The Optimal Insurance Level Without the Expected-Utility Framework

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Abstract

This article investigates the problem of the optimal insurance level without the expected-utility paradigm. For the class of insurance policy whose maximum coverage fully covers the potential loss, it is shown that (i) the first part of Mossin's Theorem (full insurance with a fair premium) holds under risk aversion and (ii) the second part of Mossin's Theorem (partial insurance with an unfair premium) needs to be modified: the optimal insurance level can be partial insurance or full insurance, depending on the asymptotic orders of the mean unreimbursed loss and the risk premium. Beyond this general result, further study is made for three specific types of policies: coinsurance, deductible insurance and upper-limit insurance. It is also demonstrated that the results derived without the expected-utility paradigm are conducive to further understanding of Mossin's Theorem in the expected-utility framework.

Keywords: Mossin's Theorem; non-expected-utility; unified approach; robustness; coinsurance; deductible insurance; upper-limit insurance.

1 Introduction

A cornerstone result in the theory of insurance demand states that (i) a risk-averse individual will purchase full insurance if the premium is (actuarially) fair, and (ii) he/she will purchase partial insurance if the premium is loaded. This theorem was first discovered in Mossin (1968) and is often known as *Mossin's Theorem*, though Smith (1968) also made important contribution independently. Mossin's Theorem has been established for several types of insurance contracts: coinsurance (Mossin 1968), deductible insurance (Mossin 1968, Schlesinger 1981), upper-limit insurance (Schlesinger 2006) and deductible insurance with an endogenous upper limit (Zhou et al. 2010)[1] . All these authors treated Mossin's Theorem for a specific type of insurance contract. Recently, Hong (2018b) employed a unified approach and showed that Mossin's Theorem holds for the class of insurance policy whose maximum coverage provides full coverage of the potential loss. Since most commonly-used types of policies belong to this class, different versions of Mossin's Theorem, such as those established by Mossin (1968), Schlesinger (1981, 2006) and Zhou et al. (2010), are all special cases of this general result.

However, all the aforementioned papers confined their investigation to the expected-utility paradigm. It is natural to ask whether Mossin's Theorem holds without the expected-utility framework. Several authors have contributed to this problem. Segal and Spivak (1990) invented the notions of risk aversion of order 1 and risk-aversion of order 2. They used a discrete loss model to show that Mossin's Theorem need not hold under risk aversion of order 1. Doherty and Eeckhoudt (1995) applied dual theory (Yaari 1987) to examine the optimal insurance problem, and found that the choice of decision rule has a significant impact on the individual's choice of insurance level. Machina (1995) applied the local expected utility analysis to show that Mossin's Theorem is robust enough to hold for coinsurance and

[1] Here we refer to the type of policy where the policyholder needs to choose both a deductible level and an upper limit. That is, the type of policy considered in Zhou et al. (2010). Cummins and Mahul (2004) studied the deductible insurance with a fixed exogenous upper limit.

deductible insurance without the expected-utility paradigm. But, as Karni (1995) pointed out, the setup in Machina (1995) implies risk-aversion of order 2. Further progress was made by Schlesinger (1997) who studied coinsurance in the non-expected-utility framework. He proved that part (i) of Mossin's Theorem holds but part (ii) needs to be modified. This paper aims to make further contribution to the problem of the optimal insurance level without the expected-utility paradigm[2]. We first investigate Mossin's Theorem for the class of insurance contract whose maximum coverage fully covers the loss. In the same spirit of Schlesinger (1997), we show that part (i) of Mossin's Theorem holds under risk aversion, and that part (ii) needs to be modified. In particular, if the order of convergence of the mean unreimbursed loss is less than that of the risk premium, a risk-averse individual will buy partial insurance. Otherwise, he/she will prefer partial insurance or full insurance. In addition, we further study three commonly used types of policies: coinsurance, deductible insurance and upper-limit insurance in non-expected-utility models and establish some new results. Finally, we demonstrate that the results derived without the expected-utility framework shed further light on the optimal insurance level problem in the expected-utility framework. In particular, we show that, if the premium is loaded, a risk-averse individual may choose full insurance if his/her utility function is only concave but not necessarily twice differentiable.

The remainder of the paper is organized as follows. Section 2 gives notation and setup. In Section 3, we establish a general theorem of the optimal insurance level without the expected-utility paradigm. Then we examine three specific types of insurance contracts in Section 4. Next, in Section 5, we show that the results in Schlesinger (1997) and this paper may contribute to further understanding of Mossin's Theorem in expected-utility models.

