The Impact of Systematic Longevity Risk on Optimal Lifecycle Portfolio Choice with Tontines

Abstract

We derive the optimal life-cycle portfolio choice and consumption pattern for a CRRA utility maximizing investor, facing risky capital market returns, systematic mortality risk and a stochastic level of living standard. In addition to stocks and bonds, the individuals have access to tontines. Tontines are cost-efficient financial contracts providing age-increasing, but volatile cash flows, generated through the pooling of mortality without guarantees, which can help to match increasing financing needs at old ages. We find that tontines can generate significant welfare gains. We find a decreasing optimal stake in tontines over the lifecycle to smooth consumption. However, higher risk aversion reduces tontine investments and increases stock investment. Depending on the level of bequest motive, the tontine is crowded out by bequeathable assets. Furthermore, we find that the value of the tontine increases in the tontine size and risk aversion. However, since stochastic mortality affects both, expected tontine returns as well as tontine payout uncertainty, the welfare gains diminish.

Keywords: Tontines, Life Insurance, Consumption, Annuities, Mortality, Retirement Planning, Consumption-portfolio choice
JEL Classification: D14, E21, G22, I31, J10, L51

1 Introduction

Demographic change and the associated shift in the age structure of the population make it increasingly difficult to secure the funding of pay-as-you-go pension systems in many Western societies. Declining birth rates and simultaneously increasing life expectancy worldwide[1] lead to an increase in benefit recipients of the statutory pension insurance with a simultaneous decrease in contributors. The associated increase in the so-called pension ratio makes it more difficult to maintain pay-as-you-go pension systems, such as the public pension system in Germany. In return, funded pension products and private pensions are gaining in importance. This challenge of an aging society is intensified by increasing funding needs in old age[2].

The medical advances of the past decades cause that a multitude of diseases and ailments can now be cured, that would have lead to death 50 years ago[3]. However, these medical measures and treatment methods are often associated with enormous costs and increase especially in old age, when afflictions pile up. Thus, e.g. a costly, elderly-friendly conversion or expansion of the home will be necessary, which allows the longest possible and independent living in the familiar environment. Furthermore, very high costs for ambulatory and inpatient care are incurred in old age. However, specialized health insurance often depends on the level of care and includes derogations so that soft factors and uninsured aspects are not covered. These include, for example, costly items to maintain the standard of living (e.g., dependence on taxi services because of visual impairment or the use of high quality meal-on-wheels services or shopping delivery services), or the delivery of high-quality nutritional services beyond the statutory level (e.g., massages or home help).

This raises the need for a pension product that is suitable to meet the rising capital requirements at retirement age at low cost. These considerations lead to the principle of tontines, which has been adapted to current conditions to meet the requirements of the 21st century.

[1] According to the Worldbank (2015), worldwide life expectancy at birth has increased between 1960 and 2015 from 52.5 to above 71.7 years. The increasing lifetime will cause the number of people over 80 years old to almost double to 9 million in Germany by the year 2060 according to forecasts by the Statistisches Bundesamt (2015). In the future, it is therefore very probable that very high ages of 100 years and even more will be achieved by a large number of people.

[2] According to the medicalisation thesis motivated by Gruenberg (1977), the additional years that people live due to demographic change are increasingly spent in bad health condition and disability. Jagger et al. (2016) find a significant increase in life expectancy between 1991 and 2011 in England. In those additional years of life, the demand for care products and medical service increases over-proportionately. Coming from 2.6 million nursing cases in Germany in 2013, Kochskämper (2015) estimates between 1.5 and 1.9 million additional nursing cases in Germany in the year 2060 due to demographic change. By the year 2030, the demand for stationary permanent care will increase by 220,000 places in Germany. A study by Standard Life (2013) shows that an 85-year-old person on average has six times higher total spending than a comparable 65-year-old person.

[3] For example, the invention of penicillin and other antibiotics in the 1940s and 1950s or the ability of medical treatment of cardiovascular diseases in the 1960s lowered the mortality rates for all age groups.

The tontine is a financial product developed by its eponymous inventor, Lorenzo de Tonti, in the 1650s for long-term public funding of the French state. In their original form, each tontine owner received a lifelong annual pension in exchange for a one-off payment to the French State. The shares of deceased tontine members are spread among the survivors, increasing their pension payments. Payments to surviving tontinists therefore increase over time. According to this mechanism, the last survivor receives the pension payments from everyone else.

A tontine thus offers an age-increasing payout structure without the need of guarantees, which is generated by the diversification of mortality between policyholders. This peculiarity makes tontines appear extremely interesting against the backdrop of an increasing need for capital in old age, since a small amount of funds can generate very high payouts in old age. However, if systematic mortality improves, the payouts per age decline.

As a result of these developments, tontines are becoming increasingly important. McKeever (2009), Milevsky (2015) and Li and Rothschild (2017) work up the historical development of tontines, Sabin (2010), Milevsky and Salisbury (2015) and Milevsky and Salisbury (2016) focus on the actuarial fair and optimal payout structure of a tontine and Chen et al. (2017) combine a tontine and an annuity into a common product. In Weinert (2017b) the tontine is supplemented with a cancellation option and its effects on the other tontinists is determined as well as the fair cancellation amount. Weinert (2017a) estimates the cost of a tontine compared to traditional life insurance products from an economic as well as an regulatory perspective. In Weinert and Gründl (2016), tontines are studied for suitability as a complement to traditional retirement products, taking into account the aforementioned demographic challenges. The authors analyze whether a tontine is suitable for serving the increasing financial needs of elderly people. However, that article aims to find the fundamental effects of tontinization and therefore, the model assumes that individuals care about gains and losses in *wealth* in a simplified framework without capital markets. To analyze this topic with a holistic approach, we include the tontine in a standard life-cycle framework[4] and identify the optimal portfolio structure considering the level of *consumption*. We incorporate over proportionally increasing costs with age to maintain a constant level of living standard by reducing the retirement income in case of frailty. We then derive the optimal consumption, saving and portfolio choice pattern for a CRRA utility maximizing investor, facing risky capital market returns and systematic mortality risk.

