Impact of Preferences on Optimal Insurance in the Presence of Multiple Policyholders

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Abstract

In the optimal insurance literature one typically studies optimal risk sharing between one insurer (or reinsurer) and one policyholder. However, the insurance business is based on diversification benefits that arise when pooling many insurance policies. In this paper, we first show that results on optimal insurance that are valid in the case of a single policyholder extend to the case of multiple policyholders, provided their insurance claims are independent. However, due to natural catastrophes, increasing life expectancy and terrorism events, insurance claims show tendency to be correlated. Interestingly, in the case of interdependent insurance policies, it may become optimal for the insurer to refuse selling insurance to some prospects, based on their attitude towards risk or due to their risk exposure characteristics. This finding calls for government policies to ensure that insurance stays available and affordable to everyone.

Key-words: Multiple policyholders, systematic risk, coinsurance, indifference price, insurance supply, public policy.

1 Introduction

Insurance business is driven by various agents who share and transfer risks among themselves. It provides an effective risk management tool for participants and plays a crucial role in economic development and social welfare. Despite the observation that the standard situation in practice is that of an insurer providing coverage to a large pool of policyholders against their individual losses, most of the literature dealing with the demand for insurance considers a theoretical setting in which there is only one policyholder and one insurer.

Assuming that the premium of the insurance contract is proportional to its expected value, Arrow (1963) shows that stop-loss insurance (also called deductible insurance) is optimal for a risk-averse policyholder if the insurer is risk neutral. Other fundamental papers on the optimal insurance demand followed such as Mossin (1968) and Smith (1968). In a similar setting as proposed in Arrow (1963), it was shown in particular that it is never optimal to purchase full insurance if the premium is strictly larger than the expected value of the policy. Phelps (1976) studies optimal coinsurance level assuming that the insurance contract provides a coverage that is proportional to the loss incurred up to some maximum value and the premium is computed from the expected value principle. Arrow's model has been extended in many directions. For instance, the objective function to be optimized has been changed from expected utility to risk measures (e.g. Kaluszka (2004) for the mean-variance setting, Cai et al. (2008) for Value-at-Risk and Tail Value-at-Risk, Cai and Weng (2016) for expectiles), and the premium principle has been generalized to the mean-variance principle (Kaluszka (2001)) and Wang's premium principle (Young (1999)). However, as already mentioned, all these studies have been carried out under the assumption that there are only two agents in the market, the insurer and one policyholder, which is not realistic.

There are not so many academic studies on optimal insurance design that explicitly consider the presence of multiple policyholders. Denuit and Vermandele (1998) obtain the optimal structure of an insurance portfolio when risks are exchangeable under the objective that the portfolio risk needs to be minimized in the sense of stop-loss order. Their results were extended by Cai and Wei (2012) who consider a portfolio of positive dependent risks. Cheung et al. (2014) provide the optimal insurance solution for an insurer who uses a law-invariant coherent risk measure and who cares about the worst-case scenario, that is when all risks are fully dependent (comonotonic). In all these papers, the authors assume some given dependence structure among the risks and moreover that premiums are proportional to the expected losses, in particular thus ignoring the impact of agents' preferences, and the fact that in practice, the insurance design may affect the premium principle (e.g., for the same expected loss an insurer is likely to charge more for a stop-loss insurance contract than for a quota share insurance). In this paper we study the effect of policyholders' personal information (risk exposure characteristics and preferences) on the insurer's decision and the way it changes in response to varying dependence, i.e., when moving from the best-case scenario of independent risks to the worst-case scenario of perfectly dependent (comonotonic) risks.

To do so, we describe the optimization problem from an insurer's perspective subject to multivariate participation constraints from a pool of policyholders. Each policyholder is characterized by her personal information, such as available wealth, risk exposure, preferences and how her risk is related to other policyholders' risk exposures. Due to different characteristics, policyholders have various abilities of accepting insurance policies. Using this information, an insurer can compare and select policyholders to form an optimal pool of policyholders in order to maximize her own objective function.

We show that when underlying risks are mutually independent[1], all policyholders receive coverage and the optimal coverage level is driven by their preferences. Even providing full insurance to all policyholders could then be a viable option for the insurance company[2]. In fact, solutions dealing with the case of one insurer and one policyholder extend well to this case of multiple policyholders bearing independent risks. Indeed, under independence the insurer effectively deals with individual cases separately, and if there exists an optimal policy design for one policyholder, then the same design works for all others.

[1]In many cases such as car insurance, health and life insurance, this independence assumption is reasonable and approximately satisfied.

[2]Note that under the independence assumption, the law of large number guarantees that premiums that are slightly higher than the expected losses can lead to a sufficient fund for future payments when the pool of policyholders is large enough.

In many - if not all - cases, individual risks are not completely independent, as they tend to be driven by common events, such as natural catastrophes, financial crises, man-made disasters and longevity risk. In property insurance, a natural catastrophe can cause hundreds of insurance claims in a short period. As pointed out by American Academy of Actuaries (2001), it has long been recognized that it is appropriate to separate the expected loss component for property insurance coverages into two parts. One component determines the provision for noncatastrophe losses and the other component develops a provision for catastrophe losses. In short, in various cases of interest the insurer cannot neglect apparently positive dependence among risks. Motivated by this phenomena, we study in this paper the case in which all individual risks are (to some extent) exposed to the same systematic risk.

We show that due to the existence of a dependent part, the insurer typically has incentive to exclude some policyholders and to target only those who are sufficiently risk averse[3] and have larger variance of their risk, as they are willing to afford a higher premium. Moreover, the stronger the dependence, the lower the number of policyholders that will be offered insurance by the insurer. As a consequence, if insurers are allowed to operate freely, the supply for insurance may not match the demand among all policyholders. In this case, regulation and government may need to step into the insurance market, and protect policyholders who are not attractive enough to insurers. Such phenomena can be observed in the catastrophe insurance market (American Academy of Actuaries (2001)). For instance, private insurance companies treat flood peril as a non-insurable risk and exclude it from homeowners insurance policies in the United States. As argued in the literature (Anderson (1974), Michel-Kerjan and Kousky (2010)), the two main reasons are that the premiums of flood insurance are too expensive to be accepted by policyholders, and that collected premiums would be insufficient for the insurer to cover future payments.

The rest of the paper is organized as follows. In Section 2 we lay out the mathematical framework and formulate the optimal allocation problem for an insurer with multivariate risks. Section 3 provides quantitative analyses under exponential utility objectives. We consider two extreme scenarios, namely when all risks are either independent or fully dependent (comonotonic), and point out the risk on insufficient insurance supply in the later one. For the intermediate case we use a normal approximation and confirm the risk on insufficient coverage in that the insurer will decrease coverage when the weight of the systematic risk component increases. In Section 4, we assess the robustness of these findings in the sense that we use a mean-variance approach instead of expected utility maximization and also assume that premiums are proportional to the expected losses. We confirm similar phenomena as in Section 3. Finally, conclusions are provided in Section 5. All proofs are relegated to an appendix.

[3]In this regard, note that along with the development of data mining and analysis technique, obtaining personal information becomes possible for an insurer.

2 Optimal insurance with multiple policyholders

2.1 The insurer's optimization problem

We assume that there is one insurer and $n \geq 1$ policyholders in the market. The insurer has initial wealth equal to w_0, whereas each policyholder has initial wealth w_i and is exposed to a risk X_i, $i = 1, \dots, n$.

Independence among individual risks is reasonable in many cases of interest and is commonly assumed by insurers. For example, a house fire is typically independent of the fire of another house located in another country. However, fires may be correlated, as they can be caused by similar risk-driven events: For instance, two houses in different cities of Florida may be damaged in the same insurance period because they are both affected by the same hurricane. Such risk-driven events impose a common factor, called *systematic risk*, on individual risk exposures, and increase both the likelihood and severity of individual insurance claims. Any two individual insurable risks are positively correlated through the systematic risk.

Mathematically, we assume that each individual risk can be decomposed into an independent part and a market-dependent part, that is

$$X_i = \sigma_i \left(\sqrt{1 - \beta_i^2} Z_i + \beta_i Y \right) + \mu_i, \quad i = 1, \dots, n$$

where $\sigma_i > 0$, $\mu_i > 0$, $\beta_i \in [0, 1]$ for $i = 1, \dots, n$, and $Z_1, \dots, Z_n, Y$ are independent random variables with zero mean and unit variance. Clearly, $\mathbb{E}[X_i] = \mu_i$ and $\mathrm{Var}(X_i) = \sigma_i^2$ for any choice of $\beta_i \in [0, 1]$. In other words, β_i characterizes the correlation between X_i and Y given the first and second moments of X_i. Specifically, when $\beta_i = 0$, the risk $X_i = \sigma_i Z_i + \mu_i$ is independent of the systematic risk and therefore of all other risks in the market; when $\beta_i = 1$, $X_i = \sigma_i Y + \mu_i$ fully depends on the systematic risk. If $\beta_i = 1$ for all $i = 1, \dots, n$, then all risks are comonotonic.[4] For the ease of notation, we may also write

$$X_i = p_i Z_i + q_i Y + \mu_i, \qquad \text{where } q_i = \sigma_i \beta_i \text{ and } p_i = \sigma_i \sqrt{1 - \beta_i^2}. \tag{1}$$

An insurance policy, denoted by (I, π), is constituted by an *indemnity function* or *ceded loss function* $I : \mathbb{R}_+ \to \mathbb{R}_+$[5] and its premium $\pi \geq 0$. To avoid moral hazard, an indemnity function I is assumed to be an increasing[6] and 1-Lipschitz continuous function satisfying $I(0) = 0$. The set of all feasible indemnity functions is denoted by $\mathcal{I}$. For policyholder i, $i = 1, \dots, n$, given an insurance policy (I, π), her terminal wealth becomes $T_i(I, \pi) = w_i - X_i + I(X_i) - \pi$. We assume that the insurer and all policyholders are risk averse agents and that their preferences are driven by expected utilities. Specifically, we define policyholders' preference functionals $V_i : \mathcal{I} \times [0, \infty) \to \mathbb{R}$

[4]A set of random variables $(X_1, \dots, X_n)$ is *comonotonic* if it can be written as $(X_1, \dots, X_n) = (f_1(X), \dots, f_n(X))$ for some increasing functions $f_1, \dots, f_n$ and random variable X.

[5]For simplicity, we include 0 in the set $\mathbb{R}_+$.

[6]In this paper, terms "increasing" and "decreasing" are in the non-strict sense.

via

$$V_i(I,\pi) = \mathbb{E}\left[u_i\left(w_i - X_i + I(X_i) - \pi\right)\right], \quad i = 1,\ldots,n,$$

where $u_i : \mathbb{R} \to \mathbb{R}$, $i = 1,\ldots,n$ are increasing and concave utility functions. A pair (I,π) is *acceptable* for the policyholder i if the objective $V_i(I,\pi)$ is no less than the objective $V_i(0,0)$ when no insurance is purchased, i.e., the policyholder prefers (I,π) to no insurance.

The insurer, who is also risk averse, is interested in designing n acceptable insurance policies, denoted by $(\vec{I},\vec{\pi})$, for all policyholders such that her objective is optimized. The insurer's objective function $V_0 : \mathcal{I}^n \times [0,\infty)^n \to \mathbb{R}$ is defined via

$$V_0(\vec{I},\vec{\pi}) = \mathbb{E}\left[u_0\left(w_0 + \sum_{i=1}^n \pi_i - \sum_{i=1}^n I_i(X_i)\right)\right],$$

where $u_0 : \mathbb{R} \to \mathbb{R}$ is an increasing and concave utility function. Thus, the optimization problem from the insurer's perspective becomes:

$$\begin{aligned} &\max_{\vec{I}\in\mathcal{I}^n,\vec{\pi}\in\mathbb{R}_+^n} V_0(\vec{I},\vec{\pi}) \\ &\qquad \text{s.t. } V_i(I_i,\pi_i) \geq V_i(0,0), \qquad \text{for all } i = 1,\ldots,n. \end{aligned} \tag{2}$$

We note that all results are established assuming such that all expectations involved are finite without further indication.

2.2 Policyholders' premium

The constraint (2) determines whether a policyholder can or cannot accept an insurance policy (I,π). It is intuitive and straightforward to show that the insurer wants to collect the largest possible premium that the policyholder is prepared to pay. It arises when equality in (2) is obtained, i.e., when she is indifferent between having insurance protection or not.