[2]To be clear, we do not touch upon the problem of the optimal insurance design in this article. For the optimal insurance design without the expected-utility model, we refer to Gollier and Schlesinger (1996), Gollier (2013), Bernard et al. (2015) and references therein. Also, we do not treat the case where the initial wealth is random. For such an extension, we refer to Doherty and Schlesinger (1983ab), Schlesinger and Doherty (1985), Hong et al. (2011) and Hong (2018ab).

Finally, we provide some concluding remarks in Section 6.

2 Notation and setup

To set the stage, let X and w be the random loss and the deterministic initial wealth of the policyholder, respectively. Unless otherwise stated, all random quantities in this paper are defined on a given probability space $(\Omega, \mathcal{F}, P)$. An insurance level will be denoted as l. We will use the letters n and m to denote "no coverage" and "maximum coverage", respectively. We assume that $n, l, m \in \mathbb{R}^k$ for some positive integer k, where $\mathbb{R}^k$ denotes the k-dimensional Euclidean space[3]. For a given insurance level l, let $I_l(x)$ denote the indemnity function, and $P(l) = (1+\lambda)E[I_l(X)]$ denote the insurance premium, where $\lambda \geq 0$ is the premium loading factor. We assume that $I_l(x)$ is continuous and increasing in l for any given loss x, and is non-decreasing in x for a given l. Let L be the supremum of the support of X. For all $0 \leq x \leq L$ and any insurance level $l \neq m$, we assume that $0 \leq I_l(x) \leq I_m(x) = x$. To avoid bankruptcy complications, we follow Schlesinger (2006, 2013) to assume that $0 \leq X \leq L + P(m) \leq w$. The cumulative distribution function of a random variable X is denoted as $F_X(x) \equiv P[X \leq x]$ for all $x \in \mathbb{R}$. We also assume that all integrals are finite. For an insurance level l, the terminal wealth of an individual is denoted as $Y_l = w - X - P(l) + I_l(X)$. For all $a \geq 0$, we will use $(X-a)^+$ to denote the positive part of $(X-a)$, i.e. ,

$$(X-a)^+ = \begin{cases} X-a, & \text{if } X > a; \\ 0, & \text{otherwise.} \end{cases}$$

[3]If $k = 1$, the usual order is a natural order on $\mathbb{R}$. When $k \geq 2$, there is no natural order on $\mathbb{R}^k$. But we do not need such an order. In fact, our argument depends on convergence. Therefore, we only need a suitable topology on $\mathbb{R}^k$. This topology is always assumed to be the Euclidean topology on $\mathbb{R}^k$ unless otherwise stated.

$X \wedge a$ will denote the value of X truncated from above by a, that is,

$$X \wedge a = \begin{cases} X, & \text{if } X \leq a; \\ a, & \text{otherwise.} \end{cases}$$

This general framework was first employed in Hong (2018b) towards a unified treatment of different types of insurance policies. To illustrate this, consider the following example.

Example 2.1. For a coinsurance policy, we may let $n = 0$, $l = \alpha$ and $m = 1$, where α is the proportion of the loss to be indemnified. For a deductible insurance, we can take $n = L$, $l = d$ and $m = 0$, where d is the deductible level[4]. For an upper-limit insurance, we may choose $n = 0$, $l = c$, and $m = L$, where c stands for the cap for the indemnity amount. Then the indemnity functions for these three types of insurance policies are respectively

$$\begin{aligned} I_\alpha(x) &= \alpha x, \quad x \geq 0, \\ I_d(x) &= (x-d)^+ = \begin{cases} 0, & x \leq d; \\ x - d, & x > d. \end{cases}, \\ I_c(x) &= x \wedge c = \begin{cases} x, & x \leq c; \\ c, & x > c. \end{cases} \end{aligned}$$

Similarly, if a policy requires the policyholder to choose both a deductible d and an upper limit c, such as the one considered in Zhou et al. (2010), we can take $n = (L, 0)$, $l = (d, c)$ and $m = (0, L)$ in $\mathbb{R}^2$. The indemnity function is

$$I_{(d,c)}(x) = \begin{cases} 0, & x \leq d; \\ x - d, & d < x \leq c; \\ c - d, & x > c. \end{cases} .$$

The preference relation for wealth of an individual, denoted as $\succeq$, is assumed to be continuous (with respect to the topology of weak convergence on the space of all real-valued