Results indicate that without a bequest motive, the tontine dominates the risk free investment in any situation. However, with a bequest motive, it is optimal to replace the tontine investment

[4] See for example Horneff et al. (2010), Hubener et al. (2013), Maurer et al. (2013) and Horneff et al. (2015).

over time by a risk free investment. Furthermore, the tontine has the characteristics of a risky investment with a time changing risk return profile and a lower bound guaranteed return of the risk free rate which in every period competes against the risk return profile of a stock investment. By including systematic mortality risk, the tontine's superiority over the stock declines.

In Section 2, we introduce the life-cycle model applied to find the optimal consumption and portfolio allocation into stocks, bonds and tontines. We solve the model numerically and present the expected results in Section 3.3. We discuss the optimal life-cycle asset allocation for different tontine size, with and without bequest motive, varying initial wealth endowment and changing risk aversion.

2 The Model

We assume that an individual has access to capital markets by investing in risk-free bonds, risky stocks and risky tontines.

Utility:

We assume that the investor's preferences are given by a time-separable CRRA utility function defined over a single non-durable consumption good. Let C_t be the consumption level at time t and the recursive definition of the value function is given by

$$V_t = \frac{C_t^{1-\rho}}{1-\rho} + \beta E_t \left[p_t V_{t+1} + (1-p_t)\, b \frac{D_{t+1}^{1-\rho}}{1-\rho} \right] \tag{1}$$

where ρ is the coefficient of relative risk aversion, $\beta > 0$ is the personal time preference discount factor, b is the strength of bequest motive and D_t is the level of bequest. In our model we assume that the person considered has just reached retirement age at the beginning of the observation horizon ($t = 0$). We furthermore assume homogeneous investors and that a tontine cohort always consists of the same characteristics, meaning that they have the same age, gender and disposable income. p_t is the individual survival probability of surviving from t to $t+1$. In the final period of life at the maximum attainable age T, $p_T = 0$ and therefore, $V_T = \frac{C_T^{1-\rho}}{1-\rho} + \beta E_T \left[b \frac{D_{T+1}^{1-\rho}}{1-\rho} \right]$.

Capital Market:

The individual has access to capital markets by investing in risk-free bonds, risky stocks (denoted by $\bullet$) and risky tontines (denoted by $\circ$). The real bond gross return is constant over time and given by R_f. The real gross risky stock return at time t is denoted by $R_t^{\bullet}$. The risky stock returns are assumed to be serially independent and identically log-normally distributed with an expected return $\mu^{\bullet}$ and volatility $\sigma^{\bullet}$. In addition to bonds and stocks, at each point in time the individual can invest in a risky one-year tontine. The tontine is set up at retirement in $t = 0$ and consists of initially N_0 people. After one year, the whole tontine pool (which grows with the risk free rate) is distributed amongst the surviving participants. As long as $N_{t-1} \geq 2$ the tontine can operate and each member can decide on how much to invest in the tontine for the next year. Thus, the tontine return in t is

$$R_t^{\circ} = \begin{cases} \frac{N_{t-1}}{N_t} R_f & \text{if alive in t} \\ 0 & \text{if dead in t} \end{cases} \tag{2}$$

and $N_t \sim \text{Bin}\left(N_{t-1} - 1, p_t\right) + 1$ with expected tontine return in t

$$\mu_t^{\circ} = \begin{cases} \frac{1-(1-p_t)^{N_{t-1}}}{p_t} R_f & \text{if alive in t} \\ 0 & \text{if dead in t} \end{cases} \tag{3}$$

and variance of tontine returns in t

$$(\sigma_t^{\circ})^2 = \begin{cases} {R_f}^2 {N_{t-1}}^2 \left((1-p_t)^{\left(N_{t-1}-1\right)} {}_3F_2\left(\left\{1, 1, -\left(N_{t-1}-1\right)\right\}, \left\{2, 2\right\}, \frac{p_t}{p_t - 1}\right) - \left(\frac{1-(1-p_t)^{N_{t-1}}}{N_{t-1} p_t}\right)^2\right) & \text{if alive in t} \\ 0 & \text{if dead in t} \end{cases} \tag{4}$$

where ${}_iF_j\left(\overline{x}; \overline{y}; z\right)$ is the the generalized hypergeometric function with $\overline{x} = \{x_1, \ldots, x_i\}$, $\overline{y} = \{y_1, \ldots, y_j\}$ and z.

Retirement Income:

The individual is endowed with an initial wealth of W_0. Furthermore, he receives a constant exogenous pension income Y_t. However, the individual can be exposed to an empirically calibrated increasing liquidity need with $P = \begin{pmatrix} 0.97 & 0.03 \\ 0 & 1 \end{pmatrix}$, which lowers the pension income (and can even lead to negative pension income)

Wealth accumulation:

At the beginning of every period, the individual can spread the wealth on hand W_t across bonds B_t, stocks S_t, tontines Υ_t and consumption C_t. The budget constraint is:

$$W_t = B_t + S_t + \Upsilon_t + C_t \tag{5}$$

$B_t + S_t + \Upsilon_t$ compounds to the financial wealth. Individual disposable wealth on hand in $t+1$ is

$$W_{t+1} = B_t R_f + S_t R_{t+1}^{\bullet} + \Upsilon_t R_{t+1}^{\circ} + Y_{t+1}. \tag{6}$$

$B_tR_f + S_tR^{\bullet}_{t+1} + \Upsilon_tR^{\circ}_{t+1}$ describes the value of financial wealth in $t+1$ and Y_{t+1} is the retirement income. Short selling is not allowed, thus

$$B_t, S_t, \Upsilon_t \geq 0. \tag{7}$$

Since in case of death, the tontine investment goes to the tontine pool and not to the heirs, the bequest in $t+1$ is

$$D_{t+1} = B_tR_f + S_tR^{\bullet}_{t+1}. \tag{8}$$

Mortality dynamics:

We employ the Cairns et al. (2006) two-factor model for stochastic mortality, where the logits of the conditional 1-year mortality rates (q^t_x) for an individual aged x in year t are presumed to evolve according to

$$\text{logit}\left(q^t_x\right) = \kappa^t_1 + \kappa^t_2\left(x - \bar{x}\right) \tag{9}$$

where $\bar{x} = \left(x_u - x_l + 1\right)^{-1}\sum_{x=x_l}^{x_u} x$ is the mean of the range of ages considered to be fitted, κ^t_1 is the 'level' of mortality and usually has a downwards trend, reflecting generally improving mortality rates over time and κ^t_2 is the 'slope' coefficient, which has a gradual upwards drift, reflecting the fact that, historically, mortality at high ages has improved at a slower rate than at younger ages. Using OLS regression, we calibrate this model to the Human Mortality Database for U.S. males ages 60-105 over the period 1933-2014.

