only corresponds to 5.0% of the premium. Similarly, the CVaR equals 13.7 for fund volatilities of 0.2 and only 3.9 for fund volatilities of 0.1.

Furthermore, I consider short-term at-the-money put options on the proxy as additional hedging instruments and compare the performances of naive delta hedging and naive delta-gamma hedging. The (at-the-money) put options used for hedging have a maturity of 0.5 years and Black and Scholes prices are used for the analysis. The results are presented in table 6. We see that – just as in the univariate case – the performance of naive delta-gamma hedging is better. For $n = 5$, the CVaR of the hedged loss can be reduced from 10.4 to 7.8. For $n = 10$ funds, we also observe a significant reduction (9.1 versus 7.0). Thus, gamma hedging still adds value for the multivariate case.

6 Robustness and Further Empirical Evidence

6.1 Bootstrapping and Out-of-Sample Analysis

In this section, I check the robustness and provide further empirical evidence for selected results. Firstly, I test whether selected results also hold true when the data is not generated based on multivariate geometric Brownian motions. Instead of using a more advanced model, I decided to use a block-bootstrap approach to generate return data without making any distributional assumptions. Notice that Augustyniak and Boudreault (2012) also use a block-bootstrap approach for the evaluation of their hedging strategy in a similar context, but without basis risk.[16] I use an approach that I call "*simultaneous block-bootstrapping*" to generate multivariate sample paths based on the data sets described in sections 4.1 and 5.2. Therefore, data-blocks are drawn *simultaneously* with replacement from the n return time series. I use a block length of 6 months, but the results are similar for other block lengths. The bootstrap approach applied here can be seen as a multivariate extension of the approach proposed by Augustyniak and Boudreault (2012). Secondly, I provide exemplary out-of-sample hedging results for the univariate and multivariate case. Thereby, I use historical return data and calculate empirical losses that would have occurred over time for standard contracts.

In this section, the standard product specifications are mostly used ($T = 10$, $P = 100$, $\kappa = 100\%$) and the risk-free rate is still 3% per year. The fair fee of section 2.1, which is calculated based on a fund volatility of 0.15 p.a., is applied. The hedging portfolios are adjusted on a monthly basis and I only consider the naive delta hedging strategy (39). Thus, even if the multivariate geometric Brownian motion is not the "true" model, I still determine the prices and hedge ratios based on these dynamics. However, this seems to be consistent with the common market practice (at least as of 2011). According to a hedging survey by *Towers Watson*, the majority of top North American VA writers use log-normal distributions for modeling equity index dynamics for pricing and hedging (Towers Watson, 2011, p. 4).

[16] Block-bootstrapping is a popular approach for the evaluation of portfolio insurance strategies (Annaert et al., 2009; Bertrand and Prigent, 2011; Dichtl et al., 2015).

Table 7
Bootstrapped data: Basis risk effect

	L_T [$x = 1$]	Hedged Loss HL_T				
		$x = 1$	$x = 0.8$	$x = 0.5$	$x = 0.3$	$x = 0$
mean	-12.8	0.7	2.2	4.6	6.6	10.4
std	10.6	2.1	3.2	6.1	8.6	13.6
$\text{CVaR}_{1\%}$	51.1	9.7	14.0	27.7	38.0	57.1

Note: The table reports characteristics of the (un-)hedged loss for different degrees of basis risk. Data is generated by simultaneous block-bootstrapping.

6.2 Standard Case

I present the results for the bootstrap and out-of-sample results for the standard case where the total premium of $P = 100$ is invested in a single fund. The bootstrap analysis is based on the data described in section 4.1. In this setting, the magnitude of basis risk can easily be controlled by the choice of x which corresponds to the weight of the S&P 500 index in the portfolio. A lower weight x means a higher magnitude of basis risk because the differences between the fund and the proxy get larger. Table 7 shows that the impact of basis risk is strong. The risk figures increase significantly when the S&P 500 fraction decreases and the correlation between the returns of the hedging instrument and the proxy gets smaller. For example, the CVaR corresponds to 9.7% of the initial premium for $x = 1$ and it corresponds to 27.7% for $x = 0.5$. If the S&P 500 index is used for hedging the variable annuity product on the MSCI World ex USA ($x = 0$), the CVaR is even larger than the CVaR for the case *without hedging* (57.1 vs. 51.1). Furthermore, the results presented in table A5 (appendix) confirm our previous result that the risk increases for higher levels of financial protection in a situation with basis risk. Without basis risk, however, the guarantee level only has a minor impact on the insurer's risk and the CVaRs for $\kappa = 100\%$ and $\kappa = 120\%$ are almost equal (9.7 versus 10.1).