[4]We can also take $n = -L$, $l = -d$, and $m = 0$. As we mentioned earlier, our argument does not reply on any order on the space of insurance levels.

random variables) and monotonic with respect to first-order stochastic dominance, that is, if X and Y are two random variables, then $F_X(x) \geq F_Y(x), \forall x \in \mathbb{R}$ implies $Y \succeq X$. When the individual is indifferent between X and Y, we write $X \sim Y$. If X is strictly preferred to Y, we denote it as $X \succ Y$. An individual is said to be *risk-averse* if he/she prefers any random variable X to a mean-preserving spread of X; see, for example, Rothschild and Stiglitz (1970) and Cohen (1995). For each X, the certainty equivalent of X, denoted as $CE(X)$, is a constant such that the individual is indifferent between receiving X and $CE(X)$. The *risk premium* for X is defined to be $\pi(X) \equiv E[X] - CE(X)$. In particular, if X can be decomposed as the sum of a non-random initial wealth part x and a random risk part $\tilde{z}$, the individual will be indifferent between receiving the certain amount $x - \pi(X)$ and the random amount $X = x + \tilde{z}$. If an individual is risk-averse, then his/her risk premium is always positive; see, for instance, Segal and Spivak (1990). We assume that $\pi(Y_l)$ is continuous in l.

For two real-valued functions $f(x)$ and $g(x)$ defined on $\mathbb{R}^k$ and a point $a \in \mathbb{R}^k$, the expression $f(x) = o(g(x)) \quad (x \to a)$ means $\lim_{x \to a} \dfrac{f(x)}{g(x)} = 0$. If in addition $\lim_{x \to a} f(x) = \lim_{x \to a} g(x) = 0$, this means that $f(x)$ decreases to zero faster than $g(x)$ as $x \to a$, i.e., the rate of convergence of f is faster than that of g.

3 A general theorem of the optimal insurance level

3.1 The fair premium case

We assume $\lambda = 0$ throughout Section 3.1. To motivate our general argument, we first examine two special cases. Consider a coinsurance policy. The indemnity function is $I_\alpha(x) = \alpha x$ for all $x \geq 0$, and the insurance premium equals $P(\alpha) = \alpha E[X]$. For an insurance level

$0 \leq \alpha \leq 1$, the terminal wealth of an individual is given by

$$
\begin{aligned}
Y_\alpha &= w - X - P(\alpha) + I_\alpha(X) \\
&= w - E[X] + (1-\alpha)(E[X] - X).
\end{aligned}
$$

Note that $Y_1 = w - E[X]$ is independent of α and Y_α is a mean-preserving spread of $Y_1 = w - E[X]$. Therefore, a risk-averse individual would prefer Y_1 over Y_α for all $0 \leq \alpha < 1$. This implies that the optimal insurance level is $\alpha^\star = 1$.

Next, consider a deductible insurance. We have $I_d(x) = (x-d)^+$ and $P(d) = E[(X-d)^+]$ for all $d \geq 0$. For an insurance level $d \geq 0$, the terminal wealth of the individual is

$$
\begin{aligned}
Y_d &= w - X - P(d) + I_d(X) \\
&= w - X - E[(X-d)^+] + (X-d)^+ \\
&= w - E[X] + (E[X \wedge d] - X \wedge d),
\end{aligned}
$$

where the last equality follows from the fact that $X = (X-a)^+ + X \wedge a$ for all $a \geq 0$. Since Y_0 is independent of the deductible level d and Y_d is a mean-preserving spread of Y_0, the optimal insurance level for a risk-averse individual is $d^\star = 0$.

The arguments for the above two cases bear some resemblance. It suggests that a similar argument might hold in the general case. The next theorem confirms this intuition.

Theorem 3.1. *If the premium is fair, a risk-averse individual will purchase full insurance.*

Proof. In the general case, the terminal wealth of the individual for an insurance level l is given by

$$
\begin{aligned}
Y_l &= w - X - P(l) + I_l(X) \\
&= w - X - E[I_l(X)] + I_l(X) \\
&= w - E[X] + (E[X - I_l(X)] - (X - I_l(X)))\,.
\end{aligned}
$$

In particular, we have $Y_m = w - E[X]$ which is deterministic and independent of the insurance level l. Since Y_l is a mean-preserving spread of Y_m for $l \neq m$, it is less preferred by a risk-averse individual. This shows that the optimal insurance level is full insurance. $\square$