Definition 2.1 (Indifference Price)**.** For policyholder i, $i = 1,\ldots,n$, the *indifference price* of an indemnity function $I \in \mathcal{I}$, denoted by $\pi_i(I)$, is the quantity leading to the equality in (2). It is also the highest premium that the policyholder i, $i = 1,\ldots,n$ is prepared to pay for this indemnity function.

Mathematically, the indifference price $\pi_i(I)$ is thus implicitly determined by the equation

$$\mathbb{E}\left[u_i\left(w_i - X_i + I(X_i) - \pi_i(I)\right)\right] = \mathbb{E}\left[u_i(w_i - X_i)\right]. \tag{3}$$

The indifference price $\pi_i(I)$ in (3) has a close relation with the *premium principle of equivalent utility*, in which the premium $P(Y)$ for a risk Y is the solution to

$$u\left(w - P(Y)\right) = \mathbb{E}\left[u(w - Y)\right], \tag{4}$$

where w is the initial wealth and u is the utility function for the policyholder. More details on the principle of equivalent utility can be found in Borch (1963), Gerber and Pafumi (1998) and references therein. The key difference between (3) and (4) is that the policyholder's wealth position may still be random after buying insurance, namely when she does not received full insurance. Thus, (3) extends the classical premium principle (4) of equivalent utility to a more general formulation. The next proposition lists a series of properties that the indifference price satisfies.

Proposition 2.2. *Let $I, J \in \mathcal{I}$. The following properties hold:*

(1) Risk loading (or no undercut): $\pi(I) \geq \mathbb{E}[I(X)]$.

(2) No unjustified risk loading: $\pi(I)$ *is a constant if* $I(X)$ *is a constant.*

(3) Maximal loss (or no rip-off): $\pi(I) \leq \text{ess-sup}(I(X))$.

(4) Translation invariance: $\pi(I + c) = \pi(I) + c$ *for a constant* $c \in \mathbb{R}$.

(5) Preserving first stochastic dominance order [7] *:* $\pi(I) \leq \pi(J)$ *if* $I(X) \preccurlyeq_{sd} J(X)$.

(6) Preserving convex order[8]*:* $\pi(I) \leq \pi(J)$ *if* $I(X) \preccurlyeq_{cx} J(X)$.

(7) Continuity: $\lim_{d\downarrow 0} \pi((I - d)^+) = \pi(I) = \lim_{d\uparrow\infty} \pi(I \wedge d)$, *where* $x^+ = \max\{x, 0\}$ *and* $x \wedge y = \min\{x, y\}$ *for all* $x, y \in \mathbb{R}$.

The indifference price satisfies most properties that are typically expected for an insurance premium principle (for a more complete list of properties that are often desired, please refer to Young (2006)). In particular, *risk loading* and *preserving convex order* are typically seen as the two most fundamental properties, and they are satisfied by all practical used insurance premium principles, such as the expected premium principle, Wang's premium principle, and the standard deviation premium principle, amongst others.

The next lemma confirms the basic intuition that the optimal insurance design from an insurer's perspective requires premiums that correspond to indifference pricing.

Lemma 2.3. $(\vec{I}^*, \vec{\pi}^*)$ *is a solution to problem (2) if and only if* $\vec{\pi}^* = \vec{\pi}(\vec{I}^*) = (\pi_1(I_1^*), \ldots, \pi_n(I_n^*))$ *and*

$$\vec{I}^* \in \underset{\vec{I} \in \mathcal{I}^n}{\arg\max}\, V_0\left(\vec{I}, \vec{\pi}(\vec{I})\right).$$

Definition 2.4 (Profitable Level)**.** A portfolio of insurance policies $(\vec{I}, \vec{\pi}) \in (\mathcal{I} \times \mathbb{R}_+)^n$ is *profitable at level* Δ for the insurer, where $\Delta \geq 0$, if

$$V_0(\vec{I}, \vec{\pi}) \geq V_0(\vec{0}, \vec{0}) + \Delta. \tag{5}$$

In particular, if $\Delta = 0$, then $(\vec{I}, \vec{\pi}) \in \mathcal{I}^n \times \mathbb{R}_+^n$ is called *at least profitable*.

[7] A random variable X is larger than a random variable Y in the first stochastic dominance sense, denoted by $Y \preccurlyeq_{sd} X$, if the survival functions of X and Y satisfy $S_X(x) \geq S_Y(x)$ for all $x \in \mathbb{R}$.

[8] A random variable X is larger than a random variable Y in the convex order sense, if $\mathbb{E}[v(Y)] \leq \mathbb{E}[v(X)]$ for all convex functions v, provided expectations exist.

3 Optimal coinsurance level with exponential utilities (CARA)

In the rest of this paper, we focus on coinsurance, which is an important insurance type in practice. In a coinsurance policy, the indemnity function for policyholder i, $i = 1, \ldots, n$ is $I_i(x) = \alpha_i x$ for some coinsurance level $\alpha_i \in [0, 1]$. Since a coinsurance indemnity function I_i is determined by a sole parameter α_i, we use α_i to represent I_i. For example, we denote the indifference price for I_i by $\pi_i(\alpha_i)$. Similarly, we write $V_i(\alpha_i, \pi_i)$ instead of $V_i(I_i, \pi_i)$ and $V_0(\vec{\alpha}, \vec{\pi})$ instead of $V_0(\vec{I}, \vec{\pi})$, where $\vec{\alpha} = (\alpha_1, \ldots, \alpha_n) \in [0, 1]^n$. Furthermore, in this section, we assume that both the insurer and policyholders adopt exponential utilities, that is $u_0(x) = 1 - e^{-a_0 x}$ and $u_i(x) = 1 - e^{-a_i x}$, $i = 1, \ldots, n$, for $x \in \mathbb{R}$ for some risk aversion coefficients $a_0 > 0$, $a_i > 0$. From Lemma 2.3, if $(\vec{\alpha}^*, \vec{\pi}^*)$ maximizes the insurer's objective as in (2), then

$$\vec{\alpha}^* \in \underset{\vec{\alpha} \in [0,1]^n}{\arg\max}\, V_0\left(\vec{\alpha}, \vec{\pi}(\vec{\alpha})\right) = \underset{\vec{\alpha} \in [0,1]^n}{\arg\max}\, \mathbb{E}\left[1 - e^{-a_0\left(w_0 + \sum_{i=1}^n \pi_i(\alpha_i) - \sum_{i=1}^n \alpha_i X_i\right)}\right], \tag{6}$$

where $\vec{\pi}(\vec{\alpha}) = (\pi_1(\alpha_1), \ldots, \pi_n(\alpha_n))$ is the indifference price vector, and $\vec{\pi}^* := \vec{\pi}(\vec{\alpha}^*)$. From the equation $V_i(\vec{\alpha}, \vec{\pi}(\vec{\alpha})) = V_i(0, 0)$, we get

$$\pi_i(\alpha_i) = \frac{1}{a_i} \ln \frac{M_{X_i}(a_i)}{M_{X_i}(a_i(1 - \alpha_i))}, \quad i = 1, \ldots, n, \tag{7}$$

where $M_{X_i}(\cdot)$ is used to reflect the moment generating function of X_i.

Proposition 3.1. *The insurer's objective $V_0(\vec{\alpha}, \vec{\pi}(\vec{\alpha}))$ is a concave function on $[0, 1]^n$. If there exists $\vec{\alpha}^* \in [0, 1]^n$ such that $\nabla V_0(\vec{\alpha}^*, \vec{\pi}(\vec{\alpha}^*)) = 0$, then $V_0(\vec{\alpha}, \vec{\pi}(\vec{\alpha}))$ achieves is global maximal at $\vec{\alpha}^*$; otherwise, the maximum is achieved at the boundary of the set $[0, 1]^n$. Moreover, full insurance cannot be the optimal for the insurer, i.e., $\alpha_i^* < 1$ for all $i = 1, \ldots, n$.*

Since $\alpha_i^* \neq 1$ $(i = 1, \ldots, n)$, it follows that if the maximum is achieved on the boundary then $\alpha_i^* = 0$ for some i and some policyholders will not receive insurance. In other words, in this case the insurer selects some policyholders but excludes some others in order to achieve the highest possible utility. In the next section we will analyze the influence of the systematic risk Y on the insurer's selection. First we will assume that all risks are independent, i.e., $\beta_i = 0$, $i = 1, \ldots, n$, and next we consider the case in which all risks are assumed to be perfectly dependent (comonotonic), i.e., $\beta_i = 1$, $i = 1, \ldots, n$. Finally, we will also asses all intermediate cases using a multivariate normal model for risks X_i, $i = 1, \ldots, n$.

3.1 Optimal coinsurance when all risks are independent

We first assume that the systematic risk has no influence on the individual risks. Given a coinsurance level vector $\vec{\alpha} \in [0, 1]^n$, we rewrite the insurer's objective as

$$V_0(\vec{\alpha}, \vec{\pi}(\vec{\alpha})) = 1 - e^{-a_0 w_0} \prod_{i=1}^n \gamma_i, \tag{8}$$

in which

$$\gamma_i = M_{X_i}(a_0\alpha_i)\frac{M_{X_i}(a_i(1-\alpha_i))^{a_0/a_i}}{M_{X_i}(a_i)^{a_0/a_i}}, \quad i = 1, \ldots, n. \tag{9}$$

We can interpret γ_i as the effect on the insurer's expected utility when offering the insurance policy $(\alpha_i, \pi_i(\alpha_i))$ to the i-th policyholder: if $\gamma_i < 1$, the policy $(\alpha_i, \pi_i(\alpha_i))$ improves the insurer's objective and is profitable for the insurer. Interestingly, effects from different policyholders on the insurer's objective are separated in the sense that a given γ_i does not depend on other policyholders' characteristics and there are no interaction effects in (8). Therefore, we intuitively expect that the optimal type of an insurance policy for one policyholder works for the others as well. This intuition is confirmed in the following proposition.

Proposition 3.2 (Optimal portfolio - independence)**.** *Assume that all individual risks are independent. The optimal coinsurance levels and the corresponding optimal premiums are given as*

$$\alpha_i^* = \frac{a_i}{a_i + a_0} \quad \text{and} \quad \pi_i^* = \frac{1}{a_i}\ln\frac{M_{X_i}(a_i)}{M_{X_i}\left(\frac{a_0 a_i}{a_0+a_i}\right)}, \quad i = 1, \ldots, n.$$

When the insurer has a pool of policyholders with mutually independent risks, the optimal coinsurance levels are determined independently from each other: the presence of other policyholders does not affect the allocation of insurance for a given policyholder. All policyholders will receive protection and no policyholders are excluded. Furthermore, the optimal coinsurance level is not affected by the policyholder's initial wealth, but is increasing in her ArrowPratt measure of absolute risk-aversion. That is, the more risk averse the policyholder is, the more insurance is offered by the insurer because a more risk avers policyholder is willing to pay a higher premium. This is consistent with economic intuition as the premium paid to the insurer increases with risk aversion.

Denote by γ_i^*, $i = 1, \ldots, n$ when α_i^* is used in the expression (9).

Proposition 3.3. *Assume that all individual risks are independent. A portfolio of coinsurance policies $(\vec{\alpha}, \vec{\pi}(\vec{\alpha}))$ is profitable at level $\Delta \in [0, e^{-a_0 w_0}]$ for the insurer if and only if $\prod_{i=1}^n \gamma_i \leq 1 - e^{a_0 w_0}\Delta$. Moreover, the following statements hold true:*

1. *If the policyholder i is more risk averse than the insurer, i.e., $0 < a_0 < a_i$, then $\gamma_i \leq 1$ for any coinsurance level $\alpha_i \in [0, 1]$, which means that offering insurance is always profitable.*

2. *For any risk aversion levels $a_0, a_i > 0$, at the optimum,*

$$\gamma_i^* = \frac{M_{X_i}\left(\frac{a_0 a_i}{a_0+a_i}\right)^{\frac{a_0}{a_i}+1}}{M_{X_i}(a_i)^{\frac{a_0}{a_i}}} < 1, \quad i = 1, \ldots, n.$$

   and γ_i^* is decreasing in a_i.