We maximize Equation (1) with respect to the optimal consumption C_t subject to Equations (5)-(8).

Tontine Characteristics

The tontine generates its age-increasing return through a mortality-credit, i.e. the redistribution of the invested capital of deceased tontine members to the survivors. Figure 2a shows the expected tontine mortality credit in t ($\frac{\mu^{\circ}_t}{R_f} - 1$), i.e. the share of the tontine return resulting from the development of pure mortality. Due to the fact that on average more and more members die with advancing age, the expected mortality credit increases with the age of the tontine members. The mortality credit per unit invested amounts to 1.3% at the age of 65 and increases at an accelerating rate to more than 121% at the age of 105. However, if the tontine consists of only very few members, the expected mortality credit decreases. If one considers the extreme case with only one member, it becomes clear that in this case no redistribution of assets can take place yielding to an expected mortality credit of 0% for any age. However, since the realized tontine development is unknown in advance, the realized mortality credit is volatile. Figure 1 shows the simulated development of tontinists over time.

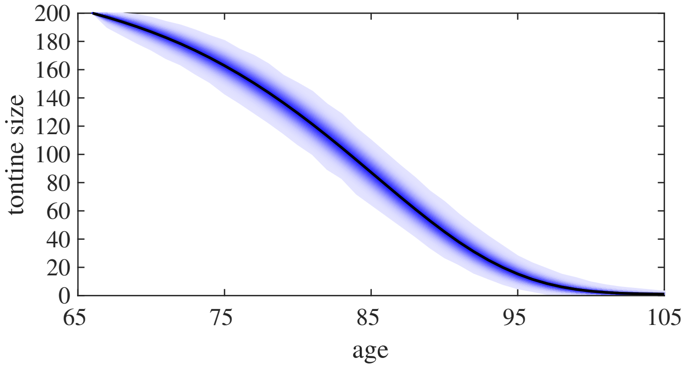

Figure 1: Simulated development of the tontine members of $N_0 = 200$ initial members aged 65 (99.95%:0.05%), 10,000 simulated paths. Darker areas represent higher probability mass.

Figure 2b shows the standard deviation of the tontine mortality credit in t ($\frac{\sigma_t^\circ}{R_f}$). The older the tontine members are, the more the realized tontine size fluctuates, which leads to a higher volatility of the mortality credit with increasing age of the tontine members. Due to the death of tontine members, however, the number of remaining tontine members also decreases from period to period. The smaller the tontine size, the more volatile the mortality credits. The standard deviation of the mortality credit for a tontine size of 50 members aged 65 is only 1.66% and increases to 11.31% for the smallest possible tontine size of 2 persons. If the tontine only consisted of one person, payments in the event of survival in the amount of the risk-free interest-bearing investment would be secure for all ages. The standard deviation of the mortality credit is greatest for a tontine consisting of 6 members with the age of 105 and decreases to 35.55% for a tontine size of 50 members at that age.

Due to the decreasing number of tontine members over time, situations can arise in which there are not enough members in a tontine founded at the beginning of retirement ($t = 0$ at the age of 65) and the tontine therefore expires. This is shown in Table 1. For a small initial tontine size of $N_0 = 200$ at the age of 65, there is a noteworthy probability that the tontine will run out of people. While the probability that the tontine has insufficient members is only 1% at the age of 96, it is 92.56% at the age of 105. However, it we consider a medium sized tontine consisting of initially $N_0 = 1,000$ members at the age of 65, the probability that the tontine has insufficient members decreases for all ages and diminishes to 65.66% at the age of 105. For a large tontine of initially $N_0 = 10,000$ at the age of 65, the probability that the tontine will still exist in the final periods is 100% and only in the very last period at the age of 105, the probability that the tontine will not exist because it has run out of members is only 1.72%.

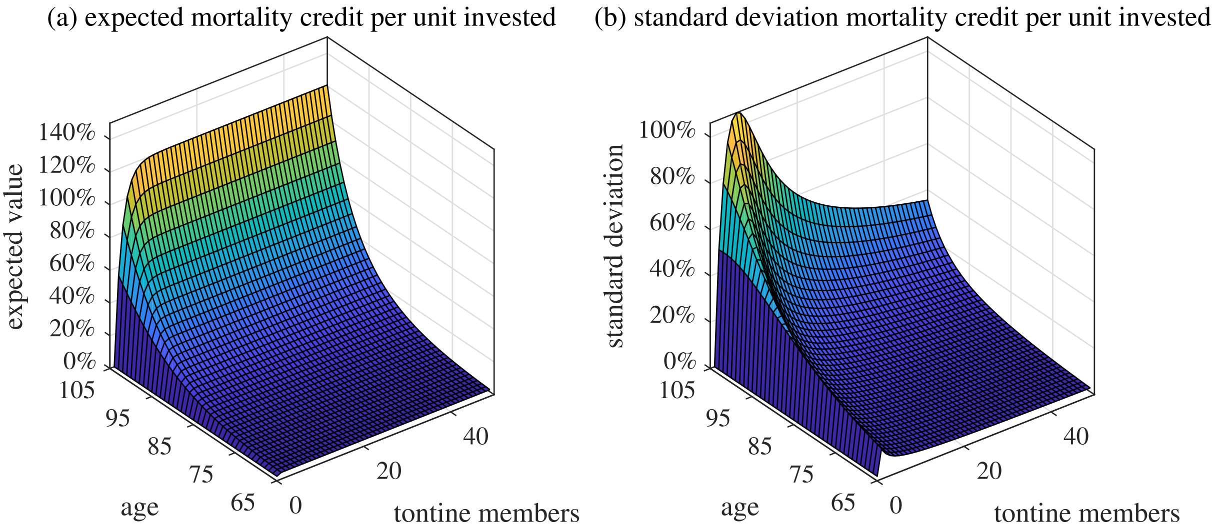

Figure 2: moments of the tontine

Initial tontine size	$Pr(N_t = 1)$ at the age of									
at the age of 65	96	97	98	99	100	101	102	103	104	105
$N_0 = 200$	0.01	0.06	0.54	3.15	11.45	28.24	50.32	69.04	83.25	92.56
$N_0 = 1,000$	0.00	0.00	0.00	0.00	0.01	0.24	2.85	15.74	39.71	65.66
$N_0 = 10,000$	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	1.72