3.2 The loaded premium case

In Section 3.2, we assume that $\lambda > 0$. Similar to the previous subsection, we first investigate coinsurance and deductible insurance to gain some intuition. For an insurance level $0 \leq \alpha \leq 1$, the premium of a coinsurance contract equals $P(\alpha) = \alpha(1+\lambda)E[X]$. Now the terminal wealth of an individual equals

$$\begin{aligned} Y_\alpha &= w - X - \alpha(1+\lambda)E[X] + \alpha X \\ &= w - (1+\lambda)E[X] + (1-\alpha)\lambda E[X] + (1-\alpha)(E[X] - X). \end{aligned}$$

It is clear that $Y_1 = w - (1+\lambda)E[X]$ is non-random and independent of α. Let $\pi(Y_\alpha)$ be the risk premium for Y_α. If the individual is risk-averse, then $\pi(Y_\alpha) > 0$ and

$$Y_\alpha \sim w - (1+\lambda)E[X] + (1-\alpha)\lambda E[X] - \pi(Y_\alpha).$$

If $\pi(Y_\alpha) = o((1-\alpha)E[X]) \quad (\alpha \to 1^-)$[5], then there exists some $0 < \delta_1 < 1$ such that $(1-\alpha)\lambda E[X] - \pi(Y_\alpha) > 0$ for all $\delta_1 < \alpha < 1$. Therefore, partial insurance is optimal. If $\pi(Y_\alpha) \neq o((1-\alpha)\lambda E[X]) \quad (\alpha \to 1^-)$, then it is possible that $(1-\alpha)\lambda E[X] - \pi(Y_\alpha) < 0$ for all $0 \leq \alpha \leq 1$ in which case full insurance is optimal. Otherwise, partial insurance will be optimal.

For an insurance level $d \geq 0$, the premium $P(d)$ of a deductible insurance equals $(1+\lambda)E[X]$ and the terminal wealth of an individual is

$$\begin{aligned} Y_d &= w - X - (1+\lambda)E[(X-d)^+] + (X-d)^+ \\ &= w - (1+\lambda)E[X] + \lambda E[X \wedge d] + (E(X \wedge d) - X \wedge d). \end{aligned}$$

[5] Multiplication by a constant does not change the asymptotic order. Therefore, $\pi(Y_\alpha) = o((1-\alpha)E[X]) \quad (\alpha \to 1^-)$ if and only if $\pi(Y_\alpha) = o(\lambda(1-\alpha)E[X]) \quad (\alpha \to 1^-)$.

Let $\pi(Y_d)$ be the risk premium of Y_d. Then

$$Y_d \sim w - (1+\lambda)E[X] + \lambda E[X \wedge d] - \pi(Y_d).$$

Similar to the coinsurance case, the optimal insurance level depends on the convergence rates of $E[X \wedge d]$ and $\pi(Y_d)$. Applying the preceding argument, we know that the optimal insurance level can be partial insurance or full insurance.

It turns out that this same line of reasoning may be applied to establish the following theorem for the general case.

Theorem 3.2. *Suppose $\lambda > 0$. For any loss random variable X, let Y_m be the final wealth of an individual at the full insurance level m. If $\pi(Y_l) = o(E[X - I_l(X)]) \quad (l \to m)$, a risk-averse individual will purchase partial insurance. Otherwise, a risk-avers individual will purchase partial insurance or full insurance.*

Proof. At an insurance level l, the insurance premium equals $P(l) = (1+\lambda)E[I_l(X)]$ and the terminal wealth of an individual is given by

$$\begin{aligned} Y_l &= w - X - P(l) + I_l(X) \\ &= w - X - (1+\lambda)E[I_l(X)] + I_l(X) \\ &= w - (1+\lambda)E[X] + \lambda E[X - I_l(X)] + \left(E[X - I_l(X)] - (X - I_l(X))\right). \end{aligned}$$

In particular, at the full insurance level m, the terminal wealth of the individual equals

$$Y_m = w - (1+\lambda)E[X].$$

Let $\pi(Y_l)$ be the risk premium for Y_l. Then we have

$$Y_l \sim w - (1+\lambda)E[X] + \lambda E[X - I_l(X)] - \pi(Y_l). \tag{1}$$

Note that $\lambda E[X - I_l(X)] \to 0$ and $\pi(Y_l) \to 0$ as $l \to m$. If $\pi(Y_l) = o(E[X - I_l(X)]) \quad (l \to m)$, there is an $l_1 \in \mathbb{R}^k$ such that $l_1 \neq m$ and $\lambda E[X - I_l(X)] - \pi(Y_l) > 0$ for all l satisfying