The out-of-sample results are shown in figure 5. I always consider overlapping 10-year contracts that always start and mature in January. I present the empirical losses (after hedging) for several degrees of basis risk. The results confirm the strong impact of imperfect correlations between the fund and the hedging instrument. Without basis risk, the empirical losses are always close to zero with a maximum loss of 4.8 for the contract that matures in January 2009 within the financial crisis. With basis risk, we observe much stronger variations. For $x = 0$, the maximum loss equals 24.1 and would have occurred for a GMAB contract with a maturity until January 2018. Interestingly, we do not observe large hedging losses for contracts maturing

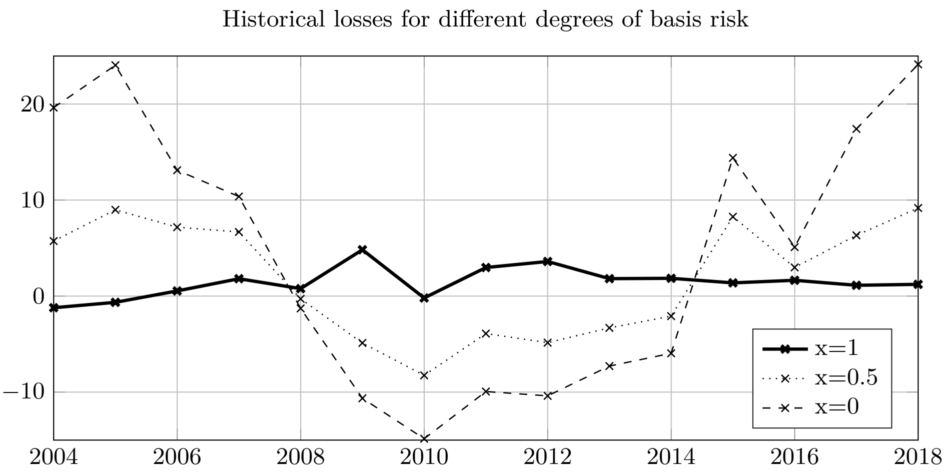

Figure 5
The figure shows empirical hedging losses for different degrees of basis risk.

during or just after the 2008 financial crisis for $x < 1$ in this empirical example.[17] Figure 8 (appendix) confirms the previous results that higher guarantee levels tend to be riskier for the insurance company.

6.3 Multivariate Case

In this section, I present the bootstrap and out-of-sample results for different degrees of diversification. More precisely, I compare the univariate case ($n = 1$) with the multivariate case, where the total premium $P = 100$ is equally distributed among ten US equity funds. Figure 6 shows the results of the bootstrap analysis. We observe that diversification adds value in eight out of ten cases. For $n = 10$, the CVaR corresponds to 14.4% of the total premium. The CVaR for $n = 1$ ranges from 8.7% of the premium for the Fidelity Contrafund to 57.1% for the White Oak Select Growth Fund offered by Oak Associates.

The out-of-sample results presented in figure 7 further confirm the benefits of diversification. If the premium is allocated among all funds, the empirical hedging errors are close to zero with a maximum loss corresponding to 4.7% of the total premium. For the univariate case, in contrast, the maximum loss corresponds to more than one third of the premium (34.1% for fund 8).

[17] Notice that this is not directly a contradiction to the statement of several authors (see introduction) that basis risk contributed to the large losses during the financial crisis. I consider aggregated empirical losses for the whole maturity of the GMAB-contracts and *not* yearly losses for a realistic "book" of VA contracts with different riders and maturities.