Table 1: Probability that the tontine consists of only one member at t depending on tontine size N_0 and age

Calibration

For the base case calibration of the model, we set the degree of risk aversion to $\rho = 4$. Following Weinert and Gründl (2016), we set the subjective discount factor to $\beta = 0.98$. We refrain from an accumulation phase and begin the analysis at retirement age 65 ($t = 0$) with an initial wealth endowment of $w_0 = 150,000$ EUR. We assume a maximum attainable age of 105 ($T = 40$). The individual receives a constant yearly retirement income of $Y_t = 24,000$ EUR. However, with probability 0.03, the individual faces an increased liquidity need, which depends on the age of the individual. In this case, the liquidity need is 12,000 EUR at age 66 ($t = 1$) and linearly increases up to 30,000 EUR at age of 105 ($T = 40$). Once the individual faces a higher liquidity need, this state will remain until death. Furthermore, the risk free bond return is $R_f = 1.02$, the expected stock return is $\mu_S = 0.06$ and the volatility of stock returns is $\sigma_S = 0.18$ as in Horneff et al. (2010). We assume a large sized tontine of $N_0 = 10,000$ in the base case scenario. Furthermore we set the bequest parameter $b = 1$. The risk aversion parameter of the bequest motive is set according to the corresponding risk aversion parameter. We assume no correlation between stock returns and income shocks. Furthermore, we vary risk aversion to $\gamma = 1$ (low) and $\gamma = 8$ (high), consider a no bequest ($b = 0$) and high bequest ($b = 3$) situation and further consider a small ($N_0 = 200$) initial tontine size. We use 10,000 realizations of stock and tontine returns for the expected utility determination and use 10,000 realizations to simulate the optimal policies.

3 Results

3.1 Consumption Smoothing with the Natural Tontine

As Milevsky and Salisbury (2015) and Sabin (2010) show, it is optimal to design the tontine in such a way that its payment profile reflects that of a constant annuity. However, if the optimal fractions in the investment portfolio can be adjusted flexible over time, no adjustment of the tontine itself is required. Then it is possible to use natural tontine with its age increasing payout structure to smooth consumption. Figure 3 shows this situation. The fractions in the tontine are reduced over time (Figure 3a), as the return of a unit invested in the tontine increases. In this way, a constant payment profile can be generated over time (Figure 3b). This is more flexible than smoothing the payment profile of a tontine itself, as it enables the individual to respond optimally to individual needs, such as increasing liquidity requirements in old age. Therefore, investment and consumption decisions can be adjusted accordingly in each period.

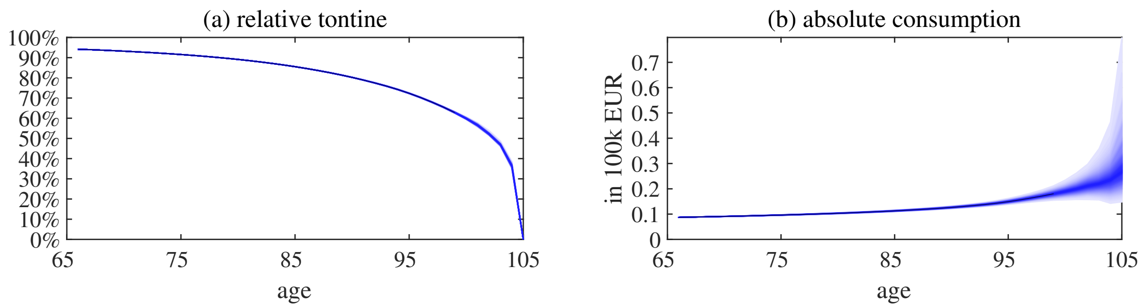

Figure 3: Large initial tontine size ($N_0 = 10,000$), with bequest motive ($b = 1$), initial wealth endowment $W_0 = 150,000$ EUR, medium CRRA risk aversion parameter ($\rho = 4$)

3.2 Optimal Lifecycle Profile

First we consider the base case calibration of the model. This is described by a large tontine ($N_0 = 10,000$), a bequest motive ($b = 1$) and a medium risk aversion parameter ($\rho = 4$). Furthermore, the initial wealth endowment at the age of 65 is $W_0 = 150,000$ EUR. Figure 4a describes the absolute wealth over time and the distribution of it to the tontine, the bond, the stock and consumption. Wealth increases up to the age of 84 and then falls again. Figure 4b shows the fractions of wealth which are consumed and saved by investing in the capital market. It can be seen that the optimal share of consumption increases over the pension period and in the last period half is consumed and invested each. This is because the utility of bequest is as large as the utility of consumption in the final period. Figure 4c shows the optimal portfolio composition over time, i.e. the composition of savings. At the age of 66, the investor optimally invests 66.48% in tontine, 33.52% in the stock and nothing in bonds. The reason is that the tontine dominates the bond in terms of the risk return profile while the bequest motive exerts only a minor influence in the first retirement years. By the age of 81, the tontine investment rises to 82.59%. The reason for this is the increasing expected return of the tontine with age and the lower bound of returns of the tontine, in contrast to the stock. The proportion in the tontine is then reduced further and finally falls to 0% at the age of 105 due to the increasing influence of the bequest motive. Since the investment in the tontine goes to the other tontine members in the event of death, and not to the heirs, and the probability of dying increases with increasing age, the tontine investment is reduced further and further. Since the model assumes a certain death at the age of 105, it is optimal not to invest anything in the tontine at this age. From the age of 81, the optimal proportion in the stock increases again and finally reaches 45.08% at the age of 105. At the same time there is also a shift into risk-free bonds, which have a proportion of 54.92% at the age of 105. The reason for this is the risk aversion, where less risky alternatives are preferred.