$0 < ||l - m|| < ||l_1 - m||$, where $|| \cdot ||$ is the Euclidean norm in $\mathbb{R}^k$. Thus, the individual will choose partial insurance. On the other hand, if $\pi(Y_l) \neq o(E[X - I_l(X)]) \quad (l \to m)$, it is possible that $\lambda E[X - I_l(X)] - \pi(Y_l) < 0$ for all $l \neq m$ in which case the optimal insurance level is full insurance. Otherwise, the optimal insurance level is partial insurance. $\square$

Combining Theorem 3.1 and Theorem 3.2, we arrive at a general theorem of the optimal insurance level without the expected-utility paradigm. Similar to Schelsinger (1997), we name it the *Modified Mossin's Theorem.*

Theorem 3.3. ***(Modified Mossin's Theorem)*** *Suppose the maximum coverage of an insurance policy provides full coverage of the potential loss.*

(i) If the premium is fair, a risk-averse individual will purchase full insurance.

(ii) If the premium is loaded, a risk-averse individual will purchase (a) partial insurance if $\pi(l) = o(E[X - I_l(X)])$ *, or (b) partial insurance or full insurance if* $\pi(l) \neq o(E[X - I_l(X)])$.

Theorem 3.3 gives a unified treatment of the optimal insurance level problem for a wide range of types of insurance contracts. In particular, the Modified Mossin's Theorem for coinsurance in Schelisnger (1997) is a special case of it. When specialized to deductible insurance, Theorem 3.3 addresses a case not covered by Machina (1995); see Section 4 for more details. To our knowledge, no work has been done to investigate upper-limit insurance or deductible insurance with an endogenous upper-limit in the non-expected utility framework. But Theorem 3.3 covers these two cases too.

We conclude this section with an interesting observation. If the individual is risk-seeking or risk-neutral, then $\pi(Y_l) < 0$ or $\pi(Y_l) = 0$, respectively. Since $E[X - I_l(X)] \geq 0$ always holds, Equation (1) shows that we will have $Y_l \succ Y_m$ for all $l \neq m$. That is, if the premium is loaded, partial insurance is optimal for risk-seeking or risk-neutral individuals.

4 Three commonly used types of policies

This section further examines coinsurance, deductible insurance and upper-limit insurance without the expected-utility framework.

4.1 Coinsurance

Coinsurance has been carefully studied by Schlesinger (1997). He used the notions of risk order 1 and risk order 2 to study the loaded premium case. In view of Segal and Spivak (1990), risk order 2 for coinsurnace means $\pi(Y_\alpha)$ and $(1-\alpha)^2$ have the same asymptotic order as $\alpha \to 1^-$ and hence $\pi(Y_\alpha) = o(1-\alpha) \quad (\alpha \to 1^-)$; risk order 1 means $\pi(Y_\alpha) \neq o(1-\alpha) \quad (\alpha \to 1^-)$. Since $E[X - I_l(X)] = (1-\alpha)\lambda E[X]$ has the same asymptotic order as $(1-\alpha)$, risk order 2 implies $\pi(Y_\alpha) = o(E[X - I_l(X)]) \quad (\alpha \to 1^-)$. Likewise, risk order 1 means $\pi(Y_\alpha) \neq o(E[X - I_l(X)]) \quad (\alpha \to 1^-)$. Therefore, our Theorem 3.3 specializes to the Modified Mossin's Theorem for coinsurance in Schlesinger (1997).

4.2 Deductible insurance

Mossin's Theorem for deductible insurance without the expected-utility framework was previously investigated by Machina (1995). His Theorem 2 showed that Mossin's Theorem holds in his setup if the cumulative distribution function $F(x)$ of the loss X is continuous. Later on, Karni (1995) argued that Machina's setup basically implies risk order 2. As a matter of fact, the special structure of deductible insurance allows us to establish a stronger result: Mossin's Theorem for deductible insurance is robust enough to hold without the expected-utility framework as far as the mild condition "$F(0) = 0$" is satisfied. Most papers in the literature assume that the loss random variable X is continuous; in this case, the condition "$F(0) = 0$" holds trivially.