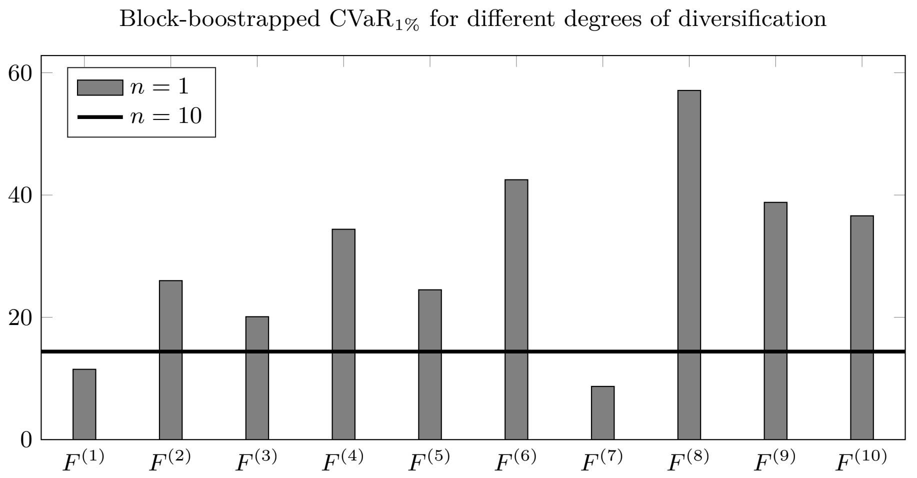

Figure 6
The figure shows the CVaRs of the hedged loss for different degrees of diversification. Data is generated by simultaneous block-bootstrapping.

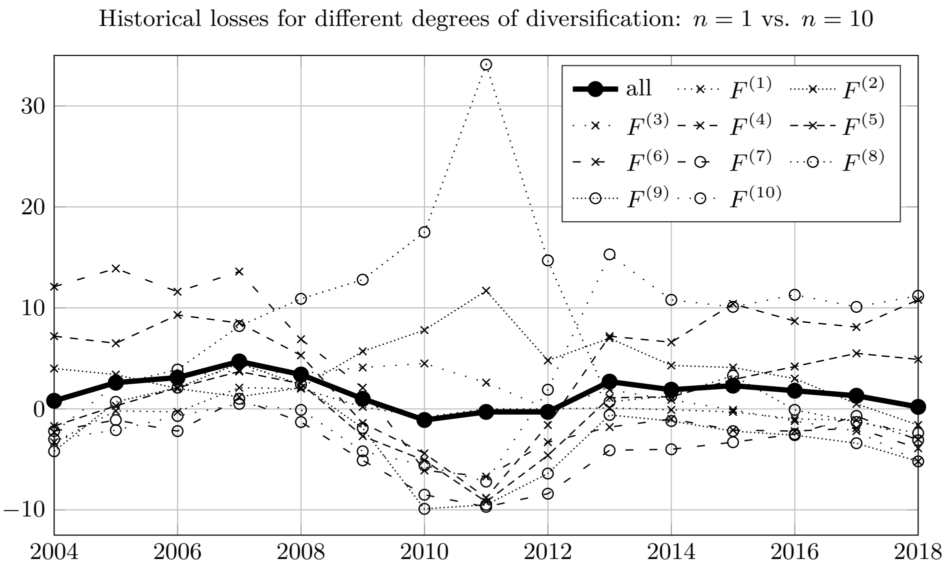

Figure 7
The figure shows empirical losses for different degrees of diversification.

Conclusion

In this paper, I studied VA hedging under basis risk. I documented surprisingly severe basis risk effects that are particularly strong for high levels of financial protection and high fund volailities. I proposed several simple hedging strategies based on the proxy asset and compared their performance with well-established approaches from mathematical finance, namely local risk minimization and mean-variance hedging. I found very similar hedging results. Furthermore, I developed a naive hedging strategy that also uses short-term plain vanilla options for delta-gamma hedging and found an improvement over naive delta-hedging. These results suggest that – from a practical point of view – using simple strategies seems to be sufficient where gamma hedging can add value. Furthermore, insurance companies could, in principle, manage basis risk by trying to avoid high levels of financial protection and by trying to restrict the fund universe to low-volatility funds.