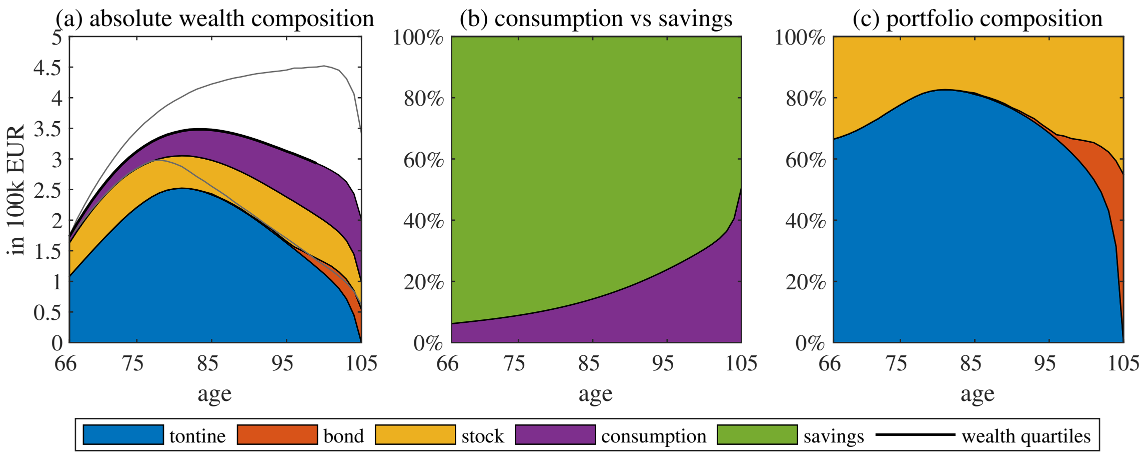

Figure 4: Large initial tontine size ($N_0 = 10,000$), with bequest motive ($b = 1$), initial wealth endowment $W_0 = 150,000$ EUR, medium CRRA risk aversion parameter ($\rho = 4$)

Figure 5 shows a variation of the base case without bequest motive ($b = 0$). As in the base case, it is optimal to initially invest 67.01% in the tontine and to slowly increase its share with age. However, because the investor does not attach any benefit to bequest, it is optimal to increase the proportion in the tontine to 100% at the age of 80 and to leave it constant until death (Figure 5c). Due to the more favorable risk/return profile of the tontine, compared to that of the stock over the lifetime, the volatility in wealth is lower than in the base case. Due to the lack of bequest motives, the available wealth is also used up faster and finally completely consumed (Figure 5a).

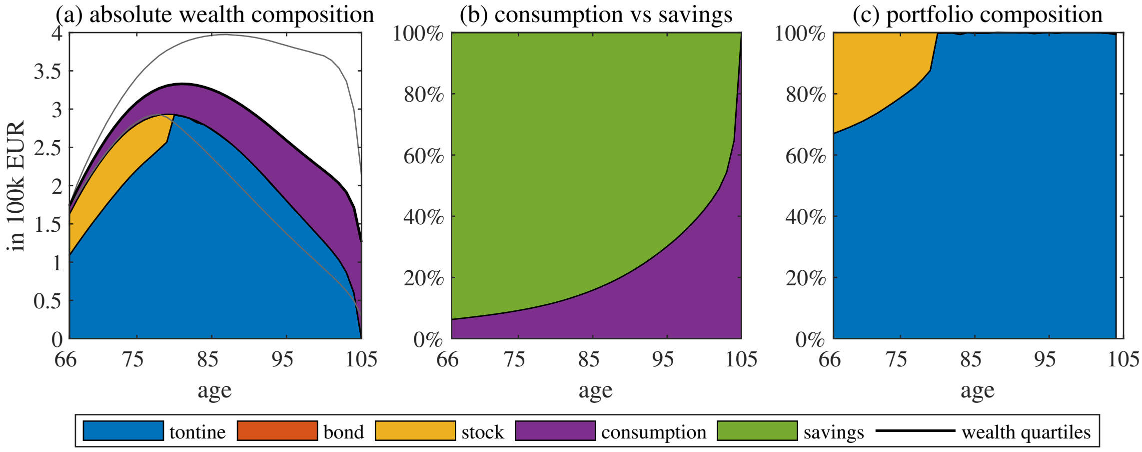

Figure 5: Large initial tontine size ($N_0 = 10,000$), without bequest motive ($b = 0$), initial wealth endowment $W_0 = 150,000$ EUR, medium CRRA risk aversion parameter ($\rho = 4$)

Figure 6 describes the same situation as in the previous Figure 5, with the difference that the tontine now initially consists of only $N_0 = 200$ members. This means that the tontine is more likely to consist of too few members in old age to be able to continue. As a result, the investor must switch to stocks and bonds (Figure 6c) in old ages. The sharply increasing volatility of tontine payments with a small tontine size is particularly noticeable in old age, which increases the volatility of the available wealth (Figure 6a) and the optimal consumption and investment policy is more subject to fluctuations (Figure 6b).

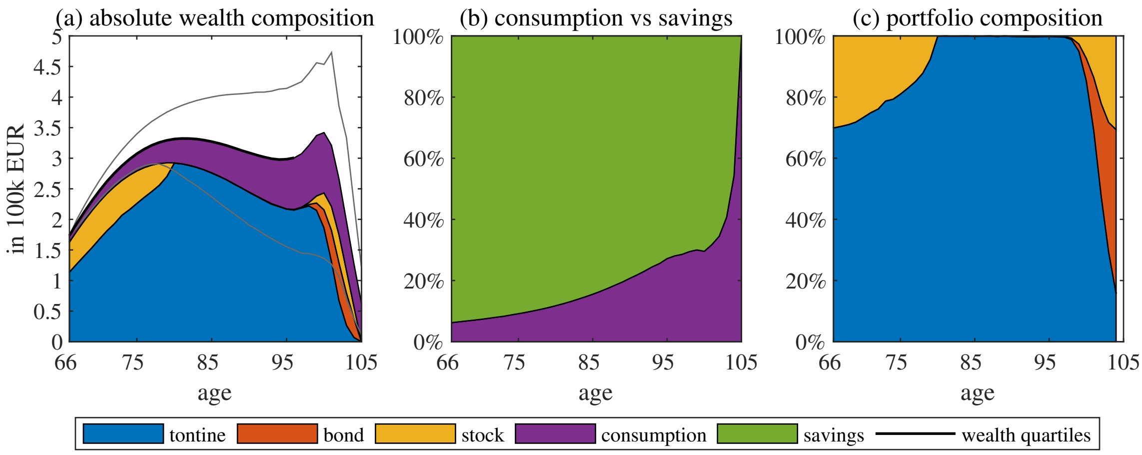

Figure 6: Small initial tontine size ($N_0 = 200$), without bequest motive ($b = 0$), initial wealth endowment $W_0 = 150,000$ EUR, medium CRRA risk aversion parameter ($\rho = 4$)