Theorem 4.1. *For a deductible insurance contract, a risk-averse individual will purchase* *(i) full insurance if the premium is fair;*

(ii) partial insurance if the premium is loaded and $F(0) = 0$.

In particular, if X is continuous, then the above two statements hold.

Proof. In view of Theorem 3.3, it suffice to prove the case $\lambda > 0$. As shown in Theorem 4.1 of Hong (2018a), if $F(0) = 0$, then there exists $0 < d_0 < L$ such that $P(d_0) + d_0 < P(0)$, where

$$P(d) = (1 + \lambda)E[(X - d)^+].$$

Since $X \wedge d \le d$ for all $0 \le d \le L$, we have

$$\begin{aligned} Y_{d_0} &= w - P(d_0) - X \wedge d_0 \\ &\succeq w - P(d_0) - d_0 \\ &\succeq w - P(0) = Y_0. \end{aligned}$$

Therefore, partial insurance is optimal. $\square$

Note that Theorem 4.1 does not require $\pi(Y_d) = o(E[X - I_d(X)]$. Also, it implies Theorem 2 in Machina (1995).

What if $F(0) > 0$? In this case, (i) of Theorem 4.1 still holds. For the loaded premium case, we can apply Theorem 3.3. For a deductible level d, we have $E[X - I_l(X)] = E[X \wedge d]$ and $\lim_{d \to 0^+} \frac{E[X \wedge d]}{d^2} = \lim_{d \to 0^+} \frac{F(d)}{d} = \infty$, i.e., $E[X \wedge d]$ decreases to d slower than d^2 as $d \to 0^+$. Now if $\pi(d)$ decreases to 0 no slower than d^2, Theorem 3.3 implies that the optimal insurance level will be partial insurance. Otherwise, the optimal insurance level can be full insurance or partial insurance.

4.3 Upper-limit insurance

It turns out that Mossin's Theorem for upper limit insurance (Schlesinger 2006) is robust enough to hold under the condition "$L > 1$". This condition is fairly mild since most losses are expected to be much larger than 1.

Theorem 4.2. *Let L be the supremum of the support of the loss X. For an upper-limit insurance policy, a risk-averse individual will purchase*

(i) full insurance if the premium is fair;

(ii) partial insurance if the premium is loaded and $L > 1$.

Proof. We only need to show (ii). Note that

$$\begin{aligned} Y_c &= w - X - P(c) + I_c(X) \\ &= w - P(c) - (X - c)^+ \\ &\geq w - P(c) - (L - c), \end{aligned}$$

and $Y_L = w - P(L)$. If $L > 1$, then there exists some $1 < c_0 < L$ such that

$$L - c_0 < (1+\lambda)c_0(L - c_0) < (1+\lambda)\int_{c_0}^{L} xdF(x),$$

which is equivalent to

$$(1+\lambda)\int_0^{c_0} xdF(x) + (L - c_0) < (1+\lambda)\int_0^{L} xdF(x).$$

That is, $P(c_0) + (L - c_0) < P(L)$. Therefore, $Y_{c_0} \succeq Y_L$. This shows that partial insurance is optimal. □

5 Back to the expected-utility paradigm

Here we apply the results in Schlesinger (1997) and this paper to provide some new insights into Mossin's Theorem in the expected-utility theory. We still focus on the three types of insurance contracts in Section 4.

5.1 Coinsurance

In the expected utility framework, it is usually assumed that the preferences of the individual are represented by the von Neumann-Morgenstern utility function u, $u > 0$, and u is twice differentiable. Under such an assumption, risk-aversion is tantamount to $u'' < 0$. In general, risk-aversion is only equivalent to saying that u is concave; see, for instance, Proposition 9 of Cohen (1995). A natural question is what we can say if we only assume the individual is risk-averse, i.e, u is concave but not necessarily twice differentiable. Clearly, Theorem 3.3 shows that a fair premium still implies full insurance. But the Modified Mossin's Theorem in Schlesinger (1997) and Proposition 2 of Segal and Spivak (1990) suggest that it might be possible that a risk-averse individual (an expected utility maximizer) will prefer full insurance when the premium is loaded. In addition, our Theorem 3.2 suggests that the utility function should not be differentiable at Y_1 for such an example. Following these "guidelines", one can find such an example like the next one.