In addition to the univariate analysis where the total premium is invested in the same mutual fund, I also considered the multivariate case. Thereby, it was assumed that the total premium is equally allocated among several funds. I observed diversification benefits, that is, the insurer's risk decreases if the number of underlying funds gets larger. I also observed higher risk for higher levels of financial protection and a superior performance of delta-gamma hedging in the multivariate case. The multivariate results suggest that it seems to be a reasonable strategy for the insurance company to try to achieve a distribution of the policyholders' premium among several funds.

The results could not only be interesting for insurance companies but also for regulators who analyze the hedge effectiveness in order to determine adequate risk capital requirements (Ruez, 2016). Further research could consider VA contracts equipped with other guarantees, study the impact of basis risk using other models for the joint dynamics of the funds and the proxy asset or include other relevant risk sources such as policyholder behavior risk or mortality/longevity risk. Additionally, the interest rate could be modeled stochastically and interest rate risk could be taken into account (Augustyniak and Boudreault, 2017). Furthermore, it would be interesting to develop and test hedging strategies for the multivariate case that take correlations between the funds and proxy assets explicitly into account.

A Proofs

Proof of Proposition 1. Let us consider the function

$$\hat{\pi}_0(\alpha) := \begin{cases} \bar{\pi}_0(\alpha) & \text{for} \quad \alpha \in [0,1) \\ e^{-rT}K - P & \text{for} \quad \alpha = 1 \end{cases} \tag{A1}$$

which is continuous on $[0,1]$. It holds that $\hat{\pi}_0(0) = \pi_{\text{put}}(0, F_0, K) > 0$ and $\hat{\pi}_0(1) = e^{-rT}K - P < 0$ due to the assumptions. Consequently, there is at least one root in $(0,1)$. Furthermore,

$$\bar{\pi}_0'(\alpha) = -Ke^{-rT}\phi(-d_2)\tfrac{\partial d_2}{\partial \alpha} + F_0\phi(-d_1)\tfrac{\partial d_1}{\partial \alpha} + F_0\Phi(-d_1) - \alpha F_0\phi(-d_1)\tfrac{\partial d_1}{\partial \alpha} - P. \tag{A2}$$

ϕ (Φ) is the density function (cumulative distribution function) for the standard normal distribution. Making use of the relations $\phi(-x) = \phi(x)$, $\frac{\partial d_1}{\partial \alpha} = \frac{\partial d_2}{\partial \alpha}$ and $\phi(d_2) = \frac{(1-\alpha)F_0}{K}e^{rT}\phi(d_1)$, which can be verified easily, it follows that

$$\bar{\pi}_0'(\alpha) = P\left(\Phi(-d_1) - 1\right) < 0 \tag{A3}$$

for $\alpha \in (0,1)$. Consequently, there is exactly one root $\alpha^* \in (0,1)$. □

Proof of Proposition 2. For the conditional variance,

$$\text{var}_t\left[\Delta_t(F_{t+h} - F_t) - \hat{x}_t(S_{t+h} - S_t)\right] \tag{A4}$$

$$= \Delta_t^2 \text{var}_t\left[F_{t+h}\right] + \hat{x}_t^2 \text{var}_t\left[S_{t+h}\right] - 2\Delta_t\hat{x}_t \,\text{cov}_t\left[S_{t+h}, F_{t+h}\right]. \tag{A5}$$

The first derivative of the function

$$f(x) := \Delta_t^2 \text{var}_t\left[F_{t+h}\right] + x^2 \text{var}_t\left[S_{t+h}\right] - 2\Delta_t x \,\text{cov}_t\left[S_{t+h}, F_{t+h}\right] \tag{A6}$$

is given by

$$f'(x) := 2\,x\,\text{var}_t(S_{t+h}) - 2\,\Delta_t\,\text{cov}_t(S_{t+h}, F_{t+h}). \tag{A7}$$