It is interesting to compare two statements in Proposition 3.3. First, if the insurer is less risk averse than a policyholder, which is a reasonable assumption, then any coinsurance policy can be profitable for the insurer. Second, if the insurer uses the policyholders' information (i.e., the respective risk aversion levels of the policyholders) to determine the optimal coinsurance level $\alpha_i^* = \frac{a_i}{a_0+a_i}$ to offer to each policyholder, from one hand, she can guarantee a profitable policy regardless the policyholder's risk attitude, from the other hand, the insurer has the incentive to target the most risk averse policyholders.

When the insurer only offers profitable polices to all policyholders, the expected utility of the insurer is always increasing with respect to the number of policyholders. For instance, assume that all policyholders have the same risk attitude $a_1 > a_0$. Then

$$V_0(\vec{\alpha}^*, \vec{\pi}^*) \leq 1 - e^{-a_0 w_0} (\gamma_1^*)^n \to 1, \qquad \text{as } n \to \infty,$$

where 1 is the maximal possible expected utility for the insurer.

Example 3.1 (Gamma Distribution)**.** To illustrate the effects of the policyholder's risk attitude a_i and of the distribution of the risk X_i, we consider an example in which all risks follow gamma distributions. Suppose $X_i \sim GAM(\lambda_i, \theta_i)$ with moment generating function $M_{X_i}(t) = (1-\lambda_i t)^{-\theta_i}$, which is well defined for $t < 1/\lambda_i$. Assume that $\lambda_i < 1/a_i$. Then,

$$\gamma_i^* = \left(1 - \lambda_i \frac{a_i a_0}{a_i + a_0}\right)^{-\theta_i \frac{a_i+a_0}{a_i}} (1-\lambda_i a_i)^{\theta_i \frac{a_0}{a_i}}.$$

It follows that

$$\begin{aligned}
\frac{\mathrm{d}\gamma_i^*}{\mathrm{d}a_i} &= \gamma_i^* \frac{\theta_i a_0}{a_i^2} \left(\frac{\lambda_i \frac{a_0 a_i}{a_0+a_i}}{1-\lambda_i \frac{a_0 a_i}{a_0+a_i}} - \frac{\lambda_i a_i}{1-\lambda_i a_i} + \ln\left(\frac{1-\lambda_i \frac{a_0 a_i}{a_0+a_i}}{1-\lambda_i a_i} \right) \right) \\
&\leq \gamma_i^* \frac{\theta_i a_0}{a_i^2} \left(\frac{\lambda_i \frac{a_0 a_i}{a_0+a_i}}{1-\lambda_i \frac{a_0 a_i}{a_0+a_i}} - \frac{\lambda_i a_i}{1-\lambda_i a_i} + \frac{\lambda_i a_i}{1-\lambda_i a_i} \frac{a_i}{a_0+a_i} \right) < 0.
\end{aligned}$$

That is, if all risks follow the same gamma distribution, the insurer can obtain larger profit from the policyholder with higher a_i, i.e. more risk averse policyholder.

Next, we consider the effect of the risk distribution. For the family of gamma distributions, the parameter λ is called a *scale* parameter and the parameter θ is called a *shape* parameter. The distribution $GAM(\lambda, \theta)$ has a heavy tail if $\theta > 1$, and a light tail if $\theta \leq 1$. In particular, $GAM(\lambda, 1)$ means the exponential distribution with mean λ. It is easy to see that

$$\frac{\mathrm{d}\gamma_i^*}{\mathrm{d}\theta_i} = \frac{\gamma_i^*}{\theta_i} \ln \gamma_i^* < 0.$$

Therefore, the insurer prefers risks with a heavier tail, i.e., bigger θ_i, given that all other parameters are the same.

3.2 Optima coinsurance when all risks are perfectly dependent

The independence among risks is a commonly used assumption in the insurance business, which typically relies on the law of large numbers. However, this assumption is increasingly criticized and challenged as it tends to be too optimistic and unrealistic in many scenarios. In a catastrophic insurance market, dependent risks are used to model the effect of a catastrophe on the insurer's loss. For property and casualty insurers, catastrophes are infrequent events that cause severe losses, injuries or property damages to a large population of exposures. It has long been recognized that it is appropriate to separate the expected loss component for property insurance coverages into two parts. One component determines the provision for noncatastrophe losses and the other component develops a provision for catastrophe losses. Increased awareness of catastrophe exposures has forced insurers and reinsurers to improve risk selection methodologies and carefully evaluate how individual risks fit into their overall aggregate exposure and capital allocation plan.

In a reinsurance market, micro correlation phenomena can be frequently observed and it suggests a reinsurer should pay attention to the dependence among aggregate risks from insurers, who bought reinsurance policies. We first quote the example of U.S. national Flood Insurance Program in Cooke et al. (2011). According to the data of flood losses, the average correlation among counties in the U.S. is 0.04, which would not be statistically distinguishable from zero at the 5% significance level. However, the average correlation among the sum of 500 randomly chosen counties is 0.71. Even if individual policyholders come to primary insurers with small correlations, the aggregate risks from multiple primary insurers may be highly correlated. In other words, micro correlation will amplify the correlation of sums of globally correlated variables, and the reinsurer may face a strong positive dependence among aggregate risks. Therefore, the independent risks assumption is inappropriate and may significantly underestimate the reinsurer's potential losses.

Among all dependent structures, comonotonicity characterizes the strongest positive correlations. In order to provide the insurer, or the reinsurer in a reinsurance contract, some insights about extreme events, we study the worst case scenario by considering a fully dependent case among the n policies, that is $X_i = \sigma_i Y + \mu_i$ for all $i = 1, \ldots, n$.

Without loss of generality, we assume $a_1\sigma_1 \leq a_2\sigma_2 \leq \cdots \leq a_n\sigma_n$. Denote

$$A_i = \frac{\sum_{j=i}^{n} \sigma_j}{\frac{1}{a_0} + \sum_{j=i}^{n} \frac{1}{a_j}}, \quad i = 1, \ldots, n, \quad \text{and } A_{n+1} = 0,$$

and

$$i^* = \min\left\{i \in \{1, \ldots, n\} : A_{i+1} < a_i\sigma_i\right\}. \tag{10}$$

Since $A_{n+1} = 0 < a_n\sigma_n$, the set in (10) is not empty and i^* is well-defined.

Proposition 3.4 (Optimal portfolio - comonotonicity). *Assume that all individual risks are comono-*

tonic, the optimal coinsurance levels are given as

$$\alpha_i^* = \begin{cases} 0, & i < i^*, \\ 1 - \frac{A_{i^*}}{a_i \sigma_i}, & i \geq i^*. \end{cases} \tag{11}$$

Moreover, the optimal coinsurance strategy $(\vec{\alpha}^*, \vec{\pi}(\vec{\alpha}^*))$ *is at most profitable for the insurer.*

Proposition 3.4 shows that in order to achieve her maximal profitable level the insurer has the incentive to exclude some policyholders, which is in sharp contrast to the situation obtained under the independence assumption. As a result, not every risk can be transferred to the insurer, and in some sense, the supply does not meet the demand. We assess this further by considering a particular example with $a_0 = 1$ and random generating all a_i and σ_i, $i = 1, \ldots, n$ in $[0.5, 1.5]$ and $[1, 2]$ respectively. Figure 1 shows the percentage of policyholders, who can receive insurance from the insurer among the entire pool, drops quickly as n increasing. Only a very small percentage of policyholders, smaller than 5% in most of cases, are offered insurance by the insurer. In practice, risks are not perfectly comonotonic thus teh real optimal percentage may be different.

Due to the existence of the systematic risk, policyholders' information (risk aversion level and risk exposure) is not only used for charging a high premium but also used to select policyholders. Figure 2 renders characteristics of policyholders selected by the insurer in a specific case when there are 10,000 policyholders with $a_i \in \{0.5 + 0.01k : k = 1, \ldots, 100\}$ (on x-axis) and $\sigma_i \in \{1 + 0.01k : k = 1, \ldots, 100\}$ (on y-axis). Policyholders selected by the insurer are concentrated in the upper right corner of the whole range. It means that only the most risk averse policyholders with the largest variances are selected by the insurer, and those policyholders only take a small percentage in the entire pool.

Our results reveal the need for regulation to ensure that everyone is offered insurance at a reasonable premium. In many cases, especially in a catastrophe insurance market, governments step in to provide alternative solutions. One example is the California Earthquake Authority (CEA) established in 1996 by the California legislature. The CEA is a privately financed, publicly managed entity, which is used to ensure the availability of residential earthquake insurance. By law, insurers writing homeowners policies in California must either offer earthquake coverage or participate financially in this program. More examples can be found in American Academy of Actuaries (2001).

Next, we focus on the group of policyholders selected by the insurer, and investigate the effect of their characteristics on insurance coverages. Without loss of generality, we assume that all policyholders can get insurance, which is equivalent to the condition $A_1 < a_i \beta_i$ for all $i = 1, \ldots, n$, or equivalently

$$a_0 \leq \min \left\{ \frac{a_i}{\sum_{j \neq i} \left(\frac{\sigma_j}{\sigma_i} - \frac{a_i}{a_j} \right)} : \frac{a_i}{\sum_{j \neq i} \left(\frac{\sigma_j}{\sigma_i} - \frac{a_i}{a_j} \right)} > 0 \text{ and } i = 1, \ldots, n \right\}$$

so that $i^* = 1$ in (10). The optimal coinsurance level vector $\vec{\alpha}^*$ given by (11) is always located in

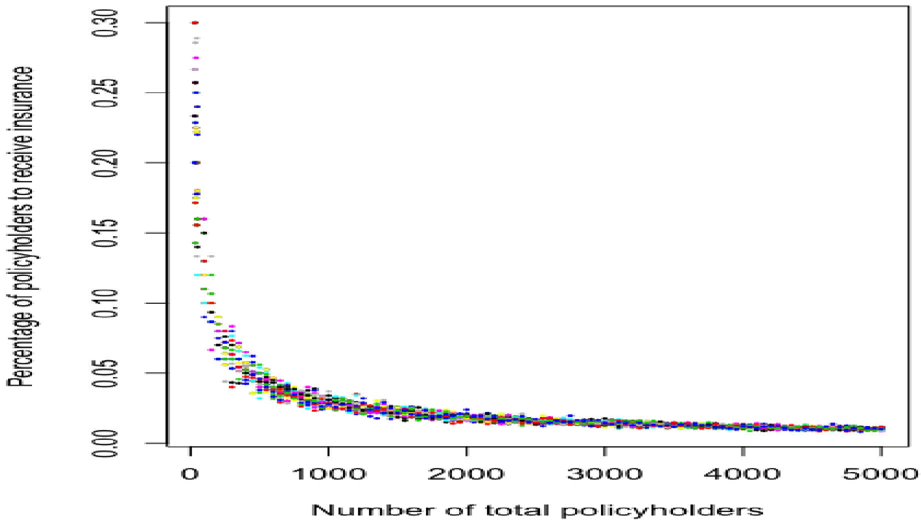

Figure 1: The percentage of policyholders receiving insurance

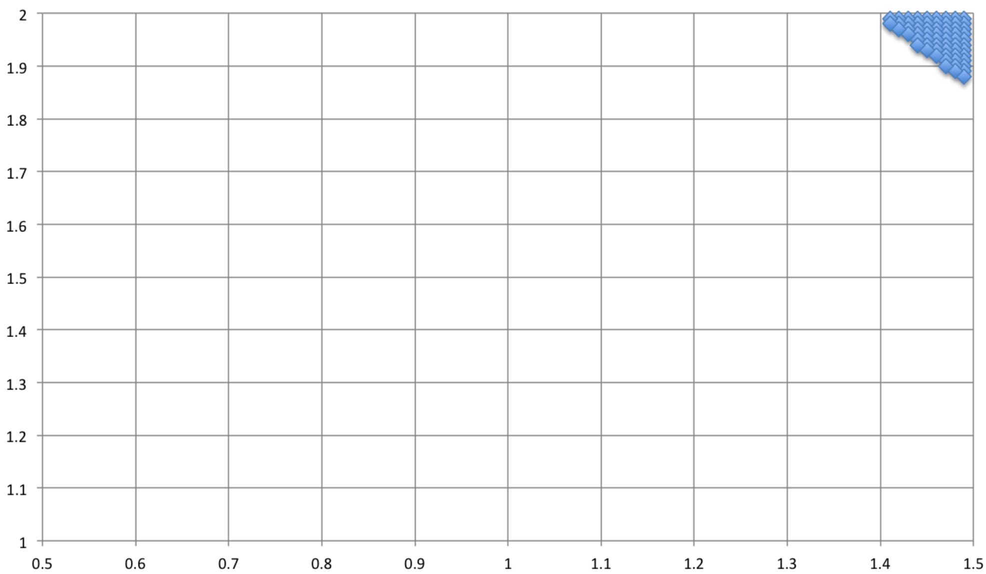

Figure 2: The characteristics of policyholders receiving insurance

$[0,1]^n$. Let we consider three special examples.