Example 5.1. Consider a coinsurance contract with the coinsurance factor $0 \le \alpha \le 1$, where α is the proportion of the loss to be reimbursed to the policyholder. Suppose $w = 40, \lambda = 0.1$, loss X has a binary distribution such that $P[X = 10] = P[X = 20] = \frac{1}{2}$, and the utility function of a risk-averse individual is

$$u(x) = \begin{cases} \sqrt{x}, & 0 \le x \le 23.5; \\ \sqrt[3]{x} + (\sqrt{23.5} - \sqrt[3]{23.5}), & x > 23.5. \end{cases}$$

It is easy to see that u is concave, increasing and differentiable everywhere except at 23.5. For an insurance level $0 \le \alpha \le 1$, we have

$$Y_\alpha = 40 - 16.5\alpha - (1 - \alpha)X.$$

Hence, the corresponding expected utility is

$$\begin{aligned} E[u(Y_\alpha)] &= \frac{1}{2}u(30 - 6.5\alpha) + \frac{1}{2}u(20 + 3.5\alpha) \\ &= \frac{1}{2}[(\sqrt{23.5} - \sqrt[3]{23.5}) + (\sqrt{20 + 3.5\alpha} + \sqrt[3]{30 - 6.5\alpha})]. \end{aligned}$$

Let $g(\alpha) = \sqrt{20+3.5\alpha} + \sqrt[3]{30-6.5\alpha}$. Then

$$\begin{aligned} g'(\alpha) &= \frac{3.5}{2\sqrt{20+3.5\alpha}} - \frac{6.5}{3(\sqrt[3]{30-6.5\alpha})^2} \\ &= \frac{10.5(\sqrt[3]{30-6.5\alpha})^2 - 13\sqrt{20+3.5\alpha}}{6\sqrt{20+3.5\alpha}(\sqrt[3]{30-6.5\alpha})^2} \\ &> 0, \quad \text{for all } 0 < \alpha < 1. \end{aligned}$$

This shows that $g(\alpha)$ is increasing on $[0,1]$ which implies $E[u(Y_\alpha)] \leq E[u(Y_1)] = 23.5$. Therefore, full insurance is optimal even if the premium is loaded.

In fact, the condition "u' is continuous and positive" is sufficient to imply that partial insurance is optimal for a risk-averse expected utility maximizer when the premium is loaded.

Theorem 5.1. *For a coinsurance contract, a risk-averse individual will purchase*

(i) full insurance if the premium is fair;

(ii) partial insurance if the premium is loaded and his/her utility function is increasing and continuously differentiable.

Proof. By Theorem 3.3, we only need to establish (ii). For all $0 < \Delta < 1$, we have

$$\begin{aligned} & E[u(Y_1)] - E[u(Y_{1-\Delta})] \\ =& \int_0^L [u(w-(1+\lambda)E[X]) - u(w-(1-\Delta)(1+\lambda)E[X] - \Delta x)]\, dF(x) \\ =& \int_0^L u'(\xi_x^\Delta)\,[-\Delta(1+\lambda)E[X] + \Delta x]\, dF(x), \end{aligned}$$

where the last step follows from the mean value theorem with $\min\{(w-(1+\lambda)E[X]), (w-(1-\Delta)(1+\lambda)E[X]-\Delta x)\} < \xi_x^\Delta < \max\{(w-(1+\lambda)E[X]), (w-(1-\Delta)(1+\lambda)E[X]-\Delta x)\}$. Since u' is continuous on the compact set $[0,L]$, it is bounded. By the Dominated Convergence Theorem, we have

$$\begin{aligned} \liminf_{\Delta\to 0^+} \frac{E[u(Y_1)] - E[u(Y_{1-\Delta})]}{\Delta} &= \int_0^L \lim_{\Delta\to 0^+} u'(\xi_x^\Delta)[-(1+\lambda)E[X] + x] dF(x) \\ &= -\lambda E[X] \int_0^L u'(w-L) dF(x) < 0. \end{aligned}$$

Therefore, partial insurance is optimal. $\square$

5.2 Deductible insurance

Theorem 4.1 holds without the expected-utility paradigm. Thus, it must hold in the expected-utility model. In particular, it shows that if X is continuous, then Mossin's Theorem will hold for deductible insurance, bypassing those involved conditions in Schlesinger (1981) and Meyer and Ormiston (1999).

In view of Example 5.1, it is natural to ask whether it is possible that a risk-averse expected utility maximizer will prefer full insurance when the premium of a deductible insurance contract is loaded. The answer is affirmative, as pointed out by Gollier and Schlesinger (1996). Theorem 4.1 shows that this is possible only if $F(0) > 0$.