Next, for the expected value conditional on the information at time t, we get

$$\mathbb{E}_t\left[\Delta_t(F_{t+h} - F_t) - \hat{x}_t(S_{t+h} - S_t)\right] \tag{A8}$$

$$= \Delta_t \mathbb{E}_t[F_{t+h}] - \Delta_t F_t - \hat{x}_t \mathbb{E}_t[S_{t+h}] - \hat{x}_t S_t\,. \tag{A9}$$

The first derivative of the function

$$g(x) := \Delta_t\, \mathbb{E}_t[F_{t+h}] - \Delta_t F_t - x\ \mathbb{E}_t[S_{t+h}] - x\, S_t \tag{A10}$$

is given by

$$g'(x) := -\mathbb{E}_t[S_{t+h}] - S_t. \tag{A11}$$

The first derivative of the function

$$h(x) := f(x) + 2\lambda\, g(x)\,. \tag{A12}$$

is equal to

$$\begin{aligned} h'(x) = 2\,x\,\mathrm{var}_t(S_{t+h}) - 2\,\Delta(t)\,\mathrm{cov}_t(S_{t+h}, F_{t+h}) \\ - 2\lambda(\mathbb{E}_t[S_{t+h}] - S_t). \end{aligned} \tag{A13}$$

From setting $h'(x) = f'(x) + 2\lambda\, g'(x) = 0$, we get

$$\hat{x}_t^* = \Delta_t \frac{\mathrm{cov}_t[S_{t+h}, F_{t+h}]}{\mathrm{var}_t[S_{t+h}]} + \lambda \frac{\mathbb{E}[S_{t+h}] - S_{t+h}}{\mathrm{var}_t[S_{t+h}]}\,. \tag{A14}$$

Furthermore, $h''(x) = 2\,\mathrm{var}_t(S_{t+h}) > 0$. Thus, $\hat{x}^*$ is a minimum point. Finally,[18]

$$\mathbb{E}_t[S_{t+h}] = S_t e^{\mu_S h}\,, \tag{A15}$$

$$\mathrm{cov}_t\left[S_{t+h}, F_{t+h}\right] = S_t F_t e^{(\mu_S + \mu_F)h}\left(e^{\rho\sigma_S\sigma_F h} - 1\right), \tag{A16}$$

$$\mathrm{var}_t[S_{t+h}] = S_t^2 e^{2\mu_S h}\left(e^{\sigma_S^2 h} - 1\right). \tag{A17}$$

This finishes the proof. □

[18] See, for example, Glasserman (2004, p. 104).

B Quadratic Hedging Approaches

In this section, I present the general ideas for the two quadratic hedging approaches, namely the local risk minimization approach (lrm) and mean-variance hedging (mvh). I follow the nice overview paper Schweizer (2001) while trying to avoid technical details. These quadratic approaches are general hedging approaches for incomplete financial markets, but I look at the special case when the incompleteness is a result of basis risk. I also present the lrm- and mvh-strategies explicitly for hedging plain vanilla put options under basis risk (Hulley and McWalter, 2015). Notice that the insurer's task is to hedge $(1-\alpha^*)$ European put options with strike $\frac{\kappa P}{1-\alpha^*}$. Thus, the results of Hulley and McWalter (2015) can directly be applied for the VA contract considered here.

Let $(\Omega, \mathcal{F}, P)$ be the probability space, the time horizon is T and the filtration is given by $(\mathcal{F}_t)_{0\leq t\leq T}$ where $\mathcal{F}_t$ describes the information available at time t. I start with the standard Black and Scholes setting with the (tradeable) risky asset F, where $dF_t = \mu_F F_t\, dt + \sigma_F F_t\, dW_t$ and risk-free asset $dB_t = rB_t\, dt$. If not stated otherwise, I assume that $r = 0$ for simplicity. A contingent claim H is defined as an $\mathcal{F}_T$-measurable random variable describing a payoff at time T, for example, $H = (X - F_T)^+$. A dynamic portfolio strategy is given by the tuple $(\xi_t, \eta_t)_{0\leq t\leq T}$, where ξ_t equals the number of units of the risky asset F hold at time t and η_t is the amount invested in the bond at time t. The corresponding value at time t is $V_t = \xi_t\, F_t + \eta_t$. The dynamic portfolio strategy $(\xi_t, \eta_t)_{0\leq t\leq T}$ is called *self-financing* if the corresponding value process is given by $V_t = V_0 + \int_0^t \xi_s\, dF_s$ or, equivalently, if the cost process C with $C_t := V_t - \int_0^t \xi_s\, dF_s$ is constant. In this model framework, any contingent claim is *attainable*, that is, there exists a self-financing trading strategy with $V_T = H$ P-a.s. for any contingent claim H. In this case, the market (model) is called *complete*.