- *(Identical policyholders.)* Assume $a_i = a$ and $\sigma_i = \sigma$ for all $i = 1, \ldots, n$.

  Since $A_n = \frac{\sigma}{a_0^{-1}+a^{-1}} < a\sigma$ always holds for $a, a_0 > 0$, we have $A_n < \cdots < A_1 < a\sigma$. Then all policyholders have the same optimal coinsurance parameter $\alpha_i^* = \frac{a/a_0}{n+a/a_0}$, which is decreasing in the number of policyholders. That is, when all policyholders are identical to each other, the insurer should equally allocate coinsurance. When the pool of policyholders becomes bigger, the optimal coinsurance level for each individual policyholder decreases, while the total insurer's coinsurance level $\sum_{i=1}^n \alpha_i^* = \frac{a/a_0}{1+a/(na_0)}$ increases to $\frac{a}{a_0}$ as $n \to \infty$.

- *(Same risk attitude, different risk sizes.)* Assume $a_i = a$ for all $i = 1, \dots, n$, $\sigma_1 \le \cdots \le \sigma_n$ where at least one inequality is strict and $i^* = 1$.

  It can be easily checked that $\alpha_i^* = 1 - \frac{\bar{\sigma}/\sigma_i}{n + a/a_0}$ for $i = 1, \dots, n$ where $\bar{\sigma} = \sum_{j=1}^n \sigma_j$ and $\alpha_1^* \le \alpha_2^* \le \cdots \le \alpha_n^*$. Among the pool of policyholders, one with largest variance will be offered the largest coinsurance level. For the policyholder i, if her variance σ_i^2 increases, then her coinsurance level α_i^* becomes larger; if another policyholder's variance, say σ_j^2, $j \ne i$ increases, then α_i^* becomes smaller.

- *(Same risk size, different risk attitudes.)* Assume that $\sigma_i = \sigma$ for $i = 1, \dots, n$, and $a_1 \le \cdots \le a_n$ with at least one strict inequality and that $i^* = 1$.

  In this case, $\alpha_i^* = 1 - \frac{n}{a_i/a_0 + a_i \sum_{j=1}^n 1/a_j}$ for $i = 1, \dots, n$, and $\alpha_1^* \le \alpha_2^* \le \cdots \le \alpha_n^*$. For each policyholder i, her optimal coinsurance level α_i^* depends on other policyholders' risk attitudes a_j, $j \ne i$. More precisely, if another policyholder j becomes more risk averse, i.e. a_j is larger, then α_i^* becomes smaller. However, if she becomes more risk averse, i.e. a_i is larger, then α_i^* becomes larger.

3.3 The intermediate case

The results presented in the previous two sections suggest that the systematic risk plays a significant role in the design of the optimal pool of insurance policies. Heavy weight of the systematic risk in individual risks leads insurers to become very cautious on the exact composition of their portfolio of policies. In this section, we investigate the effect of β_i on the composition of the pool of policies sold by the insurer.

Solving $\vec{\alpha}^*$ for the intermediate case with $\beta_i \in (0,1)$, $i = 1, \dots, n$ is difficult in general. To gain insight, we use the normal approximation on the insurer's risk model. To start with, we impose the following assumptions

$$\begin{cases} Y \text{ and } Z_i,\ i = 1, \dots, n \text{ follow the standard normal distribution;} \\ \beta_1 = \cdots = \beta_n = \beta \in [0,1]; \\ a_1\sigma_1 \le \cdots \le a_n\sigma_n. \end{cases} \tag{12}$$

Under the multivariate normal assumption (12), the insurer's objective becomes $V_0(\vec{\alpha}, \vec{\pi}(\vec{\alpha})) = 1 - e^{-a_0 w_0 + \frac{a_0}{2} f(\vec{\alpha};\beta)}$, where

$$f(\vec{\alpha};\beta) = a_0 \sum_{i=1}^n \alpha_i \sigma_i^2 (1-\beta^2) + a_0 \left(\sum_{i=1}^n \alpha_i \sigma_i \sqrt{1-\beta^2} \right)^2 - \sum_{i=1}^n a_i \sigma_i^2 (2\alpha_i - \alpha_i^2).$$

Given $\beta \in [0,1]$, maximizing $V_0(\vec{\alpha}, \vec{\pi}(\vec{\alpha}))$ is equivalent to minimizing $f(\vec{\alpha};\beta)$ for $\vec{\alpha} \in [0,1]^n$. Applying the same technique used in the proof of Proposition 3.4, we can get a comparative result.

Denote by

$$B_i(\beta) = \frac{\sum_{j=i}^{n} \frac{a_i\sigma_i}{a_0(1-\beta^2)+a_j}}{\frac{1}{a_0\beta^2} + \sum_{j=i}^{n} \frac{1}{a_0(1-\beta^2)+a_j}}, \quad i = 1, \ldots, n, \quad \text{and } B_{n+1}(\beta) = 0,$$

and $i^*(\beta) = \min\{i \in \{1, \ldots, n\} : B_{i+1}(\beta) < a_i\sigma_i\}$, which is well-defined. It is easy to check that, for each $i = 1, \ldots, n$, $\frac{\mathrm{d}}{\mathrm{d}\beta} B_i(\beta) > 0$, i.e., $B_i(\beta)$ is increasing in β.

Proposition 3.5. *Under assumptions (12), the optimal coinsurance levels are*

$$\alpha_i^* = \begin{cases} 0, & i < i^*(\beta), \\ \frac{a_i}{a_0(1-\beta^2)+a_i}\left(1 - \frac{B_{i^*}(\beta)}{a_i\sigma_i}\right), & i \geq i^*(\beta). \end{cases} \tag{13}$$

The number of policyholders selected by the insurer is $n - i^*(\beta) + 1$, and it is decreasing as β is increasing.

Since the proof of Proposition 3.5 is similar to the one of Proposition 3.4 given in the Appendix, we omit it.

In Figure 3, we choose three cases when $n = 50$, $n = 500$ and $n = 5,000$, and we calculate percentages of policyholders to receive insurance. Roughly speaking, in all three cases, the percentage is decreasing in β. When $n = 50$ is small, there is a flat range for β near zero, which means that the insurer has a tolerable range for the existence of the systematic risk if it has very light influence on individual risks and the group size is small. In $n = 500$ and $n = 5,000$ cases, the "flat range" is rarely observed, and curves become steeper. Such result is consistent with the micro correlation phenomena that correlation of sums of globally correlated variables will be amplified. Thus, a pool with larger size is more sensitive to the occurrence of the systematic risk Y. In all three cases, the correlation coefficient β has a significantly impact on the insurer's selection behaviour.

4 Optimal coinsurance with a mean-variance approach

In finance, the mean-variance approach is a popular and commonly used method to choose an investment portfolio. It is an alternative to expected utility maximization, as it is typically more tractable. In addition, this approach may be easier to motivate as the mean and variance can directly be estimated from data, whereas choosing a utility function is a controversial step to maximize expected utility. In this section, we study a special setting in which some closed-form expressions can be obtained under some assumptions. First, we assume that the insurer uses a mean-variance objective:

$$V_0(\vec{\alpha}, \vec{\pi}) = \mathbb{E}\left[T_0(\vec{\alpha}, \vec{\pi})\right] - \rho_0 \mathrm{Var}\left(T_0(\vec{\alpha}, \vec{\pi})\right), \tag{14}$$

where $T_0(\vec{\alpha}, \vec{\pi}) = w_0 + \sum_{i=1}^{n} \pi_i - \sum_{i=1}^{n} \alpha_i X_i$ and $\rho_0 > 0$ is a penalty factor of the variance. In addition, we assume that, given a coinsurance level $\alpha_i \in [0, 1]$, the premium paid by policyholder i

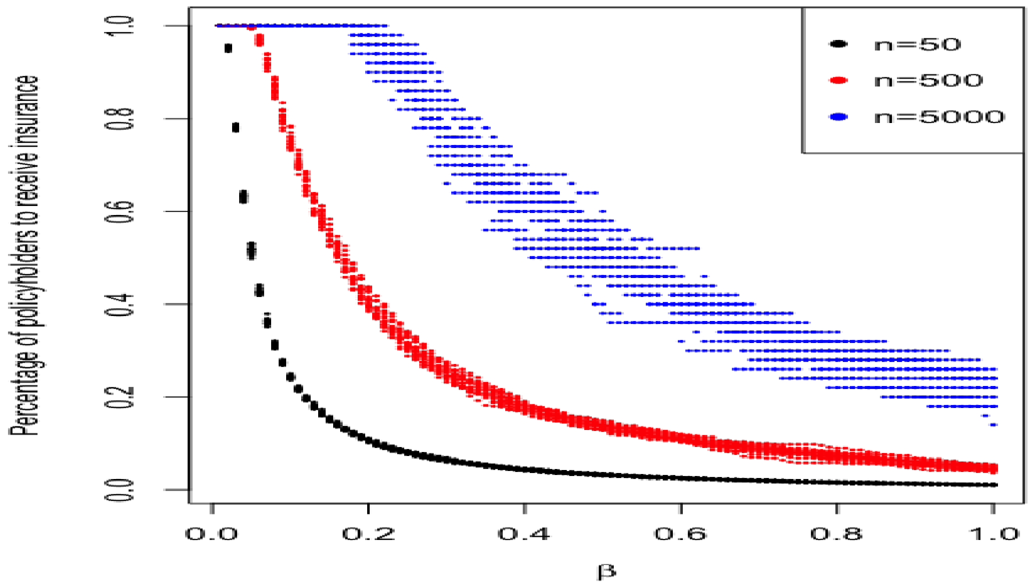

Figure 3: The characteristics of policyholders receiving insurance

is given by the expectation of the loss plus a risk loading, which is linear in α_i:

$$\tilde{\pi}_i(\alpha_i) = \mathbb{E}[\alpha_i X_i] + \delta_i \alpha_i, \quad i = 1, \ldots, n. \tag{15}$$

The premium proposed in (15) includes two widely used insurance premium principles: (i) the expected premium principle $\tilde{\pi}(\alpha_i) = (1 + \theta_i)\mathbb{E}[\alpha_i X_i]$ for some $\theta_i > 0$ (when $\delta_i = \theta_i \mu_i$), and (ii) the standard deviation premium principle $\tilde{\pi}(\alpha_i) = \mathbb{E}[\alpha_i X_i] + \rho_i \sqrt{\mathrm{Var}(\alpha_i X_i)}$ for some penalty factor $\rho_i > 0$ (when $\delta_i = \rho_i \sigma_i$). Now, the insurer's optimization problem becomes

$$\min_{\vec{\alpha} \in [0,1]^n} V_0(\vec{\alpha}, \vec{\tilde{\pi}}(\vec{\alpha})), \tag{16}$$

where $\vec{\tilde{\pi}}(\vec{\alpha}) = (\tilde{\pi}_1(\alpha_1), \ldots, \tilde{\pi}_n(\alpha_n))$ and $\tilde{\pi}_i(\alpha_i)$ is given by (15).

Proposition 4.1. *Under the mean-variance assumption, the insurer's objective $V_0(\vec{\alpha}, \vec{\tilde{\pi}}(\vec{\alpha}))$ in (16) is a concave function of $\vec{\alpha}$ on $[0,1]^n$.*

In the case of a single policyholder ($n = 1$), the problem (16) has been well-studied, see Kaluszka (2004) for example. We extend this work to the case of multiple policyholders, i.e., when $n > 1$. Due to the existence of the systematic risk Y, individual coinsurance levels may be affected by other policyholders' characteristics. The next proposition provides optimal solutions to the problem (16) in the best-scenario (independent risks) and the worst-scenario (fully dependent risks, i.e., comonotonic risks).