Figure 7 differs from the base case by a low risk aversion ($\rho = 1$). This results in a much riskier optimal investment policy. Thus the investor invests the entire savings in the stock at the beginning of retirement. As the tontine return increases with age, the proportion in the tontine rises sharply to 69.97% at the age of 88 before the influence of bequest causes the tontine proportion to drop back to 0% at the age of 105 (Figure 7c). However, such a risky

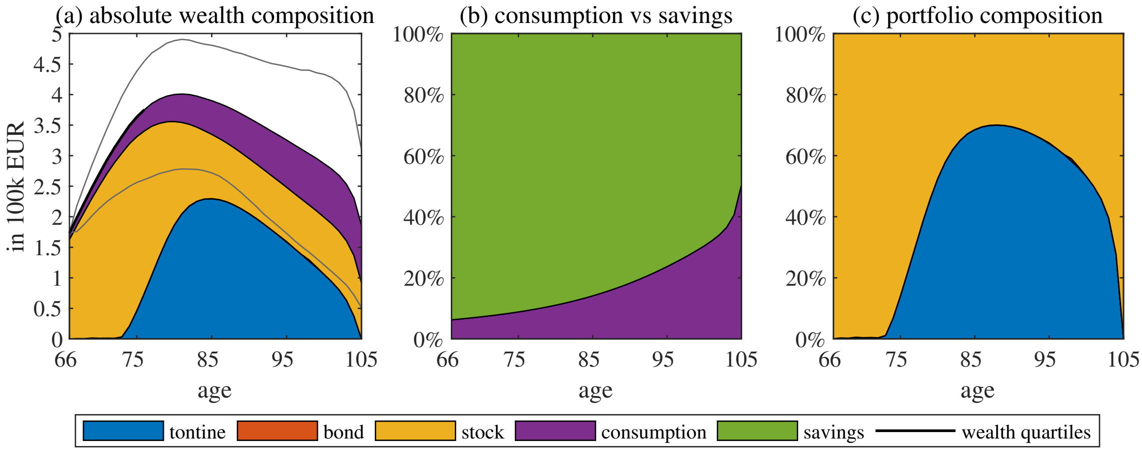

Figure 7: Large initial tontine size ($N_0 = 10,000$), with bequest motive ($b = 1$), initial wealth endowment $W_0 = 150,000$ EUR, low CRRA risk aversion parameter ($\rho = 1$)

investment strategy can lead to problems. As Table 2 shows, the lower the risk aversion is and the smaller the original tontine size is in old age, the higher the individual probability of ruin. This is amplified by the stochastic medical costs in old age, which exert unforeseen shocks on assets.

	$\rho = 1$		$\rho = 2$	
age	104	105	104	105
$N_0 = 200$	13.99%	43.38%	0.59%	7.11%
$N_0 = 300$	6.58%	33.09%	0.37%	4.37%
$N_0 = 400$	4.03%	25.76%	0.09%	2.74%
$N_0 = 500$	2.29%	19.51%	0.01%	1.02%

Table 2: Individual ruin probability depending on tontine size and risk aversion ρ

However, if the risk aversion is high ($\rho = 8$), the investor behaves very conservatively and invests a relatively constant small proportion of around 20% in the risky stock (Figure 8). In the case of high risk aversion, the influence of the bequest on the tontine investment can already be seen from an age of 76 and the portions in the tontine are substituted by the bond, which are not affected by the bequest.

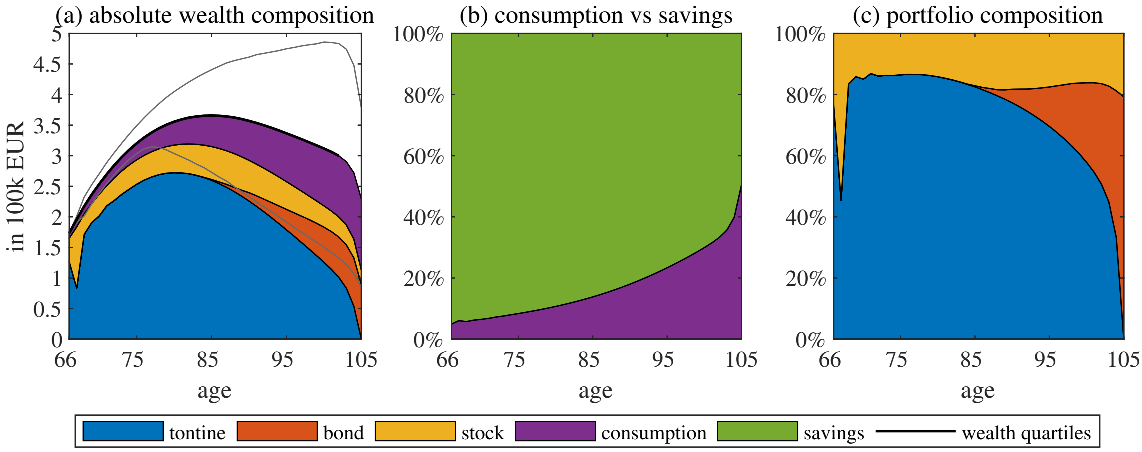

Figure 8: Large initial tontine size ($N_0 = 10,000$), with bequest motive ($b = 1$), initial wealth endowment $W_0 = 150,000$ EUR, high CRRA risk aversion parameter ($\rho = 8$)

3.3 Tontine Equivalent Wealth

The tontine equivalent wealth (TEW) describes the amount of initial wealth endowment W_0 required at the start of retirement in $t = 0$ that would give the same expected lifetime utility if no tontine existed. TEW increases in the initial tontine size N_0. This is because a higher initial tontine size implies lower volatility of tontine payments, which is valued by risk averse investors. TEW increases in the risk aversion ρ. This is because the tontine has a more favorable risk/return profile than the risky stock. The more risk averse the investor is, the more the absence of a tontine hurts in terms of risk. If expected stock return increases and the stock volatility decreases, then the TEW increases less strongly in risk aversion.