Now, consider the basis risk model introduced in section 1 with two correlated risky assets whereas only S can be traded. In this model, a European put option written on the non-traded asset F with payoff $H = (X - F_T)^+$ is in general not attainable, that is, there is no self-financing dynamic portfolio strategy with $V_T = H$. Thus, the market (model) is not complete.

The local risk minimization strategy (Föllmer and Sondermann, 1986; Schweizer, 1988; Schweizer, 1991) replicates H, that is, $V_T = H$, while the corresponding cost process is "small". More precisely, the strategy generates the payoff H while keeping "conditional variances of the hedging cost as small as possible in a local manner".[19] Given the basis risk model of section 1, Hulley and McWalter (2015, Proposition 3) derive the lrm-strategy for a plain vanilla put option with an interest rate $r > 0$. Their result is confirmed by Ankirchner and Heyne (2012, Corollary 3.3). The strategy is determined by

$$\xi_t^{\ell rm} = \rho \frac{\sigma_F}{\sigma_S} \frac{F_t}{S_t} \Delta(t, F_t) , \tag{A18a}$$

where $\xi_t^{\ell rm}$ denotes the number of the traded asset which has to be hold at time t, and

$$\Delta(t, x) = -e^{-\gamma(t-T)} \Phi[-d_1(t, x)]. \tag{A18b}$$

Thereby, Φ is the standard normal cumulative distribution function and

$$d_1(t, x) = \frac{\ln \frac{x}{X} + (r - \gamma + \frac{1}{2}\sigma_F^2)(T - t)}{\sigma_F \sqrt{T - t}}. \tag{A18c}$$

Finally,

$$\gamma = \sigma_F(\rho\, \theta_S - \theta_F) \quad \text{and} \quad \theta_x = \frac{\mu_x - r}{\sigma_x} \quad \text{for} \quad x \in \{S, Y\}. \tag{A18d}$$

The mean-variance approach goes back to Duffie and Richardson (1991) and Schweizer (1994). "In a nutshell, mean–variance hedging is the problem of approximating, with minimal mean-squared error, a given payoff by the final value of a self-financing trading strategy in a financial market."[20]

In contrast to the lrm-strategy, the resulting dynamic portfolio strategy is self-financing which means that $V_T = H$ cannot hold in general. The mean-variance strategy is such that $\mathbb{E}[(H - V_T)^2]$ is minimal where the expected value is with respect to the physical probability measure P and $V_t = v_0 + \int_0^t \xi_s \, dS_s$.

For $H = (X - F_T)^+$ and $r > 0$, the mvh-strategy is characterized by (Hulley and McWalter, 2015, Proposition 3)

$$\xi_t^{mvh} = \xi_t^{lrm} + \frac{\mu_S - r}{\sigma_S^2 e^{-rt} S_t} \cdot \left(\hat{V}_t - v - \int_0^t \xi_t^{mvh} \, d(e^{-ru} S_u) \right) , \tag{A19a}$$

[19] See Schweizer (1991); note that it is a very difficult task to make this statement precise.
[20] See Schweizer (2010).