Proposition 4.2. *Assume $\frac{\delta_1}{\sigma_1} \leq \frac{\delta_2}{\sigma_2} \leq \cdots \leq \frac{\delta_n}{\sigma_n}$ with m strict inequality signs.*

1. *Assume that all individual risks are independent. The optimal coinsurance level for the policyholder i is* $\alpha_i^* = \min\left\{1, \frac{\delta_i}{2\rho_0\sigma_i^2}\right\}$. *The portfolio* $\left(\vec{\alpha}^*, \vec{\pi}(\vec{\alpha}^*)\right)$ *is at least profitable for the insurer.*

2. *Assume that all individual risks are comonotonic. Choose* $p(i) \in \{1, \ldots, n\}$ *such that* $\frac{\delta_{p(1)}}{\sigma_{p(1)}} < \frac{\delta_{p(2)}}{\sigma_{p(2)}} < \cdots < \frac{\delta_{p(m+1)}}{\sigma_{p(m+1)}}$. *Denote by* $\Phi_i = \left\{j : \frac{\delta_j}{\sigma_j} = \frac{\delta_{p(i)}}{\sigma_{p(i)}}\right\}$ *for* $i = 1, \ldots, m+1$. *The optimal aggregate coinsurance levels for groups* Φ_i, $i = 1, \ldots, m+1$ *are*

$$\bar{\alpha}^*_{p(i)} = \sup\left\{\alpha \in [0,1] : \alpha \sum_{j\in\Phi_i} \sigma_j + \sum_{k\geq p(i+1)} \sum_{j\in\Phi_k} \sigma_j \leq \frac{\delta_{p(i)}}{2\sigma_{p(i)}\rho_0}\right\},$$

where $\bar{\alpha}^*_{p(i)} = 0$ *if the set of the right hand side is an empty set. The optimal coinsurance levels for policyholders in the group* Φ_i *are any* $\alpha_j^* \in [0,1]$ *such that* $\sum_{j\in\Phi_i} \alpha_j^*\sigma_j = \bar{\alpha}^*_{p(i)} \sum_{j\in\Phi_i} \sigma_j$.

Proposition 4.2 can be easily obtained from Proposition 4.1, and thus we omit its proof.

Remark 4.1. In the case of comonotonic risks, the insurer does not distinguish policyholders having the same risk loading-standard deviation ratio. Therefore, in Proposition 4.2 (2), we gather all policyholders who have the same value for $\frac{\delta_i}{\sigma_i}$ together and consider them as a whole group. Then, the insurer determines optimal aggregate coinsurance levels $\bar{\alpha}^*_{p(i)}$, $i = 1, \ldots, m+1$ for all $m+1$ groups, and among each group the distribution of coinsurance levels is arbitrary as long as the weighted summation of individual coinsurance levels $\sum_{j\in\Phi_i} \alpha_j^* \frac{\sigma_j}{\sum_{j\in\Phi_i}\sigma_j}$ is equal to the group's aggregate coinsurance level $\bar{\alpha}^*_{p(i)}$.

When all risks are comonotonic, Figure 4 displays the percentage of policyholders selected with respect tp the total number of policyholders. To obtain Figure 4, we use the following four assumptions: (1) $\rho_0 = 0.1$, (2) σ_i are uniformly generated in $[1, 2]$, which is consistent with Figure 1 in the exponential utility case, (3) μ_i are uniformly generated in $[1, 10]$, (4) we use the expected premium principle and the risk loadings θ_i are uniformly generated in $[0.1, 0.5]$. As shown on this figure, the percentage of policyholders who are offered insurance sharply decreases when the number of policyholders increases.

In this simplified mean-variance setting (Problem (16)) we observe the same issue of insufficient supply of insurance as in previous sections. In fact, the situation is even worse under the setting of (16) rather than under (2). This result provides us with some insights on why there could be insufficient supply of insurance in a catastrophe insurance market given that risks are likely dependent on some systematic common factors. In particular, we find that pricing catastrophe insurance policies by using the indifference price may help to increase the percentage of policyholders who get insurance because all policyholders can then afford insurance. However, an expected utility maximization can be approximated by the mean-variance objective, and thus the insurer will still only be willing to provide insurance to a very small group of policyholders.

To better understand the effect of the presence of systematic risk on the insurance supply, we do a similar analysis as what we did in Section 3.3. From Remark 4.1, the insurer only cares about

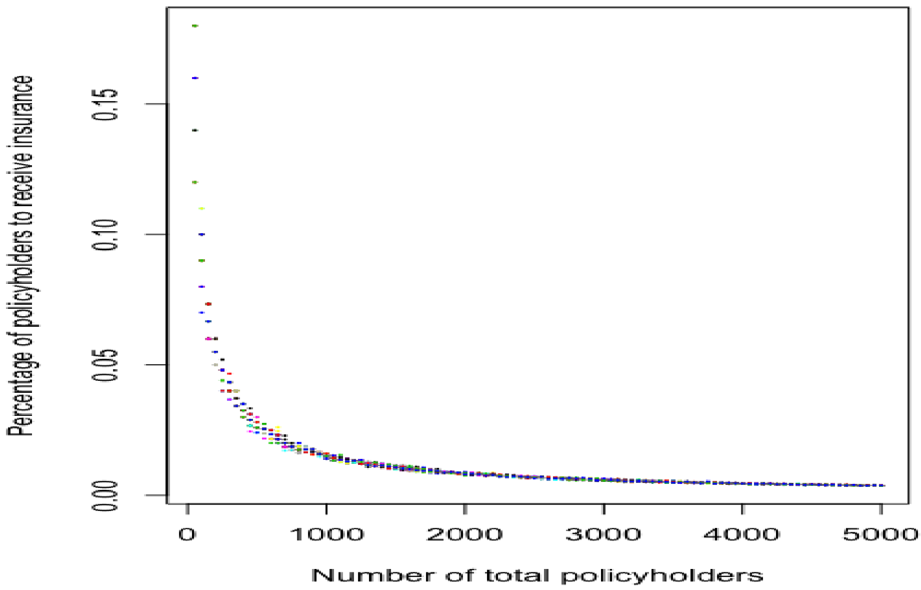

Figure 4: The worst-scenario analysis under mean-variance objective

the aggregate variance for policyholders who have the same risk loading-standard deviation ratio. Thus, for simplicity, we assume $\frac{\delta_1}{\sigma_1} < \frac{\delta_2}{\sigma_2} < \cdots < \frac{\delta_n}{\sigma_n}$.

Proposition 4.3. *Assume* $\frac{\delta_1}{\sigma_1} < \frac{\delta_2}{\sigma_2} < \cdots < \frac{\delta_n}{\sigma_n}$ *and* $\beta_1 = \cdots = \beta_n = \beta > 1$. *There exists* $i^*(\beta) \in \{1, \ldots, n\}$ *such that* $\alpha_i^*(\beta) > 0$ *for* $i \geq i^*(\beta)$ *and* $\alpha_i^*(\beta) = 0$ *for* $i < i^*(\beta)$. *The number of policyholders selected by the insurer, denoted by* $n - i^*(\beta) + 1$ *decreases when* β *increases.*

Proposition 4.3 confirms the same phenomena as in Proposition 3.5 : the insurer excludes more policyholders when the correlation coefficient β increases. The major difference of the mean-variance setting from the exponential utility setting is that, in the mean-variance setting, the insurer may prefer policyholders with small variance. To see this, we assume $\delta_1 = \theta_1\mu_1 = \cdots = \delta_n = \theta_n\mu_n$, i.e. the premium are given by the expected premium principle. Then policyholders with bigger variances have smaller risk loading-standard deviation ratio and they may not be able to get insurance, i.e., if $\sigma_1 > \cdots > \sigma_n$ then $\frac{\delta_1}{\sigma_1} < \frac{\delta_2}{\sigma_2} < \cdots < \frac{\delta_n}{\sigma_n}$.

5 Conclusion

In this paper, we study optimal risk sharing between one insurer (or reinsurer) and multiple policyholders. Using a simple toy model we show that classic results on optimal risk sharing between two agents - one being the insurance company and the other one a single policyholder - have strong limitations in that they do not seem to readily carry over as soon as policyholders are not independent (which is typically the case in the reinsurance industry or when investigating portfolios of large insurance claims).

In particular, we study the effect of systematic risk on the design of the insurer's portfolio and show that she may no longer offer insurance coverage to everyone. Moreover, it may become optimal for the insurer to exclude some policyholders based on their personal information (risk attitude) and risk exposure (fatness of the tails, variance of the losses). We focus on coinsurance and determine the optimal coinsurance levels offered by the insurer to her policyholders.

To the best of our knowledge we are among the first to study the effect of multiple dependent policyholders on optimal risk sharing in the insurance and reinsurance markets; while our model is simplified, it allows to gain some intuition and insight in the functioning of insurance markets. Our study justifies the need of public policy and possibly government intervention to ensure that insurance stays available and affordable to everyone (California Earthquake Authority (CEA) established in 1996 to ensure availability of earthquake insurance throughout California).

Appendix

Proof of Proposition 2.2.

(1) *Risk loading.* For a given $I \in \mathcal{I}$, the indifference price $\pi(I)$ satisfies $V(I, \pi(I)) = V(0,0)$. Since $(X - I(X), I(X))$ are comonotonic, we have that $(-X + I(X), I(X))$ is counter-monotonic. By Theorem 1 in Cheung et al. (2014), $-X + I(X) \preccurlyeq_{cx} -X + \mathbb{E}[I(X)]$. Adding the constant $w - \mathbb{E}[I(X)]$ on both sides, we have $w - X + I(X) - \mathbb{E}[I(X)] \preccurlyeq_{cx} w - X$ and thus $V(I, \mathbb{E}[I(X)]) \geq V(0,0) = V(I, \pi(I))$. For a given indemnity function I, we know that $V(I, \cdot)$ is a decreasing function. Thus, $\mathbb{E}[I(X)] \leq \pi(I)$.

(2) *No unjustified risk loading.* Suppose $I(X) \equiv y$ for a constant $y \in \mathbb{R}$. Take $\pi(I) = y$, and we get $V(Y, \pi(Y)) = V(0,0)$ because $w - X + I(X) - \pi(I) = w - X$. Since $\mathbb{E}[u(\cdot)]$ strictly preserves the first stochastic dominance, $\pi(I) = y$ is the only value such that $V(I, \pi(I)) = V(0,0)$ hold.

(3) *Maximal loss.* Suppose $\pi(I) > \text{ess-sup} I(X)$. There exists $\varepsilon > 0$ such that $I(X) - \pi(I) < -\varepsilon$ a.e. on Ω, and then $w - X + I(X) - \pi(I) \preccurlyeq_{sd} w - X - \varepsilon$. It follows that $V(I, \pi(I)) = \mathbb{E}[u(w - X + I(X) - \pi(I))] \leq \mathbb{E}[u(w - X - \varepsilon)] < \mathbb{E}[u(w - X)] = V(0,0)$, which contradicts to the definition of the indifference price.

(4) *Translation invariance.* Note $V(0,0) = V(I + c, \pi(I + c)) = \tilde{V}(w - X + I(X) + c - \pi(I - c)) = V(I, \pi(I + c) - c)$, which implies that $\pi(I) = \pi(I + c) - c$.

(5) *Preserving the first stochastic dominance order.* Suppose $I, f \in C(X)$ such that $I(X) \preccurlyeq_{sd} f(X)$. Then $V(f, \pi(f)) = V(0,0) = V(I, \pi(I)) \leq V(f, \pi(I))$, which implies that $\pi(f) \geq \pi(I)$.

(6) *Preserving the convex order.* Same argument in (4).