Example 5.2. Consider a deductible insurance contract with a positive loading factor $\lambda = 0.1$. Suppose that $w = 160$, the distribution of the loss random variable X is given by $P[X = 0] = P[X = 100] = 0.5$, and the utility function of a risk-averse individual is given by

$$u(x) = \begin{cases} \sqrt{x}, & 0 \leq x \leq 105; \\ \frac{1}{2}(\sqrt{x} + \sqrt{105}), & x > 105. \end{cases}$$

Evidently, u is concave, increasing and differentiable everywhere except at $x = 105$. The insurance premium at deductible level $0 \leq d \leq 100$ equals $P(d) = (1 + \lambda)E[(X - d)^+] = 55 - 0.55d$. The corresponding terminal wealth of the individual is given by

$$\begin{aligned} Y_d &= 160 - X - (55 - 0.55d) - (X - d)^+ \\ &= 105 + 0.55d - X \wedge d. \end{aligned}$$

It follows that

$$\begin{aligned} E[u(Y_d)] &= \frac{1}{2}u(105 + 0.55d) + \frac{1}{2}u(105 - 0.45d) \\ &= \frac{1}{4}(\sqrt{105} + \sqrt{105 + 0.55d} + 2\sqrt{105 - 0.45d}), \\ &\leq 105 = E[u(Y_0)]. \end{aligned}$$

where the last inequality follows from the fact that the function $h(d) = \sqrt{105 + 0.55d} + 2\sqrt{105 - 0.45d}$ is increasing for $d \geq 0$, which can be verified by straightforward calculus as in Example 5.1. Thus, the optimal insurance level is full insurance.

5.3 Upper-limit insurance

Similar to Section 5.1 and Section 5.2, we give an example to show that a risk-averse expected utility maximizer (whose utility function is concave but not necessarily twice differentiable) might choose full insurance if the premium of an upper-limit insurance is loaded. Theorem 3.2 shows that this is possible if the individual's utility function is not differentiable at Y_L. Theorem 4.2 provides further direction for finding such an example: the loss support L must be below 1.

Example 5.3. Consider an upper-limit insurance with a cap c for the indemnity amount. Assume $w = 2$, $\lambda = 0.1$, and the loss X has the distribution $P[X = 0] = P[X = 1] = 0.5$. The individual's utility function is

$$u(x) = \begin{cases} x, & 0 \leq x \leq 1.45; \\ 0.2x + 1.16, & x > 1.45. \end{cases}$$

For an upper-limit level $0 \leq c \leq 1$, the insurance premium equals $P(c) = (1 + \lambda)E[X \wedge c] = 0.55c$ and the corresponding terminal wealth is given by

$$Y_c \;=\; 2 - 0.55c + (X - c)^+.$$

Therefore,

$$\begin{aligned} E[u(Y_c)] &= \frac{1}{2}u(2 - 0.55c) + \frac{1}{2}u(1 + 0.45c) \\ &= 1.28 + 0.17c \\ &\leq 1.45 = E[u(Y_1)]. \end{aligned}$$

Hence, full insurance is optimal.

6 Concluding remarks

The problem of the optimal insurance level is one of the fundamental problems in insurance economics (Loubergé 2013). The past few decades have witnessed many exciting developments. However, the majority of the extant results were derived for a particular type of insurance contract and within the confines of the expected-utility framework. In the expected utility model, Hong (2018b) recently proposed a unified approach to investigating many different types of insurance policies under a single framework, and proved that the celebrated Mossin's Theorem holds for the class of insurance policy whose maximum coverage fully covers the potential loss. In this paper, we employed the same unified framework to investigate the problem of the optimal insurance level without the expected-utility framework. Similar to Schlesinger (1997), we found that Mossin's Theorem needs to be modified in general as we see in Theorem 3.3. Moreover, we made a further study of coinsurance, deductible insurance and upper-limit insurance. For deductible and upper-limit policies, we provided some new sufficient conditions for Mossin's Theorem to hold without the expected-utility paradigm. Finally, we applied the results in Schlesinger (1997) and this paper to gain further insights into Mossin's Theorem in the expected-utility framework. In particular, we showed that various versions of Mossin's Theorem, such as those established in Mossin (1968), Schlesinger (1981, 2006), rely heavily on the assumption that the utility function is twice differentiable. If the individual is risk-averse but his/her utility function is not necessarily twice differentiable, then the second part of Mossin's Theorem may not hold in the expected-utility framework.

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