which is the number of units of the traded asset that has to be hold at time t. Here,

$$\hat{V}_t = e^{-rt}\pi(t, F_t) \tag{A19b}$$

and

$$\pi(t, x) := Xe^{-r(T-t)}\Phi[-d_2(t, x)] - xe^{-\gamma(T-t)}\Phi[-d_1(t, x)] \tag{A19c}$$

where $d_2(t, x) = d_1(t, x) - \sqrt{T - t}$. The so-called approximation price v, which is the initial value of the self-financing mvh-strategy, is given by

$$v = \pi(0, F_0). \tag{A19d}$$

C Further Results

Table A1
Guarantee level

	guarantee level = 80%						
		Hedged Loss HL_T					
				local		quadratic	
	L_T	$\rho = 1$	n	minv	mv	lrm	mvh
mean	-3.3	0.0	0.0	-0.2	-1.2	-0.4	-0.4
std	2.9	0.2	1.7	1.6	1.8	1.6	1.3
$\text{CVaR}_{1\%}$	22.2	0.6	8.0	7.9	8.4	8.2	8.4
	100%						
				local		quadratic	
	L_T	$\rho = 1$	n	minv	mv	lrm	mvh
mean	-9.1	0.0	0.0	-0.5	-1.4	-1.0	-0.9
std	7.4	0.3	3.9	3.7	3.8	3.7	2.8
$\text{CVaR}_{1\%}$	37.0	0.9	15.6	15.3	15.5	15.6	15.3
	120%						
				local		quadratic	
	L_T	$\rho = 1$	n	minv	mv	lrm	mvh
mean	-20.1	0.0	0.1	-0.9	-1.7	-1.8	-1.5
std	17.2	0.4	7.0	6.8	6.8	6.8	5.2
$\text{CVaR}_{1\%}$	48.6	1.3	22.1	21.1	20.7	21.0	22.5

Note: The table reports characteristics of the insurer's unhedged loss L_T and hedged loss HL_T for different levels of financial protection.

Table A2
Fund volatility

	fund volatility = 10%						
		Hedged Loss HL_T					
				local		quadratic	
	L_T	$\rho = 1$	n	minv	mv	lrm	mvh
mean	-3.8	0.0	0.0	-0.2	-2.4	-0.6	-0.5
std	1.6	0.1	1.3	1.2	1.8	1.2	0.8
$\text{CVaR}_{1\%}$	9.9	0.4	5.7	5.5	5.6	5.5	4.9
	15%						
				local		quadratic	
	L_T	$\rho = 1$	n	minv	mv	lrm	mvh
mean	-9.1	0.0	0.0	-0.5	-1.4	-1.0	-0.9
std	7.4	0.3	3.9	3.7	3.8	3.7	2.8
$\text{CVaR}_{1\%}$	37.0	0.9	15.6	15.3	15.5	15.6	15.3
	20%						
				local		quadratic	
	L_T	$\rho = 1$	n	minv	mv	lrm	mvh
mean	-12.9	0.0	0.2	-0.5	-1.0	-1.0	-0.9
std	15.3	0.4	6.3	6.2	6.2	6.2	5.4
$\text{CVaR}_{1\%}$	51.4	1.5	20.9	20.6	20.6	20.8	21.7

Note: The table reports characteristics of the insurer's unhedged loss L_T and hedged loss HL_T for different fund volatilities.

Table A3
Rebalancing frequency

	daily rebalancing					
	without t.c.			with t.c. (0.2%)		
	$\rho = 1$	n	minv	$\rho = 1$	n	minv
mean	0.0	0.0	-0.5	2.5	2.6	2.0
std	0.3	3.9	3.7	1.2	4.2	4.1
$\text{CVaR}_{1\%}$	0.9	15.6	15.3	6.0	20.1	19.2

	monthly rebalancing					
	without t.c.			with t.c. (0.2%)		
	$\rho = 1$	n	minv	$\rho = 1$	n	minv
mean	0.1	0.1	-0.4	0.8	0.8	0.3
std	1.2	4.0	3.9	1.3	4.0	3.9
$\text{CVaR}_{1\%}$	4.3	15.7	15.4	5.6	16.7	16.2

	half-yearly rebalancing					
	without t.c.			with t.c. (0.2%)		
	$\rho = 1$	n	minv	$\rho = 1$	n	minv
mean	0.3	0.3	-0.1	0.7	0.7	0.2
std	3.0	4.7	4.6	3.0	4.8	4.7
$\text{CVaR}_{1\%}$	11.1	17.4	16.7	12.2	18.4	17.7

Note: The table reports characteristics of the insurer's hedged loss HL_T for different rebalancing frequencies.