(7) *Continuity.* For notation simplicity, denote $\pi_d = \pi((I - d)^+)$ for $d > 0$. We consider the non-trivial case that $\mathbb{P}(I(X) > 0) > 0$. Since $(I(X) - d)^+$ is increasing in the first stochastic dominance order as d is decreasing, by applying the property (5), the sequence $\{\pi_d : d > 0\}$

is increasing as $d \downarrow 0$. Thus, denoted by $\pi^* = \lim_{d\downarrow 0} \pi_d$, we know π^* exists and $\pi^* > 0$. Note that, for each $d > 0$, $\pi_d \le \pi(I)$ because $(I(X) - d)^+ \preccurlyeq_{sd} I(X)$. Thus, $\pi^* \le \pi(Y)$. Suppose that $\pi^* < \pi(I)$. There exists $\varepsilon > 0$ such that $\varepsilon < \pi(I) - \pi_d$ a.e. for all $d > 0$. For all $d \in (0, \varepsilon)$, $\mathbb{P}(I(X) \wedge d - \varepsilon < 0) = 1$ and then

$$\begin{aligned} w - X + I(X) - \pi(I) &= w - X + (I(X) - d)^+ - \pi_d + I(X) \wedge d - (\pi(I) - \pi_d) \\ &\preccurlyeq_{sd} w - X + (I(X) - d)^+ - \pi_d + I(X) \wedge d - \varepsilon \\ &\preccurlyeq_{sd} w - X + (I(X) - d)^+ - \pi_d. \end{aligned}$$

where the second "$\preccurlyeq_{sd}$" is in a strict sense. Therefore, we achieve a contradiction

$$V(0, 0) = V(I(X) - \pi(I)) = \tilde{V}\left(w - X + I(X) - \pi(I)\right) < \tilde{V}\left(w - X + (Y - d)^+ - \pi_d\right) = V(0, 0).$$

We conclude that $\pi^* = \pi(I)$. Similar argument can be applied to show the equality $\pi(I) = \lim_{d\uparrow\infty} \pi(I \wedge d)$.

$\square$

Proof of Lemma 2.3. From the insurer's perspective, the insurer's objective function $V_0(\vec{I}, \vec{\pi})$ is increasing in each π_i, $i = 1, \dots, n$. For each $\vec{\pi} = (\pi_1, \dots, \pi_n)$ with π_i, $i = 1, \dots, n$ satisfying (2), we know that $\pi_i \le \pi_i(I_i)$ holds, and then

$$V_0(\vec{I}, \vec{\pi}) \le V_0(\vec{I}, (\pi_1(I_1), \pi_2, \dots, \pi_n)) \le \dots \le V_0(\vec{I}, (\pi_1(I_1), \dots, \pi_{n-1}(I_{n-1}), \pi_n)) \le V_0(\vec{I}, \vec{\pi}(\vec{I})).$$

Suppose that $(\vec{I}^*, \vec{\pi}^*)$ is a solution to problem (2). Since $\vec{\pi}_i(I_i)$, $i = 1, \dots, n$ satisfies (2) for all $I_i \in \mathcal{I}$, we have $\max_{\vec{I}\in\mathcal{I}^n} V_0(\vec{I}, \vec{\pi}(\vec{I})) \le V_0(\vec{I}^*, \vec{\pi}^*)$. Meanwhile, $\vec{\pi}^* = (\pi_1(I_1^*), \dots, \pi_n(I_n^*))$; otherwise, we know that either $V_0(\vec{I}^*, \vec{\pi}^*) < (\vec{I}^*, \vec{\pi}(\vec{I}^*))$ if $\pi_i^* \le \pi_i(I_i^*)$ for all $i = 1, \dots, n$ with at least one strict inequality, or (2) is invalid if there exists one $i \in \{1, \dots, n\}$ such that $\pi_i^* > \pi_i(I_i^*)$. Therefore, $\vec{I}^* \in \arg\max_{\vec{I}\in\mathcal{I}^n} V_0(\vec{I}, \vec{\pi}(\vec{I}))$.

Suppose that $\vec{I}^{**} \in \arg\max_{\vec{I}\in\mathcal{I}^n} V_0(\vec{I}, \vec{\pi}(\vec{I}))$. For an arbitrary pair $(\vec{I}, \vec{\pi}) \in \mathcal{I}^n \times \mathbb{R}_+^n$ satisfies (2) for each $i = 1, \dots, n$, $V_0(\vec{I}^{**}, \vec{\pi}(\vec{I}^{**})) \ge V_0(\vec{I}, \vec{\pi}(\vec{I})) \ge V_0(\vec{I}, \vec{\pi})$. Therefore, $(\vec{I}^{**}, \vec{\pi}(\vec{I}^{**}))$ is a solution to problem (2). $\square$

Proof of Proposition 3.1. Given a set of coinsurance levels $\vec{\alpha} = (\alpha_1, \dots, \alpha_n) \in [0, 1]^n$, by Lemma (2.3), the highest premium acceptable for policyholder i is the indifference price $\pi_i(\alpha_i) = \frac{1}{a_i} \ln M_{X_i}(a_i) - \frac{1}{a_i} \ln M_{X_i}(a_i(1 - \alpha_i))$, $i = 1, \dots, n$ given by (7). For each $i = 1, \dots, n$, we have $M_{X_i}(t) = e^{t\mu_i} M_{Z_i}(tp_i) M_Y(tq_i)$. Next, we need to solve $\max_{\vec{\alpha}\in[0,1]^n} V_0(\vec{\alpha}, \vec{\pi}(\vec{\alpha}))$. For notation simplicity, denote

$$V_0(\vec{\alpha}, \vec{\pi}(\vec{\alpha})) = 1 - v_0(\vec{\alpha}) e^{-a_0 w_0} \prod_{i=1}^{n} M_{X_i}(a_i)^{-\frac{a_0}{a_i}}, \tag{17}$$

and $v_0(\vec{\alpha}) = \bar{v}_i(\vec{\alpha})v_i(\vec{\alpha})$, where

$$v_i(\vec{\alpha}) = e^{a_0\mu_i} M_Y\left(a_0\sum_{k=1}^n \alpha_k q_k\right) M_Y\left(q_i a_i(1-\alpha_i)\right)^{\frac{a_0}{a_i}} M_{Z_i}\left(a_i(1-\alpha_i)p_i\right)^{\frac{a_0}{a_i}} M_{Z_i}(a_0\alpha_i p_i),$$
$$\bar{v}_i(\vec{\alpha}) = e^{a_0\sum_{k\neq i}\mu_k} \prod_{k\neq i} M_Y\left(q_k a_k(1-\alpha_k)\right)^{\frac{a_0}{a_k}} M_{Z_k}\left(a_k(1-\alpha_k)p_k\right)^{\frac{a_0}{a_k}} M_{Z_k}(a_0\alpha_k p_k).$$

For $i = 1, \ldots, n$, we have

$$\frac{\partial v_0(\vec{\alpha})}{\partial \alpha_i} = \bar{v}_i(\vec{\alpha})\frac{\partial v_i(\vec{\alpha})}{\partial \alpha_i} = a_0 v_0(\vec{\alpha})h_i(\vec{\alpha}), \tag{18}$$

where

$$h_i(\vec{\alpha}) = H(p_i Z_i; a_0\alpha_i) + H(q_i Y; a_0\sum_{k=1}^n \alpha_k \frac{q_k}{q_i}) - H(p_i Z_i; a_i(1-\alpha_i)) - H(q_i Y; a_i(1-\alpha_i)) \tag{19}$$

and $H(X;h) = \frac{\mathbb{E}\left[Xe^{hX}\right]}{\mathbb{E}\left[e^{hX}\right]}$, $\quad h > 0$ is the *Esscher premium* for a random variable X.

Observe that, given a random variable X, the function $\frac{e^{hx}}{\mathbb{E}\left[e^{hX}\right]}$ is always positive for any $x \in \mathbb{R}$ and $h \geq 0$. Take $0 \leq h_1 < h_2$,

$$\frac{e^{h_2 x}}{\mathbb{E}\left[e^{h_2 X}\right]} \frac{\mathbb{E}\left[e^{h_1 X}\right]}{e^{h_1 x}} = \frac{\mathbb{E}\left[e^{h_1 X}\right]}{\mathbb{E}\left[e^{h_2 X}\right]} e^{(h_2-h_1)x}$$

is an increasing function and up-crosses the horizontal line $y = 1$ once. That is, there exists $x_0 \in \mathbb{R}$ such that $\frac{e^{h_2 x}}{\mathbb{E}\left[e^{h_2 X}\right]} < \frac{e^{h_1 x}}{\mathbb{E}\left[e^{h_1 X}\right]}$ for $x < x_0$ and $\frac{e^{h_2 x}}{\mathbb{E}\left[e^{h_2 X}\right]} > \frac{e^{h_1 x}}{\mathbb{E}\left[e^{h_1 X}\right]}$ for $x > x_0$. Therefore, $\frac{e^{h_1 X}}{\mathbb{E}\left[e^{h_1 X}\right]} \preccurlyeq_{cx} \frac{e^{h_2 X}}{\mathbb{E}\left[e^{h_2 X}\right]}$ for all $0 \leq h_1 < h_2 < \infty$. By Lemma 3.12.13 of Muller and Stoyan (2002), we know that $\mathbb{E}\left[\phi\left(X, \frac{e^{h_1 X}}{\mathbb{E}\left[e^{h_1 X}\right]}\right)\right] \leq \mathbb{E}\left[\phi\left(X, \frac{e^{h_2 X}}{\mathbb{E}\left[e^{h_2 X}\right]}\right)\right]$ holds for any directional convex function $\phi(x,y)$ such that the expectations exists. Since $\phi(x,y) = xy$ is a directional convex function, we have $H(X;h_1) \leq H(X;h_2)$ for $0 < h_1 \leq h_2 < \infty$.

Thus, each term in $h_i(\vec{\alpha})$ is increasing in α_i, and so does $h_i(\vec{\alpha})$. Depending on the boundary values of $h_i(\vec{\alpha})$ at $\alpha_i = 0$ and $\alpha_i = 1$, $h_i(\vec{\alpha})$ either up-crosses the real line once or has no crossing point. In order to determine the minimum value of $v_0(\vec{\alpha})$, it is sufficient to check the its Hassian matrix. For any $i \neq j$,

$$\frac{\partial^2 v_0(\vec{\alpha})}{\partial \alpha_i^2} = a_0 v_0(\vec{\alpha})\frac{\partial h_i(\vec{\alpha})}{\partial \alpha_i} + a_0^2 v_0(\vec{\alpha})h_i(\vec{\alpha})^2 = a_0 v_0(\vec{\alpha})\left(\frac{\partial h_i(\vec{\alpha})}{\partial \alpha_i} + a_0 h_i(\vec{\alpha})^2\right)$$
$$\frac{\partial^2 v_0(\vec{\alpha})}{\partial \alpha_i \partial \alpha_j} = a_0 v_0(\vec{\alpha})\frac{\partial h_i(\vec{\alpha})}{\partial \alpha_j} + a_0^2 v_0(\vec{\alpha})h_i(\vec{\alpha})h_j(\vec{\alpha}) = a_0 v_0(\vec{\alpha})\left(\frac{\partial h_i(\vec{\alpha})}{\partial \alpha_j} + a_0 h_i(\vec{\alpha})h_j(\vec{\alpha})\right).$$

The Hassian matrix of $v_0(\vec{\alpha})$ is $H^e(\vec{\alpha}) = a_0 v_0(\vec{\alpha}) H_1^e(\vec{\alpha}) + a_0^2 v_0 H_2^e(\vec{\alpha})$ where

$$H_1^e(\vec{\alpha}) = \begin{bmatrix} \frac{\partial h_1(\vec{\alpha})}{\partial \alpha_1} & \frac{\partial h_1(\vec{\alpha})}{\partial \alpha_2} & \cdots & \frac{\partial h_1(\vec{\alpha})}{\partial \alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial h_n(\vec{\alpha})}{\partial \alpha_2} & \frac{\partial h_n(\vec{\alpha})}{\partial \alpha_2} & \cdots & \frac{\partial h_n(\vec{\alpha})}{\partial \alpha_n} \end{bmatrix}_{n\times n}, \quad H_2^e(\vec{\alpha}) = \begin{bmatrix} h_1(\vec{\alpha})^2 & h_1(\vec{\alpha})h_2(\vec{\alpha}) & \cdots & h_1(\vec{\alpha})h_n(\vec{\alpha}) \\ \vdots & \vdots & \ddots & \vdots \\ h_n(\vec{\alpha})h_1(\vec{\alpha}) & h_n(\vec{\alpha})h_2(\vec{\alpha}) & \cdots & h_n(\vec{\alpha})^2 \end{bmatrix}_{n\times n}.$$