TEW in % of W_0	$N_0 = 10,000$			$N_0 = 200$		
	$b = 0$	$b = 1$	$b = 3$	$b = 0$	$b = 1$	$b = 3$
$\rho = 1$	106.57%	104.22%	102.47%	106.42%	104.17%	102.42%
$\rho = 4$	186.74%	188.89%	188.45%	184.84%	187.57%	187.13%

Table 3: Tontine equivalent wealth (TEW) subject to initial tontine size N_0, bequest motive b and risk aversion ρ

If the risk aversion is low, then the TEW is decreasing in the bequest b. The reason is, that the tontine does not contribute to the utility of bequest. However, for medium risk aversion, the TEW first increases in the bequest and decreases as the bequest motive becomes stronger. This results from two counteracting effects. The favorable risk return profile of the tontine allows to accumulate more wealth which can partially be bequested in future periods. This has a positive impact on the TEW. However, the stronger the bequest motive, the less suitable is the tontine because it does not contribute to the bequest. As a result, a moderate bequest motive leads to an increase in the TEW, while a strong bequest motive reduces the TEW (Table 3).

4 Conclusion

We derive the optimal life-cycle portfolio choice and consumption pattern for a CRRA utility maximizing investor, facing risky capital market returns, risky mortality risk and a stochastic level of living standard. In addition to stocks and bonds, the individual has access to tontines. Tontines are cost-efficient financial contracts providing age-increasing, but volatile cash flows, generated through the pooling of mortality without guarantees, which can help to match increasing financing needs at old ages. We find that it is not necessary to replicate a constant annuity payout pattern. Instead, the tontine holder can reproduce this optimal payout patter via capital markets even using natural tontines, providing an age increasing payout. We find that the availability of tontines can generate significant welfare gains compared to a world without tontines. We find a decreasing optimal relative stake in tontines over the lifecycle to smooth consumption. Higher risk aversion increases tontine investments and decreases stock investment. Depending on the level of bequest motive, the tontine is crowded out by bequestable assets. Furthermore, we find that the value of the tontine increases in the tontine size and risk aversion.

References

Cairns, A. J., Blake, D., and Dowd, K. (2006). A two-factor model for stochastic mortality with parameter uncertainty: theory and calibration. *Journal of Risk and Insurance*, 73(4):687–718.

Chen, A., Hieber, P., and Klein, J. (2017). Tonuity: A novel individual-oriented retirement plan. *Working paper, available at SSRN 3043013*.

Gruenberg, E. M. (1977). The failures of success. *The Milbank Memorial Fund Quarterly. Health and Society*, 55(1):3–24.

Horneff, V., Maurer, R., Mitchell, O. S., and Rogalla, R. (2015). Optimal life cycle portfolio choice with variable annuities offering liquidity and investment downside protection. *Insurance: Mathematics and Economics*, 63:91–107.

Horneff, W., Maurer, R., and Rogalla, R. (2010). Dynamic portfolio choice with deferred annuities. *Journal of Banking & Finance*, 34(11):2652–2664.

Hubener, A., Maurer, R., and Rogalla, R. (2013). Optimal portfolio choice with annuities and life insurance for retired couples. *Review of Finance*, 18(1):147–188.

Jagger, C., Matthews, F. E., Wohland, P., Fouweather, T., Stephan, B. C., Robinson, L., Arthur, A., Brayne, C., Function, M. R. C. C., Collaboration, A., et al. (2016). A comparison of health expectancies over two decades in England: results of the cognitive function and ageing study i and ii. *The Lancet*, 387(10020):779–786.

Kochskämper, Susanna und Pimpertz, J. (2015). Herausforderungen an diePflegeinfrastruktur. *IW-Trends Vierteljahresschrift zur empirischen Wirtschaftsforschung*, 42(3).

Li, Y. and Rothschild, C. (2017). Adverse selection and redistribution in the irish tontines of 1773, 1775 and 1777. *Working Paper*.

Maurer, R., Mitchell, O. S., Rogalla, R., and Kartashov, V. (2013). Lifecycle portfolio choice with systematic longevity risk and variable investment-linked deferred annuities. *Journal of Risk and Insurance*, 80(3):649–676.

McKeever, K. (2009). Short history of tontines, a. *Fordham J. Corp. & Fin. L.*, 15:491.

Milevsky, M. A. (2015). *King William's tontine: why the retirement annuity of the future should resemble its past*. Cambridge University Press.

Milevsky, M. A. and Salisbury, T. S. (2015). Optimal retirement income tontines. *Insurance: Mathematics and Economics*, 64:91–105.

Milevsky, M. A. and Salisbury, T. S. (2016). Equitable retirement income tontines: Mixing cohorts without discriminating. *ASTIN Bulletin: The Journal of the IAA*, 46(3):571–604.

Sabin, M. J. (2010). Fair tontine annuity. *Social Science Research Network Working Paper Series*.

Standard Life (2013). The retirement smile. `http://ukgroup.standardlife.com/content/news/new_articles/2013/260613NewDrawdownTransferOption.xml`.

Statistisches Bundesamt (2015). Bevölkerung Deutschlands bis 2060: 13. koordinierte Bevölkerungsvorausberechnung. `https://www.destatis.de/DE/Publikationen/Thematisch/Bevoelkerung/VorausberechnungBevoelkerung/BevoelkerungDeutschland2060Presse5124204159004.pdf?__blob=publicationFile`.

Weinert, J.-H. (2017a). Comparing the cost of a tontine with a tontine replicating annuity. *ICIR Working Paper Series*, (31/2017).

Weinert, J.-H. (2017b). The fair surrender value of a tontine. *ICIR Working Paper Series*, (26/2017).

Weinert, J.-H. and Gründl, H. (2016). The modern tontine: An innovative instrument for longevity risk management in an aging society. *ICIR Working Paper Series*, (22/2016).

Worldbank (2015). Databank. Life expectancy at birth. `http://data.worldbank.org/indicator/SP.DYN.LE00.IN`.