Table A4
Estimated correlation matrix

	S	$F^{(1)}$	$F^{(2)}$	$F^{(3)}$	$F^{(4)}$	$F^{(5)}$	$F^{(6)}$	$F^{(7)}$	$F^{(8)}$	$F^{(9)}$	$F^{(10)}$
S	1.000	0.951	0.947	0.944	0.901	0.899	0.898	0.884	0.871	0.860	0.820
$F^{(1)}$	0.951	1.000	0.958	0.968	0.857	0.832	0.924	0.914	0.886	0.796	0.752
$F^{(2)}$	0.947	0.958	1.000	0.959	0.827	0.825	0.915	0.925	0.858	0.774	0.743
$F^{(3)}$	0.944	0.968	0.959	1.000	0.844	0.808	0.925	0.918	0.894	0.771	0.726
$F^{(4)}$	0.901	0.857	0.827	0.844	1.000	0.896	0.839	0.767	0.799	0.867	0.809
$F^{(5)}$	0.899	0.832	0.825	0.808	0.896	1.000	0.805	0.797	0.701	0.930	0.887
$F^{(6)}$	0.898	0.924	0.915	0.925	0.839	0.805	1.000	0.916	0.854	0.736	0.675
$F^{(7)}$	0.884	0.914	0.925	0.918	0.767	0.797	0.916	1.000	0.783	0.711	0.690
$F^{(8)}$	0.871	0.886	0.858	0.894	0.799	0.701	0.854	0.783	1.000	0.650	0.613
$F^{(9)}$	0.860	0.796	0.774	0.771	0.867	0.930	0.736	0.711	0.650	1.000	0.895
$F^{(10)}$	0.820	0.752	0.743	0.726	0.809	0.887	0.675	0.690	0.613	0.895	1.000

Note: The table presents the maximum likelihood estimate of the correlation matrix R for the multivariate geometric Brownian motion. See Remillard (2013, section 2.2) for details on maximum likelihood estimation for multivariate geometric Brownian motions. Monthly log-returns of the S&P 500 and the funds from 06:1993 - 06:2018 are used.

Table A5
Bootstrapped data: Guarantee level

	$x = 1$			$x = 0.5$		
	80%	100%	120%	80%	100%	120%
mean	0.5	0.7	0.1	2.0	4.6	9.0
std	1.3	2.1	2.7	3.3	6.1	11.0
$\text{CVaR}_{1\%}$	7.2	9.7	10.1	18.7	27.7	41.5

Note: The table reports characteristics of the hedged loss for different levels of financial protection. Data is generated by simultaneous block-bootstrapping ($x = 0.5$).

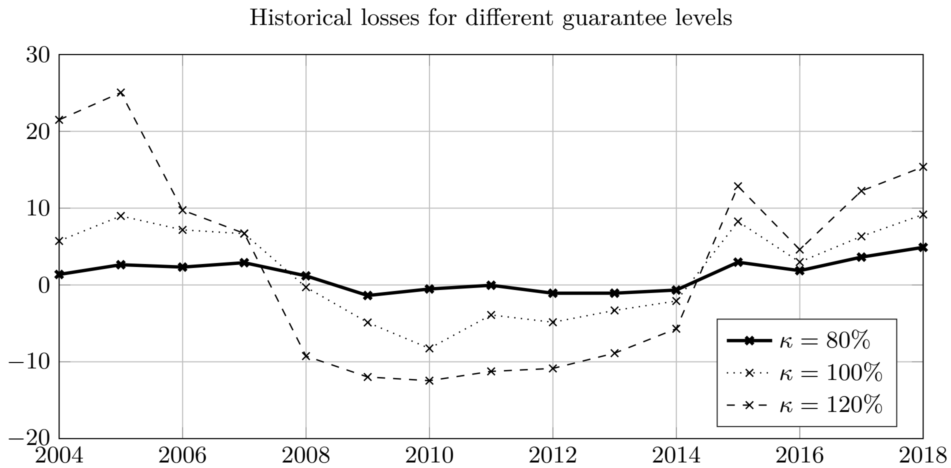

Figure 8
The figure shows empirical hedging losses for different guarantee levels ($x = 0.5$).

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