It is easy to see that $H_2^e(\vec{\alpha})$ is a positive definite matrix. Denote $h_i(\vec{\alpha}) = \hat{h}_i(\vec{\alpha}) + \tilde{h}_i(\vec{\alpha})$, where $\tilde{h}_i(\vec{\alpha}) = H(p_i Z_i; a_0\alpha_i) - H(p_i Z_i; a_i(1-\alpha_i)) - H(q_i Y; a_i(1-\alpha_i))$ is increasing in α_i and $\hat{h}_i(\vec{\alpha}) = H(q_i Y; a_0 \sum_{k=1}^n \alpha_k \frac{q_k}{q_i})$ is increasing in each α_k, $k = 1, \ldots, n$. Thus, for $i, k = 1, \ldots, n$, $\frac{\partial \hat{h}_i(\vec{\alpha})}{\partial \alpha_k} = q_i q_k \left(\frac{1}{q_k} \frac{\partial}{\partial \alpha_k} H(Y; a_0 \sum_{k=1}^n \alpha_k q_k) \right) \geq 0$, and note that for all $k = 1, \ldots, n$

$$\frac{1}{q_k}\frac{\partial}{\partial \alpha_k} H(Y; a_0 \sum_{k=1}^n \alpha_k q_k) = \frac{\mathbb{E}\left[a_0 Y^2 e^{a_0 \sum_{k=1}^n \alpha_k q_k Y}\right]}{M_Y(a_0 \sum_{k=1}^n \alpha_k q_k)} - a_0 \left(\frac{\mathbb{E}\left[Y e^{a_0 \sum_{k=1}^n \alpha_k q_k Y}\right]}{M_Y(a_0 \sum_{k=1}^n \alpha_k q_k)} \right)^2$$

are same. Therefore, H_1^e can be further decomposed into the sum of the following two matrixes

$$\begin{bmatrix} \frac{\partial}{\partial \alpha_1}\tilde{h}_1(\vec{\alpha}) & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{\partial}{\partial \alpha_n}\tilde{h}_n(\vec{\alpha}) \end{bmatrix}_{n\times n},$$

which is a semi-positive definite matrix because its diagonal entries are all non-negative, and

$$\begin{bmatrix} \frac{\partial \hat{h}_1(\vec{\alpha})}{\partial \alpha_1} & \frac{\partial \hat{h}_1(\vec{\alpha})}{\partial \alpha_2} & \cdots & \frac{\partial \hat{h}_1(\vec{\alpha})}{\partial \alpha_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial \hat{h}_n(\vec{\alpha})}{\partial \alpha_1} & \frac{\partial \hat{h}_n(\vec{\alpha})}{\partial \alpha_2} & \cdots & \frac{\partial \hat{h}_n(\vec{\alpha})}{\partial \alpha_n} \end{bmatrix}_{n\times n} = \frac{1}{q_k}\frac{\partial}{\partial \alpha_k} H(Y; a_0 \sum_{k=1}^n \alpha_k q_k) \begin{bmatrix} q_1^2 & q_1 q_2 & \cdots & q_1 q_n \\ \vdots & \vdots & \ddots & \vdots \\ q_n \beta_1 & q_n q_2 & \cdots & q_n^2 \end{bmatrix}_{n\times n},$$

is also a semi-positive definite matrix. As a consequence, the Hassian matrix $H^e(\vec{\alpha})$ is semi-positive definite, which implies that $v_0(\vec{\alpha})$ is convex and $V_0(\vec{\alpha}, \vec{\pi}(\vec{\alpha}))$ is concave in $\vec{\alpha} \in [0,1]^n$. □

Proof of Proposition 3.2. For independent $X_1, \ldots, X_n$, substitute $q_i = 0$, $i = 1, \ldots, n$ into (17)-(19). Then, $V_0(\vec{\alpha}, \vec{\pi}(\vec{\alpha})) = 1 - e^{-a_0 w_0} \prod_{i=1}^n M_{X_i}(a_i)^{-\frac{a_0}{a_i}} v_0(\vec{\alpha})$, where

$$v_0(\vec{\alpha}) = \prod_{i=1}^n \left(\mathbb{E}\left[e^{a_i(1-\alpha_i)X_i}\right]^{\frac{a_0}{a_i}} \mathbb{E}\left[e^{a_0 \alpha_i X_i}\right] \right)$$

and, for $i = 1, \ldots, n$, $\frac{\partial}{\partial \alpha_i} v_0(\vec{\alpha}) = a_0 v_0(\vec{\alpha}) \left(H(X_i; a_i(1-\alpha_i) - H(X_i; a_0\alpha_i) \right)$. From the proof of Proposition 3.1, $H(X_i; a_0\alpha_i) - H(X_i; a_i(1-\alpha_i)$ is increasing in α_i, $i = 1, \ldots, n$. Thus, $\frac{\partial}{\partial \alpha_i} v_0(\vec{\alpha})$ is negative when $\alpha_i \leq \frac{a_i}{a_i + a_0}$ and is positive otherwise. Since $V_0(\vec{\alpha}, \vec{\pi}(\vec{\alpha}))$ is concave on $[0,1]^n$, the first derivative condition implies that $V_0(\vec{\alpha}, \vec{\pi}(\vec{\alpha}))$ is maximized at $\vec{\alpha}^* = \left(\frac{a_1}{a_1 + a_0}, \ldots, \frac{a_n}{a_n + a_0} \right)$. As a consequence, the optimal premium can be calculated from (7). This completes the proof. □

Proof of Proposition 3.3. Given a coinsurance level vector $\vec{\alpha} \in [0,1]^n$ and its corresponding indifference price vector $\vec{\pi}(\vec{\alpha})$, the insurer's objective is $V_0(\vec{\alpha}, \vec{\pi}(\vec{\alpha})) = 1 - e^{-a_0 w_0} \prod_{i=1}^n \gamma_i$. By Definition 2.4, $(\vec{\alpha}, \vec{\pi}(\vec{\alpha}))$ is profitable for the insurer at level $\Delta \geq 0$ if $V_0(\vec{\alpha}, \vec{\pi}(\vec{\alpha})) - V_0(\vec{0}, \vec{0}) = (1 - \prod_{i=1}^n \gamma_i) e^{-a_0 w_0} \geq \Delta$, which is equivalent to the inequality $\prod_{i=1}^n \gamma_i \leq 1 - e^{a_0 w_0} \Delta$. Here, we note that $\Delta \leq e^{-a_0 w_0}$, that is a profitable level Δ is bounded by $e^{-a_0 w_0}$.

1. It is easy to check that $\gamma_i = \gamma_i(a_i)$ is a convex function of $a_i > 0$. When $a_i = 0$, then $\gamma_i(0) = 1$; when $a_i = 1$, Hölder's inequality leads to $\gamma_i(1) = \frac{M_{X_i}(a_0)}{M_{X_i}(a_i)^{a_0/a_i}} \leq 1$. Thus, for any $a_i \in [0, 1]$, we have $\gamma_i(a_i) \leq 1$.

2. Now, we consider the optimal coinsurance policies $(\vec{\alpha}^*, \vec{\pi}^*)$. Note that $a_0 \alpha_i^* = \frac{a_0 a_i}{a_0 + a_i} = a_i(1 - \alpha_i^*)$, then we get $\gamma_i^* = \frac{M_{X_i}(a_0 \alpha_i^*)^{\frac{a_0}{a_i}+1}}{M_{X_i}(a_i)^{\frac{a_0}{a_i}}}$ and $V_0(\vec{\alpha}^*, \vec{\pi}^*) = 1 - e^{-a_0 w_0} \prod_{i=1}^n \gamma_i^*$. Since $a_0 \alpha_i^* = \frac{a_0 a_i}{a_0 + a_i} < a_i$, $i = 1, \ldots, n$, by applying the Hölder's inequality we get

$$M_{X_i}(a_0 \alpha_i^*)^{\frac{a_0}{a_i}+1} = \mathbb{E}\left[e^{\frac{a_0 a_i}{a_0 + a_i} X_i}\right]^{\frac{a_0 + a_i}{a_i}} \leq \left(\mathbb{E}\left[e^{a_i X_i}\right]^{\frac{1}{a_i}}\right)^{a_0} = M_{X_i}(a_i)^{\frac{a_0}{a_i}}.$$

Thus, $\gamma_i^* \leq 1$, $i = 1, \ldots, n$ and $\Delta = e^{-a_0 w_0} (1 - \prod_{i=1}^n \gamma_n^*) \geq 0$, i.e. the optimal portfolio coinsurance policies is profitable for the insurer at level Δ. For each $i = 1, \ldots, n$, consider $\gamma_i^* = \gamma_i^*(a_i)$ as a function of a_i. For $a_i \in (0, \infty)$ such that $\gamma_i^*(a_i)$ is well defined and finite, the function $\gamma_i^*(a_i)$ is continuous in a_i.

The first partial derivative of $\gamma_i^*(a_i)$ with respect to a_i is

$$\frac{\partial \gamma_i^*(a_i)}{\partial a_i} = \gamma_i^*(a_i) \frac{a_0}{a_i^2} \left(H\left(\frac{a_0 a_i}{a_0 + a_i} X_i; 1\right) - \ln \mathbb{E}\left[e^{\frac{a_0 a_i}{a_0 + a_i} X_i}\right] - H(a_i X_i; 1) + \ln \mathbb{E}\left[e^{a_i X_i}\right] \right).$$

Given a positive random variable $X \in \mathcal{X}$, define $g(t; X) = H(tX; 1) - \ln \mathbb{E}[e^{tX}]$ for $t > 0$ such that all expectations involved are finite. To check the monotonicity of the function $g(t)$, we consider its first derivative

$$\frac{\mathrm{d}g(t; X)}{\mathrm{d}t} = \frac{t}{\mathbb{E}\left[e^{tX}\right]^2} \left(\mathbb{E}\left[X^2 e^{hX}\right] \mathbb{E}\left[e^{hX}\right] - \mathbb{E}\left[X e^{hX}\right]^2 \right).$$

By Hölder's inequality, we have

$$\mathbb{E}\left[X e^{hX}\right] \leq \mathbb{E}\left[|X| e^{hX}\right] = \mathbb{E}\left[\left(X^2 e^{2hX}\right)^{1/2}\right] \leq \mathbb{E}\left[X^2 e^{hX}\right]^{1/2} \mathbb{E}\left[e^{hX}\right]^{1/2}.$$

Thus, $\frac{\mathrm{d}g(t;X)}{\mathrm{d}t} > 0$, or equivalently, $g(t; X)$ is increasing in t. Since $X_i \geq 0$ and $\frac{a_0 a_i}{a_0 + a_i} < a_i$, we have $g\left(\frac{a_0 a_i}{a_0 + a_i}; X_i\right) \leq g(a_i; X_i)$. Finally $\frac{\partial \gamma_i^*(a_i)}{\partial a_i} \leq 0$ implies that γ_i^* is decreasing in a_i.

□

Proof of Proposition 3.4. The following claims are useful.

1. If $A_{i+1} < a_i \sigma_i$, then $A_i > A_{i+1}$.

2. The inequalities $A_i > a_i\sigma_i$ and $A_{i+1} > a_i\sigma_i$ are equivalent. The sign " $>$ " can be replaced by " $=$ " and " $<$ ".

3. $a_{i^*-1}\sigma_{i^*-1} \le A_{i^*} < a_{i^*}\sigma_{i^*}$.

4. For $i \in \mathbb{Z}^+$ such that $i^* < i \le n$, if any, we have $A_{i+1} < A_i < a_{i-1}\sigma_{i-1} \le a_i\sigma_i$. For $i \in \mathbb{Z}^+$ such that $1 \le i < i^*$, if any, we have $a_i\sigma_i \le A_i < A_{i+1}$.

Next, we derive the optimal coinsurance levels α_i^*, $i = 1, \ldots, n$ given in (11). For $X_i = \sigma_i Y + \mu_i$, $i = 1, \ldots, n$, the insurer's objective (17) becomes $V_0(\vec{\alpha}, \vec{\pi}(\vec{\alpha})) = 1 - e^{-a_0 w_0} \prod_{i=1}^n M_Y(\beta_i a_i)^{-\frac{a_0}{a_i}} v_0(\vec{\alpha})$, where

$$v_0(\vec{\alpha}) = e^{a_0 \sum_{i=1}^n \mu_i} M_Y\left(a_0 \sum_{i=1}^n \alpha_i \sigma_i\right) \prod_{i=1}^n M_Y\left(a_i(1-\alpha_i)\sigma_i\right)^{\frac{a_0}{a_i}}.$$

From (18)-(19), for $i = 1, \ldots, n$,

$$\frac{\partial}{\partial \alpha_i} v_0(\vec{\alpha}) = e^{a_0 \sum_{i=1}^n \mu_i} a_0 v_0(\vec{\alpha}) \sigma_i \left(H\left(Y; a_0 \sum_{k=1}^n \alpha_k \sigma_k\right) - H\left(Y; a_i(1-\alpha_i)\sigma_i\right)\right).$$

Since $v_0(\vec{\alpha})$ is a continuous, differentiable and convex function on $[0,1]^n$, it has an interior minimizer if and only if its first order condition is satisfied, i.e. $\frac{\partial}{\partial \alpha_i} v_0(\vec{\alpha}^*) = 0$, $i = 1, \ldots, n$.