A Appendix

A.1 Proofs

A.1.1 Proof of the expected tontine return

The expected tontine return in t based on the information available in $t-1$ if the individual is alive in t is

$$\mu_t^\circ = E_{t-1}\left(\frac{N_{t-1}}{N_t}R_f\right) = R_f N_{t-1} E_{t-1}\left(\frac{1}{\hat{N}_t+1}\right)$$

where $N_t \sim \text{Bin}\left(N_{t-1}-1, p_t\right)+1$ and $\hat{N}_t \sim \text{Bin}\left(\hat{N}_{t-1}, p_t\right)$ with $\hat{N}_{t-1} = N_{t-1}-1$.

$$\begin{aligned}
\mu_t^\circ &= R_f N_{t-1} \sum_{k=0}^{\hat{N}_{t-1}} \frac{1}{1+k}\binom{\hat{N}_{t-1}}{k} {p_t}^k \left(1-p_t\right)^{\left(\hat{N}_{t-1}-k\right)} \\
&= R_f N_{t-1} \frac{1}{\left(\hat{N}_{t-1}+1\right)p_t} \sum_{k=0}^{\hat{N}_{t-1}} \binom{\hat{N}_{t-1}+1}{k+1} {p_t}^{(k+1)} \left(1-p_t\right)^{\left(\hat{N}_{t-1}-k\right)}.
\end{aligned}$$

By substituting $l=k+1$

$$\begin{aligned}
\mu_t^\circ &= R_f N_{t-1} \frac{1}{\left(\hat{N}_{t-1}+1\right)p_t} \sum_{l=1}^{\hat{N}_{t-1}} \binom{\hat{N}_{t-1}+1}{l} {p_t}^l \left(1-p_t\right)^{\left(\hat{N}_{t-1}-l+1\right)} \\
&= R_f N_{t-1} \frac{1}{\left(\hat{N}_{t-1}+1\right)p_t} \left[\sum_{l=0}^{\hat{N}_{t-1}+1} \binom{\hat{N}_{t-1}+1}{l} {p_t}^l \left(1-p_t\right)^{\left(\hat{N}_{t-1}-l+1\right)} - \left(1-p_t\right)^{\left(\hat{N}_{t-1}+1\right)}\right] \\
&= R_f N_{t-1} \frac{1}{\left(\hat{N}_{t-1}+1\right)p_t} \left[\left(p_t+1-p_t\right)^{\left(\hat{N}_{t-1}+1\right)} - \left(1-p_t\right)^{\left(\hat{N}_{t-1}+1\right)}\right] \\
&= R_f N_{t-1} \frac{1}{\left(\hat{N}_{t-1}+1\right)p_t} \left[1 - \left(1-p_t\right)^{\left(\hat{N}_{t-1}+1\right)}\right] \\
&= R_f N_{t-1} \frac{1-\left(1-p_t\right)^{\left(N_{t-1}-1+1\right)}}{\left(N_{t-1}-1+1\right)p_t} \\
&= \frac{1-\left(1-p_t\right)^{N_{t-1}}}{p_t} R_f.
\end{aligned}$$

A.1.2 Proof of the tontine volatility

The variance of tontine return in t based on the information available in $t-1$ if the individual is alive in t is

$$\left(\sigma_t^\circ\right)^2 = Var\left(\frac{N_{t-1}}{N_t}R_f\right) = {R_f}^2 {N_{t-1}}^2 Var\left(\frac{1}{\hat{N}_t+1}\right)$$

where $N_t \sim \text{Bin}\left(N_{t-1}-1, p_t\right)+1$ and $\hat{N}_t \sim \text{Bin}\left(\hat{N}_{t-1}, p_t\right)$ with $\hat{N}_{t-1} = N_{t-1}-1$. From the linearity of expected values and the definition of variance follows

$$
\begin{aligned}
(\sigma_t^\circ)^2 &= {R_f}^2 {N_{t-1}}^2 \left(E\left(\frac{1}{\left(\hat{N}_t+1\right)^2} \right) - E\left(\frac{1}{\hat{N}_t+1} \right)^2 \right) \\
&= {R_f}^2 {N_{t-1}}^2 \left(E\left(\frac{1}{\left(\hat{N}_t+1\right)^2} \right) - \left(\frac{1-(1-p_t)^{\hat{N}_{t-1}+1}}{\left(\hat{N}_{t-1}+1\right) p_t} \right)^2 \right).
\end{aligned}
$$

$E\left(\frac{1}{\left(\hat{N}_t+1\right)^2}\right)$ can be written as

$$
\begin{aligned}
E\left(\frac{1}{\left(\hat{N}_t+1\right)^2} \right) &= \sum_{k=0}^{\hat{N}_{t-1}} \frac{1}{(1+k)^2} \binom{\hat{N}_{t-1}}{k} {p_t}^k (1-p_t)^{\left(\hat{N}_{t-1}-k\right)} \\
&= \sum_{k=0}^{\hat{N}_{t-1}} \frac{k+2}{k+1} \frac{1}{\hat{N}_{t-1}+1} \frac{1}{\hat{N}_{t-1}+2} \binom{\hat{N}_{t-1}+2}{k+2} {p_t}^k (1-p_t)^{\left(\hat{N}_{t-1}-k\right)} \\
&= (1-p_t)^{\hat{N}_{t-1}} {}_iF_j\left(\overline{x};\, \overline{y};\, z\right)
\end{aligned}
$$

where ${}_iF_j\left(\overline{x};\, \overline{y};\, z\right)$ is the the generalized hypergeometric function with $\overline{x} = \{x_1, \ldots, x_i\} = \{1, 1, -\left(N_{t-1}-1\right)\}$, $\overline{y} = \{y_1, \ldots, y_j\} = \{2, 2\}$ and $z = \frac{p_t}{p_t-1}$. Finally,

$$
(\sigma_t^\circ)^2 = {R_f}^2 {N_{t-1}}^2 \left((1-p_t)^{(N_{t-1}-1)} {}_3F_2\left(1, 1, -\left(N_{t-1}-1\right);\, 2, 2;\, \frac{p_t}{p_t-1}\right) - \left(\frac{1-(1-p_t)^{N_{t-1}}}{N_{t-1} p_t} \right)^2 \right).
$$