We first consider a simple case when $i^* = 1$. Then $A_1 < a_i\sigma_i$ and $\alpha_i^* = 1 - \frac{A_1}{a_i\sigma_i} \in (0,1)$ for all $i = 1, \ldots, n$. It follows that $\sum_{i=1}^n \alpha_i^* \sigma_i = \frac{\sum_{i=1}^n \sigma_i}{1 + a_0 \sum_{i=1}^n a_i^{-1}} = A_1/a_0 = a_i\sigma_i(1-\alpha_i^*)/a_0$, which means the first order condition for $v_0(\vec{\alpha})$ is satisfied. Therefore, $\alpha^* = (\alpha_i^*, \ldots, n\alpha_n^*)$ given in (11) minimizes the function $v_0(\vec{\alpha})$.

Suppose $2 \le i^* \le n$. It is easy to check that the first order condition cannot be satisfied by any $\vec{\alpha} \in (0,1)^n$. That is $v_0(\vec{\alpha})$ achieves its minimal value at the boundary. Note that the global minimal is also a local minimal. Obviously, $\vec{0}$ is not a local minimizer. If $\alpha_i = 1$, then $\frac{\partial}{\partial \alpha_i} v_0(\vec{\alpha}) > 0$ because $a_0 \sum_{k=1}^n \alpha_k \sigma_k > 0 = a_i(1-\alpha_i)\sigma_i$. Thus, every element in the minimizer is strictly smaller than one. Now, take $\tilde{\vec{\alpha}} \in [0,1)^n$ with $\tilde{\alpha}_i = 0$ if $i \in M \subsetneqq \{1, \ldots, n\}$ and $M \ne \emptyset$. Denote $M^c = \{1, \ldots, n\} \setminus M$. The vector $\tilde{\vec{\alpha}}$ is a local minimizer if and only if

$$a_j \sigma_j \le \frac{\sum_{k \in M^c} \sigma_k}{a_0^{-1} + \sum_{k \in M^c} a_k^{-1}} < a_i \sigma_i \text{ for } i \in M^c \text{ and } j \in M \tag{20}$$

From previous results, (20) holds if and only if $M = \{1, \ldots, i^* - 1\}$. If follows that $\vec{\alpha}^*$ given in (11) is only local minimizer on the boundary. Thus, it is the minimizer of $v_0(\vec{\alpha})$ on $[0,1]^n$.

Under the optimal coinsurance strategy $(\vec{\alpha}^*, \vec{\pi}(\vec{\alpha}^*))$, denote

$$\Delta^* = \prod_{i=1}^n M_{X_i}(a_i)^{-\frac{a_0}{a_i}} v_0(\vec{\alpha}^*) = M_Y\left(a_0 \sum_{i=1}^n \alpha_i^* \sigma_i\right) \prod_{i=1}^n M_Y\left(a_i(1-\alpha_i^*)\sigma_i\right)^{\frac{a_0}{a_i}} \prod_{i=1}^n M_Y(\sigma_i a_i)^{-\frac{a_0}{a_i}},$$

and then the insurer's objective (17) becomes $V_0(\vec{\alpha}, \vec{\pi}(\vec{\alpha})) = 1 - e^{-a_0 w_0} \Delta^*$. Note that we can derive from (11) that $a_0 \sum_{k\geq i^*} \alpha_k^* \sigma_k = a_j(1-\alpha_j^*)\sigma_j = A_{i^*}$ for any $j \geq i^*$. Thus,

$$\begin{aligned}\Delta^* &= \left(M_Y(A_{i^*})^{\frac{a_{i^*}\sigma_{i^*}}{A_{i^*}}}\right)^{\frac{a_0\sum_{k\geq i^*}\sigma_k}{a_{i^*}\sigma_{i^*}}} \prod_{k\geq i^*}^{n} \left(M_Y\left(a_k\sigma_k\right)^{\frac{1}{a_k\sigma_k}}\right)^{-a_0\sigma_k} \\ &\leq \left(M_Y(A_{i^*})^{\frac{a_{i^*}\sigma_{i^*}}{A_{i^*}}}\right)^{\frac{a_0\sum_{k\geq i^*}\sigma_k}{a_{i^*}\sigma_{i^*}}} \prod_{k\geq i^*}^{n} \left(M_Y\left(a_{i^*}\sigma_{i^*}\right)^{\frac{1}{a_{i^*}\sigma_{i^*}}}\right)^{-a_0\sigma_k} \\ &< M_Y\left(a_{i^*}\sigma_{i^*}\right)^{\frac{a_0\sum_{k\geq i^*}\sigma_k}{a_{i^*}\sigma_{i^*}}} \prod_{k\geq i^*}^{n} \left(M_Y\left(a_{i^*}\sigma_{i^*}\right)^{\frac{1}{a_{i^*}\sigma_{i^*}}}\right)^{-a_0\sigma_k} = 1,\end{aligned}$$

where the last inequality sign is guaranteed by $A_{i^*} < a_{i^*}\sigma_{i^*}$. Therefore, $V_0\left(\vec{\alpha}^*, \vec{\pi}^*\right) - V_0(\vec{0},\vec{0}) = e^{a_0 w_0}\left(1-\Delta^*\right) > 0$, which means that $(\vec{\alpha}^*, \vec{\pi}(\vec{\alpha}^*)$ is at least profitable for the insurer. $\square$

Proof of Proposition 4.1. Substitute (15) and $X_i = p_i Z_i + q_i Y + \mu_i$ for $i = 1, \ldots, n$ into (16), and we get $V_0(\vec{\alpha}, \vec{\bar{\pi}}(\vec{\alpha})) = w_0 + \sum_{i=1}^n \delta_i \alpha_i - \rho_0 \left(\sum_{n=1}^n \alpha_i^2 p_i^2 + (\sum_{i=1}^n \alpha_i q_i)^2\right)$ The first derivative of

$$V_0(\vec{\alpha}, \vec{\pi}(\vec{\alpha})) \tag{21}$$

with respect to α_i is $\frac{\partial}{\partial \alpha_i} V_0(\vec{\alpha}, \vec{\pi}(\vec{\alpha})) = \delta_i - 2\rho_0 p_i^2 \alpha_i - 2\rho_0 q_i \sum_{k=1}^n \alpha_k q_k$, $i = 1, \ldots, n$, and the Hessian matrix of $V_0(\vec{\alpha}, \vec{\pi}(\vec{\alpha}))$ is

$$H^{\mathrm{Var}}(\vec{\alpha}) = -2\rho_0 \begin{bmatrix} q_i^2 + p_1^2, & q_1 q_2, & \cdots, & q_i q_n \\ q_1 q_2, & q_2^2 + p_2^2, & \cdots, & q_2 q_n \\ \vdots & \vdots & & \vdots \\ q_1 q_n, & q_2 q_n & \cdots, & q_n^2 + p_n^2 \end{bmatrix}$$

which is a negative semi-definite matrix. Therefore, $V_0(\vec{\alpha}, \vec{\pi}(\vec{\alpha}))$ is a concave function on $[0,1]^n$. $\square$

Proof of Proposition 4.3 . For any $\beta \in (0,1)$, denote $\alpha_j^*(\beta)$, $i = 1 \ldots, n$ to be the optimal coinsurance levels. From the first derivative (21), we can get a necessary condition for $\alpha_j^*(\beta)$ that $\frac{\delta_i}{2\rho_0\sigma_i} \geq \sigma_i(1-\beta^2) + \beta^2 \sum_{k=1}^n \alpha_k^*(\beta)\sigma_k$ if $\alpha_i^*(\beta) = 1$, $\frac{\delta_i}{2\rho_0\sigma_i} = \sigma_i(1-\beta^2)\alpha_i^*(\beta) + \beta^2 \sum_{k=1}^n \alpha_k^*(\beta)\sigma_k$ if $\alpha_i^*(\beta) \in (0,1)$, and $\frac{\delta_i}{2\rho_0\sigma_i} \leq \beta^2 \sum_{k=1}^n \alpha_k^*(\beta)\sigma_k$ if $\alpha_i^*(\beta) = 0$. Suppose there exists $j < i$ such that $\alpha_j^*(\beta) > 0 = \alpha_i^*(\beta)$. Note that $\frac{\delta_j}{\sigma_j} < \frac{\delta_i}{\sigma_i}$. Then we get the following contradiction

$$\frac{\delta_i}{2\rho_0\sigma_i} \leq \beta^2 \sum_{k=1}^n \alpha_k^*(\beta)\sigma_k < \sigma_j(1-\beta^2)\alpha_j^*(\beta) + \beta^2 \sum_{k=1}^n \alpha_k^*(\beta)\sigma_k \leq \frac{\delta_j}{2\rho_0\sigma_j} < \frac{\delta_i}{2\rho_0\sigma_i}.$$

Therefore, there exists $i^*(\beta) \in \{1, \ldots, n\}$ such that $\alpha_k^*(\beta) > 0$ for $k \geq i^*(\beta)$ and $\alpha_k^*(\beta) = 0$ for $k < i^*(\beta)$. The number of policyholders receiving insurance is $n - i^*(\beta) + 1$.

Take $\beta_1 < \beta_2$. Suppose $i^*(\beta_2) < i^*(\beta_1)$. Then, for any $i \geq i^*(\beta_1)$,

$$\frac{\delta_i}{2\rho_0\sigma_i} \geq \sigma_i(1-\beta_1^2)\alpha_i^*(\beta_1) + \beta_1^2 \sum_{k\geq i^*(\beta_1)} \alpha_k^*(\beta_1)\sigma_k > \beta_1^2 \sum_{k\geq i^*(\beta_1)} \alpha_k^*(\beta_1)\sigma_k \geq \frac{\delta_{i^*(\beta_2)}}{2\rho_0\sigma_{i^*(\delta_2)}} \geq \beta_2^2 \sum_{l\geq i^*(\beta_2)} \alpha_l^*(\beta_2)\sigma_l.$$

It follows that $\sum_{l\geq i^*(\beta_2)} \alpha_l^*(\beta_2)\sigma_l \leq \sum_{k\geq i^*(\beta_1)} \alpha_k^*(\beta_1)\sigma_k$. We want to show that $\alpha_i^*(\beta_1) \leq \alpha_i^*(\beta_2)$. If $\alpha_i^*(\beta_2) = 1$, then we always have $\alpha_i^*(\beta_1) \leq \alpha_i^*(\beta_2)$. If $0 < \alpha_i^*(\beta_2) < 1$, then

$$\sigma_i(1-\beta_2^2)\alpha_i^*(\beta_2) + \beta_2^2 \sum_{l\geq i^*(\beta_2)} \alpha_l^*(\beta_2)\sigma_l = \frac{\delta_i}{2\rho_0\sigma_i} \geq \sigma_i(1-\beta_1^2)\alpha_i^*(\beta_1) + \beta_1^2 \sum_{k\geq i^*(\beta_1)} \alpha_k^*(\beta_1)\sigma_k.$$

It follows that $(1-\beta_2^2)\alpha_i^*(\beta_2) \geq (1-\beta_1^2)\alpha_i^*(\beta_1)$. Together with $1-\beta_2^2 < 1-\beta_1^2$, we have $\alpha_i^*(\beta_1) \leq \alpha_i^*(\beta_2)$ for all $i \geq i^*(\beta_1)$. Therefore, we get the contradiction

$$\sum_{k\geq i^*(\beta_1)} \alpha_k^*(\beta_1)\sigma_k \leq \sum_{k\geq i^*(\beta_1)} \alpha_k^*(\beta_2)\sigma_k < \sum_{k\geq i^*(\beta_2)} \alpha_k^*(\beta_2)\sigma_k \leq \sum_{k\geq i^*(\beta_1)} \alpha_k^*(\beta_1)\sigma_k.$$

We conclude that $i^*(\beta_1) \leq i^*(\beta_2)$ for any $\beta_1 < \beta_2$. In other words, the number of policyholders receiving insurance $n - i^*(\beta) + 1$ decrease in β. □

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